Bull. Korean Math. Soc. **50** (2013), No. 3, pp. 839–865
http://dx.doi.org/10.4134/BKMS.2013.50.3.839

POSITIVE RADIAL SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS CONCENTRATING ON SPHERES WITH POTENTIAL DECAY

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Abstract. We deal with the existence of positive radial solutions concentrating on spheres for the following class of elliptic system

$$(S) \qquad \begin{cases} -\varepsilon^2\Delta u + V_1(x)u = K(x)Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V_2(x)v = K(x)Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in W^{1,2}(\mathbb{R}^N),\ u, v > 0 \text{ in } \mathbb{R}^N, \end{cases}$$

where ε is a small positive parameter; $V_1,\ V_2 \in C^0(\mathbb{R}^N,[0,\infty))$ and $K \in C^0(\mathbb{R}^N,(0,\infty))$ are radially symmetric potentials; Q is a $(p+1)$-homogeneous function and p is subcritical, that is, $1 < p < 2^*-1$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent for $N \geq 3$.

1. Introduction

This work has been motivated by some papers appeared in recent years concerning the Schrödinger equation

$$(NLS) \qquad i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + K(x)\left|\psi\right|^{p-1}\psi = 0,\ x \in \mathbb{R}^N,$$

where $\hbar$ denotes the Plank constant, i is the imaginary unit and $p \in (1, \frac{N+2}{N-2})$. This equation appears in many fields of physic, in particular, when we describe Bose-Einstein condensates (see [30] and [34]) and the propagation of light in some nonlinear optical material (see [35]).

For application or motivation, we can cite also, for instance, [32, 33] where are studied the evolution of two orthogonal pulse envelope in birefringent optical fibers, see also [29]. System of type (S) is also important for industrial applications in fiber communications systems [27, 28]. Finally we would to recall that system of type (S) can describe other physical phenomena, such

Received March 18, 2012.
2010 *Mathematics Subject Classification.* 35J50, 35B06, 35A15, 35B25.
Key words and phrases. Schrödinger operator, radial solution, variational method, singular perturbation.
The second author was partially supported by Fapemig/Brazil.
The third author was partially supported by CNPq/Brazil and INCTMat/Brazil.

as Kerr-like photorefractive media in optics, (cf. [1, 20, 21, 22]), Hartree-Fock theory for double condensate [25]. See [31] and [37] for more applications in physical and chemical phenomenas.

Here we are concerned with the existence of standing waves (semiclassical states) of the nonlinear Schrödinger equations for small ε, that is, solutions of the form $\psi(x,t) = \exp(-iEt/\varepsilon)u(x)$. Notice that after a simple rescaling and putting $V(x) - E = V(x)$, ψ satisfies (NLS) if and only if u solve the elliptic equation

$$(NLS)_\varepsilon \qquad -\varepsilon^2 \Delta u + V(x)u = K(x)u^p,\ u > 0,\ x \in \mathbb{R}^N.$$

The most characteristic feature of $(NLS)_\varepsilon$ is that its solution u_ε concentrate as $\varepsilon \to 0$. When this concentration set is a single point (resp. finite points), these solutions are called, in the literature, spike solution (resp. multi-bump solutions). When the potential $V > 0$, beginning from the pioneering paper by Floer and Weinstein [24], a great number of work has been devoted to study spike or multi-bump solutions for $(NLS)_\varepsilon$ (see [5] and references there in). Studying in this case ($V > 0$), Ambrosetti-Malchiodi-Ni in [6] constructed solutions concentrating on spheres for $(NLS)_\varepsilon$. Ambrosetti-Ruiz in [9] extended this result to the case of decaying potentials. See also [4], [7], [10], [12], [13], [15] and [23]. In the critical frequence, that means $\inf_{\mathbb{R}^N} V(x) = 0$, spike solutions have been constructed in [16], [17], [18] and [19], which concentrate on the zero of the potential V as $\varepsilon \to 0$. In those papers also are constructed "small" solutions concentrating on spheres near zeroes of the potentials. On the other hand, Alves [2] and Alves-Soares [3] studied, by using the Mountain Pass Theorem due to Ambrosetti-Rabinowitz [8], the elliptic system (S), when V_1 and V_2 are globally lower bounded away from zero. The authors showed that the solution $(u_\varepsilon, v_\varepsilon)$ concentrates around local minima of the potentials V_1 and V_2.

Motivated by the above papers, we are going to construct solutions concentrating on spheres for a class of the elliptic system with decaying potentials, where V_1, V_2 and K are radially symmetric potentials satisfying:

(V) V_1, $V_2 \in C^0(\mathbb{R}^N, [0,\infty))$ are such that

$$\liminf_{|x|\to\infty} |x|^2 V(x) \equiv 4\lambda > 0,$$

where $V(x) = \min\{V_1(x), V_2(x)\}$ and the zero set of V, $Z = \left\{x \in \mathbb{R}^N : V(x) = 0\right\}$ is non-empty;

(K) $K \in C^0(\mathbb{R}^N, (0,\infty))$ is limited.

The function $Q \in C^1([0,+\infty) \times [0,+\infty), \mathbb{R})$ is a homogeneous function of degree $p+1$, with $1 < p < \frac{N+2}{N-2}$, $N \geq 3$ and verify:

(Q_1) There exists $C > 0$ such that

$$\begin{cases} |Q_u(u,v)| \leq C(|u|^p + |v|^p), \forall u, v \geq 0, \\ |Q_v(u,v)| \leq C(|u|^p + |v|^p), \forall u, v \geq 0; \end{cases}$$

(Q_2) There exist $\eta_1, \eta_2 > 0$ such that

$$\eta_1(|u|^{p+1} + |v|^{p+1}) \le Q(u,v) \le \eta_2(|u|^{p+1} + |v|^{p+1}) \ \forall u, v > 0;$$

(Q_3) $Q_u(0,1), Q_v(1,0) > 0$;
(Q_4) $Q(u,v) > 0 \ \forall u, v > 0$;
(Q_5) $Q_u(u,v), Q_v(u,v) \ge 0 \ \forall u, v \ge 0$.

Remark 1. (a) Since Q is a C^1 homogeneous function of degree $p+1$, then $(p+1)Q(u,v) = uQ_u(u,v) + vQ_v(u,v)$ and ∇Q is a homogeneous function of degree p.

(b) Note that the right hand side of (Q_2) can be obtained from (Q_1), (a) and the Young inequality.

(c) These kind of hypotheses were introduced for instance in [2] and [36].

(d) Our prototype of Q is $Q(u,v) = (au + bv)^{p+1}$, u, $v \ge 0$ and a, $b > 0$.

Our main result is the following.

Theorem 1. *Suppose that* (Q_1)-(Q_5), (V) *and* (K) *hold. Let* $A \subset Z$ *be an isolated compact subset of* Z *such that* $0 \notin A$ *and* $V_1 \equiv V_2$ *in* A. *Then for* ε *sufficienthly small,* (S) *has a solution* $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\mathbb{R}^N) \times W^{1,2}(\mathbb{R}^N)$, u_ε *and* v_ε *radially symmetric functions, such that*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = \lim_{\varepsilon \to 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0 \tag{1}$$

and

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2/(p-1)} \|u_\varepsilon + v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0. \tag{2}$$

Moreover, for each $\delta > 0$, *there are constants* C, $c > 0$ *such that*

$$u_\varepsilon(x), v_\varepsilon(x) \le C \exp(-c/\varepsilon) \left[1 + (|x|/2R_0)^{\omega_\varepsilon}\right] \forall x \in \mathbb{R}^N \backslash A^{4\delta}, \tag{3}$$

where $\omega_\varepsilon \equiv -\frac{(N-2)+\sqrt{(N-2)^2+4\lambda/\varepsilon^2}}{2}$, $A^d \equiv \left\{x \in \mathbb{R}^N \mid d(x,A) \le d\right\}$ *and* R_0 *is a positive constant given by* (V).

The proof of Theorem 1 is made adapting closely arguments used in [16] and [17], more exactly, the minimization techniques with two constraints in order to construct the spike solutions concentrating on sphere near of the zeros of V_1 and V_2. Actually, one of the constraints represents a type of the penalization of the nonlinearity. The proof of the decay estimate of the solution is slightly different those made in [16] and [17]. Here, in our case, we use some ideas in [11], as well as, those in [16] and [17], combining Moser iterations, classical elliptic estimates and comparison principle we obtain the decay estimate of the solutions desired.

2. Proof of Theorem 1

First of all by a scaling we see that system (S) is equivalent to

$$(\tilde{S}) \qquad \begin{cases} -\Delta u + V_1(\varepsilon x)u = K(\varepsilon x)Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V_2(\varepsilon x)v = K(\varepsilon x)Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in W^{1,2}(\mathbb{R}^N),\ u, v > 0 \text{ in } \mathbb{R}^N. \end{cases}$$

Let A be the isolated compact subset of Z as assumed in the theorem. We choose $\delta > 0$ such that $0 \notin A^{8\delta}$, and $A^{8\delta} \cap (Z \backslash A) = \varnothing$, where $A^\delta \equiv \{x \in \mathbb{R}^N \mid d(x,A) \leq \delta\}$. We define $A^\delta_\varepsilon \equiv \{x \in \mathbb{R}^N \mid \varepsilon x \in A^\delta\}$. Let $C^\infty_{0,rad}(\mathbb{R}^N)$ be the class of radially symmetric functions in $C^\infty_0(\mathbb{R}^N)$, where $C^\infty_0(\mathbb{R}^N)$ is the set of functions on $C^\infty(\mathbb{R}^N)$ with compact support. Let E_ε the completion of $C^\infty_{0,rad}(\mathbb{R}^N) \times C^\infty_{0,rad}(\mathbb{R}^N)$ with respect to the norm

$$\|(u,v)\|_\varepsilon = \left(\int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla v|^2 + V_1(\varepsilon x)u^2 + V_2(\varepsilon x)v^2]dx \right)^{1/2}.$$

We observe that $E_\varepsilon = E_{V_1,\varepsilon} \times E_{V_2,\varepsilon}$, where $E_{V_i,\varepsilon}$ is the completion of $C^\infty_{0,rad}(\mathbb{R}^N)$ with the norm $\|u\|_{V_i,\varepsilon} = \left(\int_{\mathbb{R}^N} [|\nabla u|^2 + V_i(\varepsilon x)u^2]dx \right)^{1/2}$, $i = 1,\ 2$. Thus, $\|(u,v)\|^2_\varepsilon = \|u\|^2_{V_1,\varepsilon} + \|v\|^2_{V_2,\varepsilon}$.

We fix a constant γ with $\gamma(p-1)/(p+1) > 2$. We define a function χ_ε by

$$\chi_\varepsilon(x) = \begin{cases} \varepsilon^{-(N-1)-3(p+1)/(p-1)} & \text{if } \ |x| \leq R_0/\varepsilon,\ x \notin A^{4\delta}_\varepsilon, \\ (|x|/\varepsilon)^\gamma & \text{if } \ |x| \geq R_0/\varepsilon, \\ 0 & \text{if } \ x \in A^{4\delta}_\varepsilon, \end{cases}$$

where $R_0 \geq 1$ is fixed so that $V(x) > 0$ for $|x| \geq R_0$ and $Z^{8\delta} \subset B(0,R_0)$.

Now we consider the following minimization problem

$$(4) \qquad M_\varepsilon = \inf \left\{ \|(u,v)\|^2_\varepsilon \ \Big| \ \int_{\mathbb{R}^N} K(\varepsilon x)Q(u,v)dx = 1, \right. \\ \left. \int_{\mathbb{R}^N} \chi_\varepsilon(x)Q(u,v)dx \leq 1,\ (u,v) \in E_\varepsilon \right\}.$$

First, using the same type of arguments developed in [16], we have the following lemma.

Lemma 2. $\lim_{\varepsilon \to 0} \varepsilon^{(N-1)(p-1)/(p+1)} M_\varepsilon = 0$.

Proof. Let $x_0 \in A$. Then, for any $a > 0$, there exists $b > 0$ such that $V_1(x), V_2(x) \in [0,a)$ for $|x - x_0| \leq b$. Without loss of generality, we can assume $|x_0| = 1$ so that $S^\delta_\varepsilon \subset A^\delta_\varepsilon$, where S is the unit sphere in $\mathbb{R}^N$. Then, using change of variables (polar coordenates) and setting $u(r + 1/\varepsilon) = \bar{u}(r)$, $v(r + 1/\varepsilon) = \bar{v}(r)$, we obtain that

$$M_\varepsilon \leq C_0 \frac{\int_{S^\delta_\varepsilon} [|\nabla u(x)|^2 + |\nabla v(x)|^2 + a((u(x))^2 + (v(x))^2)]dx}{\left(\int_{S^\delta_\varepsilon} Q(u(x), v(x))dx \right)^{2/(p+1)}}$$

$$\le C\varepsilon^{-(N-1)(p-1)/(p+1)} \times \frac{\int_{-\delta/\varepsilon}^{\delta/\varepsilon}[(\bar{u}'(r))^2+(\bar{v}'(r))^2+a((\bar{u}(r))^2+(\bar{v}(r))^2)]dr}{\left(\int_{-\delta/\varepsilon}^{\delta/\varepsilon}Q(\bar{u}(r),\bar{v}(r))dr\right)^{2/(p+1)}},$$

where C_0 and C are positive constants independent of ε. Here was used that $\chi_\varepsilon(x)=0,\ \forall\ x\in S_\varepsilon^\delta$; V_1 and V_2 are radially symmetric, and $V_1(\varepsilon x)$, $V_2(\varepsilon x)<a$, $\forall x\in S_\varepsilon^\delta$. Setting $\bar{u}(r)=u(\sqrt{a}r)$ and $\bar{v}(r)=v(\sqrt{a}r)$, where $a>0$ is arbitrary, we obtain,

$$\lim_{\varepsilon\to 0}\ \varepsilon^{(N-1)(p-1)/(p+1)}M_\varepsilon \le C\ a^{(p+3)/2(p+1)}\inf_{u,v\in C_0^\infty(-\infty,\infty)}\frac{\int_{-\infty}^{\infty}[(u')^2+(v')^2+u^2+v^2]dr}{(\int_{-\infty}^{\infty}Q(u,v)dr)^{2/(p+1)}}.$$

Then, since a is arbitrary and the last infimun is bounded, the lemma follows. □

Lemma 3. *For sufficiently small $\varepsilon>0$, M_ε is achieved at $(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\in E_\varepsilon$ which satisfies for some $\alpha_\varepsilon>0\ge\beta_\varepsilon$,*

$(S_{\alpha_\varepsilon,\ \beta\varepsilon})$

$$\begin{cases} -\Delta\bar{u}_\varepsilon+V_1(\varepsilon x)\bar{u}_\varepsilon=\alpha_\varepsilon K(\varepsilon x)Q_u(\bar{u}_\varepsilon,\bar{v}_\varepsilon)+\beta_\varepsilon\chi_\varepsilon(x)Q_u(\bar{u}_\varepsilon,\bar{v}_\varepsilon) & in\quad \mathbb{R}^N,\\ -\Delta\bar{v}_\varepsilon+V_2(\varepsilon x)\bar{v}_\varepsilon=\alpha_\varepsilon K(\varepsilon x)Q_v(\bar{u}_\varepsilon,\bar{v}_\varepsilon)+\beta_\varepsilon\chi_\varepsilon(x)Q_v(\bar{u}_\varepsilon,\bar{v}_\varepsilon) & in\quad \mathbb{R}^N,\\ \bar{u}_\varepsilon\ge 0,\ \bar{v}_\varepsilon\ge 0 & in\quad \mathbb{R}^N.\end{cases}$$

Proof. Let $\{(\bar{u}_\varepsilon^j,\bar{v}_\varepsilon^j)\}_j\subset E_\varepsilon$ be a minimizing sequence for M_ε. We can assume $\{(\bar{u}_\varepsilon^j,\bar{v}_\varepsilon^j)\}_j\subset C_{0,rad}^\infty(\mathbb{R}^N)\times C_{0,rad}^\infty(\mathbb{R}^N)$, since $C_{0,rad}^\infty(\mathbb{R}^N)\times C_{0,rad}^\infty(\mathbb{R}^N)$ is dense in E_ε. We take $R_j>0$ such that $\mathrm{supp}(\bar{u}_\varepsilon^j)\subset B(0,R_j)$ and $\mathrm{supp}(\bar{v}_\varepsilon^j)\subset B(0,R_j)$, $j\ge 1$. For a fixed $\varepsilon>0$, we can assume that $R_0/\varepsilon<R_1<R_2<\cdots$ and $\lim_{m\to\infty}R_m=\infty$. We define

$$E_\varepsilon^m\equiv E_\varepsilon\cap\left(W_0^{1,2}\left(B(0,R_m)\right)\times W_0^{1,2}\left(B(0,R_m)\right)\right).$$

We consider a restricted minimization problem

$$M_\varepsilon^m=\inf\left\{\|(u,v)\|_\varepsilon^2\ \Big|\ \int_{\mathbb{R}^N}K(\varepsilon x)Q(u,v)dx=1,\right.$$
$$\left.\int_{\mathbb{R}^N}\chi_\varepsilon(x)Q(u,v)dx\le 1,(u,v)\in E_\varepsilon^m\right\}. \tag{5}$$

Now, we will prove that there exists a non-negative minimizer $(u_\varepsilon^m,v_\varepsilon^m)$ of M_ε^m such that $M_\varepsilon\le M_\varepsilon^m$ and $\lim_{m\to\infty}M_\varepsilon^m=M_\varepsilon$. Indeed, let $\left\{(u_\varepsilon^k,v_\varepsilon^k)\right\}_k$ be a minimizing sequence for M_ε^m. Then it follows that $\left\{(u_\varepsilon^k,v_\varepsilon^k)\right\}_k$ is bounded. Since E_ε^m is reflexive, there exists $(u_\varepsilon^m,v_\varepsilon^m)\in E_\varepsilon^m$ such that $\left\{(u_\varepsilon^k,v_\varepsilon^k)\right\}_k$ is weakly convergent to $(u_\varepsilon^m,v_\varepsilon^m)$, up to subsequence. Thus, $u_\varepsilon^k\rightharpoonup u_\varepsilon^m$ weakly in $E_{V_1,\varepsilon}$ and $v_\varepsilon^k\rightharpoonup v_\varepsilon^m$ weakly in $E_{V_2,\varepsilon}$ as $k\to\infty$. Since $E_{V_i,\varepsilon}\cap W_0^{1,2}B((0,R_m))$

is compactly imbedded in $L^{p+1}(B(0,R_m))$, with $i = 1,\ 2$ and $2 < p+1 < 2^*$, from (Q_2) we have

$$\int_{B(0,R_m)} K(\varepsilon x)Q(u_\varepsilon^m, v_\varepsilon^m)dx = \lim_{k\to\infty}\int_{B(0,R_m)} K(\varepsilon x)Q(u_\varepsilon^k, v_\varepsilon^k)dx = 1, \tag{6}$$

and

$$\int_{B(0,R_m)} \chi_\varepsilon Q(u_\varepsilon^m, v_\varepsilon^m)dx = \lim_{k\to\infty}\int_{B(0,R_m)} \chi_\varepsilon Q(u_\varepsilon^k, v_\varepsilon^k)dx \le 1.$$

Since $\left\{(u_\varepsilon^k, v_\varepsilon^k)\right\}_k$ is weakly convergent to $(u_\varepsilon^m, v_\varepsilon^m)$, we have

$$\|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 \le \liminf_{k\to\infty}\left\|(u_\varepsilon^k, v_\varepsilon^k)\right\|_\varepsilon^2 = M_\varepsilon^m \le \|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 .$$

Thus, $(u_\varepsilon^m, v_\varepsilon^m)$ is a minimizer for M_ε^m. Since $|\nabla |u_\varepsilon^m|| = |\nabla u_\varepsilon^m|$ and $|\nabla |v_\varepsilon^m|| = |\nabla v_\varepsilon^m|$ we see that $\|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 = \|(|u_\varepsilon^m|, |v_\varepsilon^m|)\|_\varepsilon^2$. Then, there exists a non-negative minimizer $(u_\varepsilon^m, v_\varepsilon^m)$ of M_ε^m. Now, we observe that for any $j \ge 1$,

$$\lim_{k\to\infty}\left\|(u_\varepsilon^k, v_\varepsilon^k)\right\|_\varepsilon^2 \le ||(u_\varepsilon^j, v_\varepsilon^j)||_\varepsilon^2.$$

In fact, for any $j \le k$, $B(0,R_j) \subset B(0,R_k)$. Thus,

$$W_0^{1,2}\left(B(0,R_j)\right) \subset W_0^{1,2}\left(B(0,R_k)\right).$$

Consequently, $E_\varepsilon^j \subset E_\varepsilon^k$. This implies that $M_\varepsilon^j = ||(u_\varepsilon^j, v_\varepsilon^j)||_\varepsilon^2 \ge \left\|(u_\varepsilon^k, v_\varepsilon^k)\right\|_\varepsilon^2 = M_\varepsilon^k$. We note that

$$M_\varepsilon \le \lim_{j\to\infty} M_\varepsilon^j = \lim_{j\to\infty} ||(u_\varepsilon^j, v_\varepsilon^j)||_\varepsilon^2 \le \lim_{j\to\infty} ||(\bar u_\varepsilon^j, \bar v_\varepsilon^j)||_\varepsilon^2 = M_\varepsilon.$$

Therefore, $M_\varepsilon^m \to M_\varepsilon$ as $m \to \infty$. Thus $\{(u_\varepsilon^m, v_\varepsilon^m)\}_m$ is a minimizing sequence for M_ε.

Since $(u_\varepsilon^m, v_\varepsilon^m)$ is a minimizer for M_ε^m, there exist Lagrange multipliers α_ε^m, $\beta_\varepsilon^m \in \mathbb{R}$ such that $(u_\varepsilon^m, v_\varepsilon^m)$ satisfies the system $(S_{\alpha_\varepsilon^m,\ \beta_\varepsilon^m})$ in $B(0,R_m)$. Taking a subsequence if necessary, we can assume that for some $(\bar u_\varepsilon, \bar v_\varepsilon) \in E_\varepsilon$, $\{(u_\varepsilon^m, v_\varepsilon^m)\}_m$ converges weakly to $(\bar u_\varepsilon, \bar v_\varepsilon)$ in E_ε as $m \to \infty$. Since

$$\int_{\mathbb{R}^N} \chi_\varepsilon Q(u_\varepsilon^m, v_\varepsilon^m)dx \le 1,$$

it follows that for any $R \ge \frac{R_0}{\varepsilon}$,

$$\int_{\mathbb{R}^N\setminus B(0,R)} K(\varepsilon x)Q(u_\varepsilon^m, v_\varepsilon^m)dx \le C(\varepsilon/R)^\gamma$$

for some $C > 0$. By the Dominated Convergence Theorem of Lebesgue, we obtain $\int_{B(0,R)} K(\varepsilon x)Q(\bar u_\varepsilon, \bar v_\varepsilon)dx \ge 1 - C\left(\varepsilon/R\right)^\gamma$. This implies that

$$\int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar u_\varepsilon, \bar v_\varepsilon)dx = \lim_{R\to\infty}\int_{B(0,R)} K(\varepsilon x)Q(\bar u_\varepsilon, \bar v_\varepsilon)dx \ge 1.$$

We claim that

$$\int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx = 1. \tag{7}$$

In fact, arguing by contradiction, we assume that $\int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx > 1$. Then there exists $\bar{R} > 0$ so that $\int_{B(0,\bar{R})} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx > 1$. Hence, we get $\lim_{m\to\infty} \int_{B(0,\bar{R})} K(\varepsilon x)Q(u_\varepsilon^m, v_\varepsilon^m)dx = \int_{B(0,\bar{R})} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx > 1$. Thus, there exists $m_0 \in \mathbb{N}$ such that $\int_{B(0,\bar{R})} K(\varepsilon x)Q(u_\varepsilon^{m_0}, v_\varepsilon^{m_0})dx > 1$. But this is impossible, since $\int_{\mathbb{R}^N} K(\varepsilon x)Q(u_\varepsilon^m, v_\varepsilon^m)dx = 1$ for all $m \in \mathbb{N}$.

Since $\int_{B(0,T)} \chi_\varepsilon Q(u_\varepsilon^m, v_\varepsilon^m)dx \le 1$ for each $T > 0$ we get, again using the Dominated Convergence Theorem of Lebesgue, that

$$\int_{B(0,T)} \chi_\varepsilon Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx \le 1$$

for each $T > 0$. Consequently,

$$\int_{\mathbb{R}^N} \chi_\varepsilon Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx \le 1. \tag{8}$$

Since $\|(\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\varepsilon^2 \le \liminf_{m\to\infty} \|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 = M_\varepsilon$, we infer that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ is a minimizer of M_ε.

Now, we will prove that in system $(S_{\alpha_\varepsilon^m,\ \beta_\varepsilon^m})$, $\alpha_\varepsilon^m > 0 \ge \beta_\varepsilon^m$. In fact, using same ideas in [14], we take ξ_0, $\xi_1 \in C_0^\infty(\mathbb{R}^N)$ non-negative radially symmetric functions with $\operatorname{supp}(\xi_0) \subset \operatorname{int}(A_\varepsilon^{4\delta})$ and $\operatorname{supp}(\xi_1) \subset \{x \in \mathbb{R}^N \mid |x| < d(0, A_\varepsilon^{4\delta})\}$. Define

$$D(s,t) \equiv \int_{B(0,R_m)} K(\varepsilon x)Q((1 + t\xi_0 - s\xi_1)(u_\varepsilon^m, v_\varepsilon^m))dx.$$

The function D is continuously differentiable in a neighborhood of $(0,0)$. We note that $D(0,0) = 1$ and $\frac{\partial}{\partial t}D(0,0) = (p+1)\int_{B(0,R_m)} K(\varepsilon x)\xi_0 Q\left(u_\varepsilon^m, v_\varepsilon^m\right) dx > 0$. By the implicit function theorem, for small $\tau > 0$ there exists $t \in C^1(-\tau,\tau)$ such that

$$t(0) = 0 \text{ and } D(s,t(s)) = 1 \text{ for all } s \in (-\tau,\tau).$$

Hence

$$(p+1)\int_{B(0,R_m)} K(\varepsilon x)(t'(0)\xi_0 - \xi_1)Q\left(u_\varepsilon^m, v_\varepsilon^m\right) dx = 0. \tag{9}$$

Moreover, using the definition of χ_ε and the fact that $\chi_\varepsilon\xi_0 \equiv 0$ in $B(0, R_m)$, we obtain

$$\frac{d}{ds}\Big|_{s=0} \int_{B(0,R_m)} \chi_\varepsilon Q\left((1 + t(s)\xi_0 - s\xi_1)(u_\varepsilon^m, v_\varepsilon^m)\right) dx$$

$$= -(p+1)\varepsilon^{-(N-1)-3(p+1)/(p-1)} \int_{\operatorname{supp}(\xi_1)} \xi_1 Q(u_\varepsilon^m, v_\varepsilon^m)dx < 0. \tag{10}$$

This implies that there exists $c > 0$ such that for any $s \in (0, c)$,

$$\int_{B(0,R_m)} \chi_\varepsilon Q\left((1 + t(s)\xi_0 - s\xi_1)(u_\varepsilon^m, v_\varepsilon^m)\right) dx < 1.$$

Since $(u_\varepsilon^m, v_\varepsilon^m)$ is a minimizer for M_ε^m, we have

(11)

$$\begin{aligned} 0 \leq \frac{d}{ds}\Big|_{s=0} \int_{B(0,R_m)} &[|\nabla\left((1 + t(s)\xi_0 - s\xi_1)\, u_\varepsilon^m\right)|^2 + |\nabla\left((1 + t(s)\xi_0 - s\xi_1) v_\varepsilon^m\right)|^2 \\ &+ (1 + t(s)\xi_0 - s\xi_1)^2 \left(V_1(\varepsilon x)(u_\varepsilon^m)^2 + V_2(\varepsilon x)(v_\varepsilon^m)^2\right)] dx \\ = 2\int_{B(0,R_m)} &[\nabla u_\varepsilon^m.\nabla((t'(0)\xi_0 - \xi_1)u_\varepsilon^m) + \nabla v_\varepsilon^m.\nabla((t'(0)\xi_0 - \xi_1)v_\varepsilon^m) \\ &+ (t'(0)\xi_0 - \xi_1)\left(V_1(\varepsilon x)(u_\varepsilon^m)^2 + V_2(\varepsilon x)(v_\varepsilon^m)^2\right)] dx. \end{aligned}$$

Using $(t'(0)\xi_0 - \xi_1)(u_\varepsilon^m, v_\varepsilon^m)$ as test function in $(S_{\alpha_\varepsilon^m,\ \beta_\varepsilon^m})$, the homogeneity of Q, the definition of χ_ε and (9), we deduce that

$$\begin{aligned} 0 \leq \int_{B(0,R_m)} &[\nabla u_\varepsilon^m.\nabla((t'(0)\xi_0 - \xi_1)u_\varepsilon^m) + \nabla v_\varepsilon^m.\nabla((t'(0)\xi_0 - \xi_1)v_\varepsilon^m) \\ &+ (t'(0)\xi_0 - \xi_1)\left(V_1(\varepsilon x)(u_\varepsilon^m)^2 + V_2(\varepsilon x)(v_\varepsilon^m)^2\right)] dx \\ = (p+1)\alpha_\varepsilon^m &\int_{B(0,R_m)} (t'(0)\xi_0 - \xi_1) K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx \\ &+ (p+1)\beta_\varepsilon^m \int_{B(0,R_m)} \chi_\varepsilon (t'(0)\xi_0 - \xi_1) Q(u_\varepsilon^m, v_\varepsilon^m) dx \\ = \ - (p+1)&\beta_\varepsilon^m \varepsilon^{-(N-1)-3(p+1)/(p-1)} \int_{\mathrm{supp}(\xi_1)} \xi_1 Q(u_\varepsilon^m, v_\varepsilon^m) dx. \end{aligned}$$

By (10) and (11) we conclude that $\beta_\varepsilon^m \leq 0$.

Now, taking $(u_\varepsilon^m, v_\varepsilon^m)$ as test function in $(S_{\alpha_\varepsilon^m,\ \beta_\varepsilon^m})$ and using (6) we obtain

$$\|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 = (p+1)\alpha_\varepsilon^m + (p+1)\beta_\varepsilon^m \int_{B(0,R_m)} \chi_\varepsilon Q(u_\varepsilon^m, v_\varepsilon^m) dx. \tag{12}$$

This implies that $\alpha_\varepsilon^m > 0$.

Now we will show that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies the system $(S_{\alpha_\varepsilon,\ \beta_\varepsilon})$. We claim that $\{\alpha_\varepsilon^m\}_m$ is bounded for small $\varepsilon > 0$. Indeed, arguing by contradiction assume, without loss of generality, that $\lim_{m\to\infty} \alpha_\varepsilon^m = \infty$. For any $\sigma > 0$, choose a function $\phi_\sigma \in C_0^\infty(\mathrm{int}(A_\varepsilon^{4\delta}))$ such that $0 \leq \phi_\sigma \leq 1$, $\phi_\sigma(x) = 1$ for $d(x, \partial A_\varepsilon^{4\delta}) \geq \sigma$, and $|\nabla\phi_\sigma| \leq 2/\sigma$. Using $\phi_\sigma(u_\varepsilon^m, v_\varepsilon^m)$ as test function in $(S_{\alpha_\varepsilon^m,\ \beta_\varepsilon^m})$ and that $\chi_\varepsilon\phi_\sigma \equiv 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} &[|\nabla u_\varepsilon^m|^2 \phi_\sigma + \nabla u_\varepsilon^m.\nabla\phi_\sigma u_\varepsilon^m + |\nabla v_\varepsilon^m|^2 \phi_\sigma + \nabla v_\varepsilon^m.\nabla\phi_\sigma v_\varepsilon^m \\ &+ \phi_\sigma(V_1(\varepsilon x)\,(u_\varepsilon^m)^2 + V_2(\varepsilon x)\,(v_\varepsilon^m)^2)] dx \end{aligned}$$

$$= (p+1)\alpha_\varepsilon^m \int_{\mathbb{R}^N} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m)\phi_\sigma dx. \tag{13}$$

From $\inf_{x\in \operatorname{supp}(|\nabla\phi_\sigma|)} V(\varepsilon x) > 0$ and the properties of ϕ_σ, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} [|\nabla u_\varepsilon^m|^2 \phi_\sigma + \nabla u_\varepsilon^m.\nabla\phi_\sigma u_\varepsilon^m + |\nabla v_\varepsilon^m|^2 \phi_\sigma + \nabla v_\varepsilon^m.\nabla\phi_\sigma v_\varepsilon^m \\ &+ \phi_\sigma(V_1(\varepsilon x)(u_\varepsilon^m)^2 + V_2(\varepsilon x)(v_\varepsilon^m)^2)]dx \\ &\le C\,\|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2 \end{aligned}\tag{14}$$

for some $C>0$, independent of m. By (13), (14) and the fact that

$$\{\|(u_\varepsilon^m, v_\varepsilon^m)\|_\varepsilon^2\}_m$$

is a bounded sequence, we see that for some $C>0$, independent of m,

$$\int_{\mathbb{R}^N} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m)\phi_\sigma dx \le C/\alpha_\varepsilon^m.$$

Thus,

$$\lim_{m\to\infty} \int_{\{x\in A_\varepsilon^{4\delta}\ |\ d(x,\partial A_\varepsilon^{4\delta})\ge\sigma\}} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx = 0. \tag{15}$$

From the condition $\int_{\mathbb{R}^N} \chi_\varepsilon Q(u_\varepsilon^m, v_\varepsilon^m)dx \le 1$ and from the definition of χ_ε, we have

$$\int_{\mathbb{R}^N\setminus B(0,R_0/\varepsilon)} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx \le C\,(\varepsilon/R_0)^\gamma \tag{16}$$

and

$$\int_{B(0,R_0/\varepsilon)\setminus A_\varepsilon^{4\delta}} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx \le C\varepsilon^{(N-1)+3(p+1)/(p-1)} \tag{17}$$

for some positive constant C. Now, using $\int_{\mathbb{R}^N} K(\varepsilon x)Q(u_\varepsilon^m, v_\varepsilon^m)dx = 1$, (15), (16) and (17) we infer that

$$\begin{aligned}&\liminf_{m\to\infty} \int_{\{x\in A_\varepsilon^{4\delta}\ |\ d(x,\partial A_\varepsilon^{4\delta})\le\sigma\}} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx \\ &\ge 1 - C\varepsilon^{(N-1)+3(p+1)/(p-1)} - C\,(\varepsilon/R_0)^\gamma > 0\end{aligned}$$

for small $\varepsilon>0$ and for each $\sigma>0$. Then for each $\sigma>0$ there is a sequence $\{x_m\}_m$ in $A_\varepsilon^{4\delta}$ such that $\lim_{m\to\infty} d(x_m, \partial A_\varepsilon^{4\delta}) = 0$ and $Q(u_\varepsilon^m(x_m), v_\varepsilon^m(x_m)) = 1$. Since $A_\varepsilon^{4\delta}$ is an compact subset of $\mathbb{R}^N$, we see that $\lim_{m\to\infty} x_m = x_0 \in A_\varepsilon^{4\delta}$, up to subsequence. This implies that $x_0 \in \partial A_\varepsilon^{4\delta}$ and $\lim_{m\to\infty}|x_m| = |x_0| = r_0 > 0$ so that for each $\sigma>0$

$$\liminf_{m\to\infty} \int_{D_{r_0}^\sigma} K(\varepsilon x) Q(u_\varepsilon^m, v_\varepsilon^m) dx > 0, \tag{18}$$

where $D_{r_0}^{\sigma}$ is defined by $D_{r_0}^{\sigma} \equiv \left\{x \in \mathbb{R}^N \mid r_0 - \sigma \le |x| \le r_0 + \sigma\right\}$. To reach a contradiction of (18), we will prove the following statements:

$$\int_{D_{r_0}^{\sigma}} \left[((u_\varepsilon^m - 1)_+)^2 + ((v_\varepsilon^m - 1)_+)^2\right] dx \le C\sigma^{2/N} \left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2 \tag{19}$$

for m large and some positive constant C, independent of σ;

$$\begin{aligned} &\int_{D_{r_0}^{\sigma}} [|\nabla(u_\varepsilon^m - 1)_+|^2 + |\nabla(v_\varepsilon^m - 1)_+|^2 + V_1(\varepsilon x)((u_\varepsilon^m - 1)_+)^2 \\ &\qquad + V_2(\varepsilon x)((v_\varepsilon^m - 1)_+)^2]dx \\ &\le \left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2, \end{aligned} \tag{20}$$

$$\int_{D_{r_0}^{\sigma}} K(\varepsilon x)Q((u_\varepsilon^m - 1)_+, (v_\varepsilon^m - 1)_+)dx \le C\sigma^{s(p+1)/N} \tag{21}$$

for some $s \in (0,1)$ and $C > 0$, C independent of σ; and

$$\int_{D_{r_0}^{\sigma}} [(u_\varepsilon^m)^{p+1} + (v_\varepsilon^m)^{p+1}]dx \le C_0 \int_{D_{r_0}^{\sigma}} K(\varepsilon x)Q((u_\varepsilon^m - 1)_+, (v_\varepsilon^m - 1)_+)dx + C_0\sigma \tag{22}$$

for some positive constant C_0.

To prove the assertion (19) note that, by the Poincaré inequality, there is a positive constant C_1 so that

$$\begin{aligned} \int_{D_{r_0}^{\sigma}} ((u_\varepsilon^m - 1)_+)^2 dx &\le (\left|D_{r_0}^{\sigma}\right| / \omega_N)^{2/N} \int_{D_{r_0}^{\sigma}} |\nabla(u_\varepsilon^m - 1)_+|^2 \, dx \\ &\le C_1 \sigma^{2/N} \int_{D_{r_0}^{\sigma}} |\nabla u_\varepsilon^m|^2 \, dx \\ &\le C_1 \sigma^{2/N} \left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2 . \end{aligned}$$

Similarly $\int_{D_{r_0}^{\sigma}} ((v_\varepsilon^m - 1)_+)^2 dx \le C_2 \sigma^{2/N} \left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2$ for some constant $C_2 > 0$ and the inequality (19) follows.

The verification of (20) is immediate.

For the statement (21), we use the interpolation inequality, Sobolev inequality, (19) and (20) to find

$$\begin{aligned} &\int_{D_{r_0}^{\sigma}} ((u_\varepsilon^m - 1)_+)^{p+1} dx \\ &\le C_0 (\int_{D_{r_0}^{\sigma}} ((u_\varepsilon^m - 1)_+)^2 dx)^{s_1(p+1)/2} \times (\int_{D_{r_0}^{\sigma}} |\nabla(u_\varepsilon^m - 1)_+|^2 \, dx)^{(1-s_1)(p+1)/2} \\ &\le C_1 (\sigma^{2/N} \left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2)^{s_1(p+1)/2} \times (\left\|(u_\varepsilon^m, v_\varepsilon^m)\right\|_\varepsilon^2)^{(1-s_1)(p+1)/2} \\ &\le C_2 \sigma^{s_1(p+1)/N} \end{aligned}$$

for $s_1 \in (0,1)$ and for some constants $C_0, C_1, C_2 > 0$, independent of σ. Similarly, we get $\int_{D^\sigma_{r_0}} ((v^m_\varepsilon - 1)_+)^{p+1} dx \le C_3 \sigma^{s_2(p+1)/N}$ for some constants $C_3 > 0$ and $s_2 \in (0,1)$. Using this information and (Q_2), (21) follows.

Finally, to obtain (22), we note that

$$\begin{aligned}\int_{D^\sigma_{r_0}} (u^m_\varepsilon)^{p+1} dx &\le \int_{D^\sigma_{r_0} \cap \{u^m_\varepsilon \ge 1\}} ((u^m_\varepsilon - 1)_+ + 1)^{p+1} dx + \left| D^\sigma_{r_0} \right| \\ &\le 2^p \int_{D^\sigma_{r_0}} ((u^m_\varepsilon - 1)_+)^{p+1} dx + (2^p + 1) \left| D^\sigma_{r_0} \right|.\end{aligned}$$

Also, $\int_{D^\sigma_{r_0}} (v^m_\varepsilon)^{p+1} dx \le 2^p \int_{D^\sigma_{r_0}} ((v^m_\varepsilon - 1)_+)^{p+1} dx + (2^p + 1) \left| D^\sigma_{r_0} \right|$. Therefore

$$\begin{aligned}&\int_{D^\sigma_{r_0}} [(u^m_\varepsilon)^{p+1} + (v^m_\varepsilon)^{p+1}] dx \\ &\le 2^p \int_{D^\sigma_{r_0}} [((u^m_\varepsilon - 1)_+)^{p+1} + ((v^m_\varepsilon - 1)_+)^{p+1}] dx + 2(2^p + 1) \left| D^\sigma_{r_0} \right|.\end{aligned} \tag{23}$$

Using (23), (Q_2) and the fact that $\left| D^\sigma_{r_0} \right| \le C\sigma$ for all smal $\sigma > 0$ and for some positive constant C, we obtain (22). From (Q_2), (21) and (22) it follows that

$$\begin{aligned}\int_{D^\sigma_{r_0}} K(\varepsilon x) Q(u^m_\varepsilon, v^m_\varepsilon) dx &\le C_0 \eta_2 \int_{D^\sigma_{r_0}} [(u^m_\varepsilon)^{p+1} + (v^m_\varepsilon)^{p+1}] dx \\ &\le C_1 \int_{D^\sigma_{r_0}} K(\varepsilon x) Q((u^m_\varepsilon - 1)_+, (v^m_\varepsilon - 1)_+) dx + C_1 \sigma \\ &\le C(\sigma^{s(p+1)/N} + \sigma)\end{aligned}$$

for some $s \in (0,1)$ and for some constants $C_0, C_1, C > 0$, independent of σ and m. Therefore,

$$\liminf_{m \to \infty} \int_{D^\sigma_{r_0}} K(\varepsilon x) Q(u^m_\varepsilon, v^m_\varepsilon) dx \le C(\sigma^{s(p+1)/N} + \sigma)$$

for all $\sigma > 0$ small. But this contradicts (18), given the arbitrariness of $\sigma > 0$. Thus, we conclude that $\{\alpha^m_\varepsilon\}_m$ is bounded. This implies that $\lim_{m \to \infty} \alpha^m_\varepsilon = \alpha_\varepsilon \ge 0$, up to subsequence. Using (12) and the fact that

$$0 \le \int_{\mathbb{R}^N} \chi_\varepsilon(x) Q(u^m_\varepsilon, v^m_\varepsilon) dx \le 1$$

for all $m \in \mathbb{N}$, we get $\lim_{m \to \infty} \beta^m_\varepsilon = \beta_\varepsilon \le 0$. Since $(u^m_\varepsilon, v^m_\varepsilon)$ is solution of $(S_{\alpha^m_\varepsilon,\, \beta^m_\varepsilon})$ we have that

$$\begin{aligned}&\int_{\mathbb{R}^N} [\nabla u^m_\varepsilon . \nabla \varphi + \nabla v^m_\varepsilon . \nabla \psi + V_1(\varepsilon x) u^m_\varepsilon \varphi + V_2(\varepsilon x) v^m_\varepsilon \psi] dx \\ &= \alpha^m_\varepsilon \int_{\mathbb{R}^N} K(\varepsilon x) [\varphi Q_u(u^m_\varepsilon, v^m_\varepsilon) + \psi Q_v(u^m_\varepsilon, v^m_\varepsilon)] dx\end{aligned}$$

$$+\beta_\varepsilon^m \int_{\mathbb{R}^N} \chi_\varepsilon[\varphi Q_u(u_\varepsilon^m, v_\varepsilon^m) + \psi Q_v(u_\varepsilon^m, v_\varepsilon^m)]dx \tag{24}$$

for any φ, $\psi \in C_{0,rad}^\infty(\mathbb{R}^N)$. Finally, taking the limit in (24) as $m \to \infty$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} [\nabla \bar{u}_\varepsilon.\nabla\varphi + \nabla\bar{v}_\varepsilon.\nabla\psi + V_1(\varepsilon x)\varphi\bar{u}_\varepsilon + V_2(\varepsilon x)\psi\bar{v}_\varepsilon)]dx \\
&= \alpha_\varepsilon \int_{\mathbb{R}^N} K(\varepsilon x)[\varphi Q_u(\bar{u}_\varepsilon, \bar{v}_\varepsilon) + \psi Q_v(\bar{u}_\varepsilon, \bar{v}_\varepsilon)]dx \\
&\quad + \beta_\varepsilon \int_{\mathbb{R}^N} \chi_\varepsilon[\varphi Q_u(\bar{u}_\varepsilon, \bar{v}_\varepsilon) + \psi Q_v(\bar{u}_\varepsilon, \bar{v}_\varepsilon)]dx
\end{aligned}$$

for any φ, $\psi \in C_{0,rad}^\infty(\mathbb{R}^N)$. Therefore, $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies $(S_{\alpha_\varepsilon,\, \beta_\varepsilon})$. From (7), $\beta_\varepsilon \le 0$, the homogeneity of Q and the fact that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ is solution of $(S_{\alpha_\varepsilon,\, \beta_\varepsilon})$, we conclude that $\|(\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\varepsilon^2 \le (p+1)\alpha_\varepsilon$ and therefore $\alpha_\varepsilon > 0$. This completes the proof of lemma. □

Claim: For ε small,

$$\int_{\mathbb{R}^N} \chi_\varepsilon Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx < 1. \tag{25}$$

This claim is one of the crucial setps of our work. We will postpone its proof for while. If this is the case, for any φ, $\psi \in C_{0,rad}^\infty(\mathbb{R}^N\ )$, we define

$$\varphi_s \equiv (\bar{u}_\varepsilon + s\varphi)\left(\int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar{u}_\varepsilon + s\varphi, \bar{v}_\varepsilon + s\psi)dx\right)^{-1/(p+1)}$$

and

$$\psi_s \equiv (\bar{v}_\varepsilon + s\psi)\left(\int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar{u}_\varepsilon + s\varphi, \bar{v}_\varepsilon + s\psi)dx\right)^{-1/(p+1)}.$$

From (7) we conclude that $(\varphi_0, \psi_0) = (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$. Since Q is homogeneous of degree $p+1$, we obtain $\int_{\mathbb{R}^N} K(\varepsilon x)Q(\varphi_s, \psi_s)dx = 1$. Also, by (25), $\int_{\mathbb{R}^N} \chi_\varepsilon Q(\varphi_s, \psi_s)dx < 1$ for small $|s|$.

Thus,

$$\begin{aligned}
0 &= \frac{d}{ds}\|(\varphi_s, \psi_s)\|_\varepsilon^2\,|_{s=0} \\
&= -2M_\varepsilon/(p+1)\int_{\mathbb{R}^N} K(\varepsilon x)[\varphi Q_u(\bar{u}_\varepsilon, \bar{v}_\varepsilon) + \psi Q_v(\bar{u}_\varepsilon, \bar{v}_\varepsilon)]dx \\
&\quad + 2\int_{\mathbb{R}^N} [\nabla\bar{u}_\varepsilon.\nabla\varphi + \nabla\bar{v}_\varepsilon.\nabla\psi + V_1(\varepsilon x)\bar{u}_\varepsilon\varphi + V_2(\varepsilon x)\bar{v}_\varepsilon\psi]dx.
\end{aligned}$$

This implies that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies the system $(S_{M_\varepsilon/(p+1),\, 0})$. Then, as the functions Q_u and Q_v are homogeneous of degree p, we deduce that $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$, where $\tilde{u}_\varepsilon = (M_\varepsilon/(p+1))^{1/(p-1)}\,\bar{u}_\varepsilon$ and $\tilde{v}_\varepsilon = (M_\varepsilon/(p+1))^{1/(p-1)}\,\bar{v}_\varepsilon$, is a solution of $(\tilde{S})$.

Lemma 4. $\lim_{\varepsilon\to 0} \varepsilon^{(N-1)(p-1)/(p+1)}\alpha_\varepsilon = 0.$

Proof. Assume by contradiction, taking a subsequence if necessary, that

$$\lim_{\varepsilon\to 0}\varepsilon^{(N-1)(p-1)/(p+1)}\alpha_\varepsilon=\alpha\in(0,\infty]\,.$$

For any $\sigma>0$, we choose $\phi_\sigma\in C_0^\infty\left(\mathrm{int}(A_\varepsilon^{4\delta})\right)$ satisfying $0\le\phi_\sigma\le 1$, $\phi_\sigma(x)=1$ for $d(x,\partial A_\varepsilon^{4\delta})\ge\sigma$, and $|\nabla\phi_\sigma|\le 2/\sigma$. Using $\phi_\sigma(\bar u_\varepsilon,\bar v_\varepsilon)$ as test function in the system $(S_{\alpha_\varepsilon,\,\beta_\varepsilon})$ and the fact that $\chi_\varepsilon\phi_\sigma\equiv 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}[\left|\nabla\bar u_\varepsilon\right|^2\phi_\sigma+\nabla\bar u_\varepsilon.\nabla\phi_\sigma\bar u_\varepsilon+\left|\nabla\bar v_\varepsilon\right|^2\phi_\sigma+\nabla\bar v_\varepsilon.\nabla\phi_\sigma\bar v_\varepsilon\\&+\phi_\sigma\left(V_1(\varepsilon x)(\bar u_\varepsilon)^2+V_2(\varepsilon x)(\bar v_\varepsilon)^2\right)]dx\\&=(p+1)\alpha_\varepsilon\int_{\mathbb{R}^N}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)\phi_\sigma dx.\end{aligned}\tag{26}$$

From $\inf_{x\in\mathrm{supp}(|\nabla\phi_\sigma|)}V(\varepsilon x)>0$ and the properties of ϕ_σ, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}[\left|\nabla\bar u_\varepsilon\right|^2\phi_\sigma+\nabla\bar u_\varepsilon.\nabla\phi_\sigma\bar u_\varepsilon+\left|\nabla\bar v_\varepsilon\right|^2\phi_\sigma+\nabla\bar v_\varepsilon.\nabla\phi_\sigma\bar v_\varepsilon\\&+\phi_\sigma\left(V_1(\varepsilon x)(\bar u_\varepsilon)^2+V_2(\varepsilon x)(\bar v_\varepsilon)^2\right)]dx\\&\le C\left\|(\bar u_\varepsilon,\bar v_\varepsilon)\right\|_\varepsilon^2\end{aligned}\tag{27}$$

for some $C>0$, independent of $\varepsilon>0$. From (26) and (27) it follows that $\int_{\mathbb{R}^N}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)\phi_\sigma dx\le C\left\|(\bar u_\varepsilon,\bar v_\varepsilon)\right\|_\varepsilon^2/\alpha_\varepsilon$ for some positive constant C, independent of $\varepsilon>0$. By Lemma 2, for each $\sigma>0$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)\phi_\sigma dx=0.$$

Then

$$\lim_{\varepsilon\to 0}\int_{\{x\in A_\varepsilon^{4\delta}\,|\,d(x,\partial A_\varepsilon^{4\delta})\ge\sigma\}}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)dx=0.\tag{28}$$

From (8) and the definition of χ_ε, we get

$$\int_{\mathbb{R}^N\setminus B(0,R_0/\varepsilon)}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)dx\le C\left(\varepsilon/R_0\right)^\gamma\tag{29}$$

and

$$\int_{B(0,R_0/\varepsilon)\setminus A_\varepsilon^{4\delta}}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)dx\le C\varepsilon^{(N-1)+3(p+1)/(p-1)}\tag{30}$$

for some $C>0$. From (7), (28), (29) and (30), we see that for each $\sigma>0$,

$$\liminf_{\varepsilon\to 0}\int_{\{x\in A_\varepsilon^{4\delta}\,|\,d(x,\partial A_\varepsilon^{4\delta})\le\sigma\}}K(\varepsilon x)Q(\bar u_\varepsilon,\bar v_\varepsilon)dx>0.\tag{31}$$

From (31), for every $\sigma>0$, there exists a sequence $\{x_m\}_m$ in $A_\varepsilon^{4\delta}$ such that $\lim_{m\to\infty}d(x_m,\partial A_\varepsilon^{4\delta})=0$. Therefore, there exists some $x_0\in\partial A^{4\delta}$, with

$\lim_{m\to\infty} x_m = \frac{x_0}{\varepsilon}$, and $\omega > 0$ such that for any $\sigma > 0$,

$$\liminf_{\varepsilon\to 0} \int_{\{x\in\mathbb{R}^N \mid ||x|-|x_0|/\varepsilon|\le\sigma\}} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx \ge \omega, \tag{32}$$

otherwise we would have $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} K(\varepsilon x)Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon)dx = 0$. But this is impossible because it contradicts (7).

We fix $\sigma > 0$ and choose a radially symmetric function $\psi_\sigma \in C_0^\infty$ so that

$$\psi_\sigma(x) = \begin{cases} 0 & \text{if} \quad ||x| - |x_0|/\varepsilon| \ge 2\sigma, \\ 1 & \text{if} \quad ||x| - |x_0|/\varepsilon| \le \sigma, \end{cases}$$

$0 \le \psi_\sigma \le 1$ and $|\nabla\psi_\sigma| \le 3/\sigma$. From (32) it follows that

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} K(\varepsilon x)Q(\psi_\sigma\bar{u}_\varepsilon, \psi_\sigma\bar{v}_\varepsilon)dx \ge \omega. \tag{33}$$

Now, we claim that

$$\lim_{\varepsilon\to 0} \varepsilon^{(N-1)(p-1)/(p+1)} \|(\psi_\sigma\bar{u}_\varepsilon, \psi_\sigma\bar{v}_\varepsilon)\|_\varepsilon^2 = 0. \tag{34}$$

Indeed, by the Cauchy-Schwarz inequality, the boundedness of the gradient of ψ_σ, and by the fact that $\alpha_0 = \inf_{x\in\text{supp}(\psi_\sigma)} V(\varepsilon x) > 0$, we have

$$\begin{aligned} &\|(\psi_\sigma\bar{u}_\varepsilon, \psi_\sigma\bar{v}_\varepsilon)\|_\varepsilon^2 \\ &\le C_1 \int_{\text{supp}(\psi_\sigma)} [|\nabla\bar{u}_\varepsilon|^2 + |\nabla\bar{v}_\varepsilon|^2 + (\bar{u}_\varepsilon)^2 + (\bar{v}_\varepsilon)^2 + V_1(\varepsilon x)(\bar{u}_\varepsilon)^2 \\ &\quad + V_2(\varepsilon x)(\bar{v}_\varepsilon)^2]dx \\ &\le C_1 \int_{\text{supp}(\psi_\sigma)} [|\nabla\bar{u}_\varepsilon|^2 + |\nabla\bar{v}_\varepsilon|^2 + \frac{1}{\alpha_0}V_1(\varepsilon x)(\bar{u}_\varepsilon)^2 + \frac{1}{\alpha_0}V_2(\varepsilon x)(\bar{v}_\varepsilon)^2 \\ &\quad + V_1(\varepsilon x)(\bar{u}_\varepsilon)^2 + V_2(\varepsilon x)(\bar{v}_\varepsilon)^2]dx \\ &\le C_2 \|(\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\varepsilon^2 = C_2 M_\varepsilon \end{aligned}$$

for some positive constants C_1 and C_2, independent of $\varepsilon > 0$. By Lemma 2, (34) follows.

On other hand, putting $D_\varepsilon \equiv \left\{x \in \mathbb{R}^N \mid |x_0|/\varepsilon - 2\sigma \le |x| \le |x_0|/\varepsilon + 2\sigma\right\}$, we see that

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \varepsilon^{(N-1)(p-1)/(p+1)} \|(\psi_\sigma\bar{u}_\varepsilon, \psi_\sigma\bar{v}_\varepsilon)\|_\varepsilon^2 \\ &\ge \liminf_{\varepsilon\to 0} \left(\left[\int_{\mathbb{R}^N} K(\varepsilon x)Q(\psi_\sigma\bar{u}_\varepsilon, \psi_\sigma\bar{v}_\varepsilon)dx \right]^{2/(p+1)} \right) \\ &\quad \times \liminf_{\varepsilon\to 0} \left(\varepsilon^{(N-1)(p-1)/(p+1)} \inf_{u,v\in C_0^1(D_\varepsilon)} \frac{\|(u,v)\|_\varepsilon^2}{[\int_{\mathbb{R}^N} K(\varepsilon x)Q(u,v)dx]^{2/(p+1)}} \right) \\ &\ge C\omega^{2/(p+1)} \liminf_{\varepsilon\to 0} ((|x_0| - 2\sigma\varepsilon)^{N-1}(|x_0| + 2\sigma\varepsilon)^{-2(N-1)/(p+1)})J_\sigma \\ &= C\omega^{2/(p+1)} |x_0|^{(N-1)(p-1)/(p+1)} J_\sigma > 0 \end{aligned} \tag{35}$$

for some $C > 0$, where

$$J_\sigma \equiv \inf_{g_0, g_1 \in C_0^1(-2\sigma, 2\sigma)} \frac{\int_{-2\sigma}^{2\sigma} [(g_0'(s))^2 + (g_1'(s))^2 + \alpha_0((g_0(s))^2 + (g_1(s))^2)] ds}{[\int_{-2\sigma}^{2\sigma} |g_0(s)|^{p+1} ds]^{2/(p+1)} + [\int_{-2\sigma}^{2\sigma} |g_1(s)|^{p+1} ds]^{2/(p+1)}}.$$

From (34) and (35) we have a contradiction. So, to conclude the proof of the lemma, we will have a verification for (35).

Using (Q_2), change of variables and setting $g_0(s + |x_0|/\varepsilon) = \bar{g}_0(s)$, $g_1(s + |x_0|/\varepsilon) = \bar{g}_1(s)$, we deduce that

$$\begin{aligned} &\frac{\|(u,v)\|_\varepsilon^2}{[\int_{\mathbb{R}^N} K(\varepsilon x) Q(u,v) dx]^{2/(p+1)}} \\ &\geq C \varepsilon^{-(N-1)(p-1)/(p+1)} (|x_0| - 2\sigma\varepsilon)^{N-1} (|x_0| + 2\sigma\varepsilon)^{-2(N-1)/(p+1)} \\ &\quad \times \frac{\int_{-2\sigma}^{2\sigma} [(\bar{g}_0'(s))^2 + (\bar{g}_1'(s))^2 + \alpha_0((\bar{g}_0(s))^2 + (\bar{g}_1(s))^2)] ds}{\left(\int_{-2\sigma}^{2\sigma} |\bar{g}_0(s)|^{p+1} ds\right)^{2/(p+1)} + \left(\int_{-2\sigma}^{2\sigma} |\bar{g}_1(s)|^{p+1} ds\right)^{2/(p+1)}} \\ &\geq C \varepsilon^{-(N-1)(p-1)/(p+1)} (|x_0| - 2\sigma\varepsilon)^{N-1} (|x_0| + 2\sigma\varepsilon)^{-2(N-1)/(p+1)} J_\sigma \end{aligned}$$

for some positive constant C. Then

$$\begin{aligned} &\varepsilon^{(N-1)(p-1)/(p+1)} \inf_{u,v \in C_0^1(D_\varepsilon)} \frac{\|(u,v)\|_\varepsilon^2}{[\int_{\mathbb{R}^N} K(\varepsilon x) Q(u,v) dx]^{2/(p+1)}} \\ &\geq C (|x_0| - 2\sigma\varepsilon)^{N-1} (|x_0| + 2\sigma\varepsilon)^{-2(N-1)/(p+1)} J_\sigma. \end{aligned} \tag{36}$$

Combining (33) and (36) we obtain (35). The proof of the lemma is complete. □

Lemma 5. *If* $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ *and* α_ε *are as above, then*

$$\lim_{\varepsilon \to 0} || (\alpha_\varepsilon)^{1/(p-1)} \bar{u}_\varepsilon ||_{L^\infty(\mathbb{R}^N)} = \lim_{\varepsilon \to 0} || (\alpha_\varepsilon)^{1/(p-1)} \bar{v}_\varepsilon ||_{L^\infty(\mathbb{R}^N)} = 0.$$

Proof. Let $w_\varepsilon = (\alpha_\varepsilon)^{1/(p-1)} (\bar{u}_\varepsilon + \bar{v}_\varepsilon)$. By (Q_1), (Q_5) and the fact that $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ is solution $(S_{\alpha_\varepsilon,\, \beta_\varepsilon})$ it follows that

$$-\Delta w_\varepsilon + V(\varepsilon x) w_\varepsilon \leq C K(\varepsilon x) (w_\varepsilon)^p \text{ in } \mathbb{R}^N \tag{37}$$

for some positive constant C.

Now, we claim that

$$\lim_{\varepsilon \to 0} ||w_\varepsilon||_{L^\infty(\{x \in \mathbb{R}^N \mid |y|/\varepsilon - 1 \leq |x| \leq |y|/\varepsilon + 1\})} = 0 \tag{38}$$

for all $y \in \mathbb{R}^N \setminus \{0\}$ and

$$\lim_{\varepsilon \to 0} ||w_\varepsilon||_{L^\infty(B(0, r_0/\varepsilon))} = 0 \tag{39}$$

for some constant $r_0 > 0$.

Proof of (38): First of all we will show that

$$\lim_{\varepsilon\to 0}\int_{B(y/\varepsilon,2)}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx=0, \tag{40}$$

all $y\in\mathbb{R}^N\setminus\{0\}$. Suppose that

$$\liminf_{\varepsilon\to 0}\int_{B(y/\varepsilon,2)}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx>0$$

for some $y\in\mathbb{R}^N\setminus\{0\}$. As K, $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ are radially symmetric functions, it follows that

$$\begin{aligned}&(\varepsilon/|y|)^{N-1}\int_{\{x\in\mathbb{R}^N\ |\ |y|/\varepsilon-2\le|x|\le|y|/\varepsilon+2\}}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx\\ \ge\ &C\int_{B(y/\varepsilon,2)}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx.\end{aligned}$$

This implies that

$$\liminf_{\varepsilon\to 0}(\varepsilon/|y|)^{N-1}\int_{\{x\in\mathbb{R}^N\ |\ |y|/\varepsilon-2\le|x|\le|y|/\varepsilon+2\}}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx>0.$$

In view of (7) and Lemma 4, we have a contradiction. Similarly,

$$\limsup_{\varepsilon\to 0}\int_{B(y/\varepsilon,2)}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx=0,\forall y\in\mathbb{R}^N\setminus\{0\}$$

and the proof of (40) is complete.

Note that

$$\int_{B(y/\varepsilon,2)}(w_\varepsilon)^{p+1}\,dx\le C\int_{B(y/\varepsilon,2)}(\alpha_\varepsilon)^{(p+1)/(p-1)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)\,dx$$

for some $C>0$. From this and (40) we see that

$$\lim_{\varepsilon\to 0}\int_{B(y/\varepsilon,2)}(w_\varepsilon)^{p+1}\,dx=0 \tag{41}$$

for all $y\in\mathbb{R}^N\setminus\{0\}$. Now we fix $\varepsilon>0$. Using (37) and the fact that w_ε is radially symmetric we deduce, by the Moser iteration argument (see Theorem 9.20 in [26]), that

$$||w_\varepsilon||_{L^\infty(\{x\in\mathbb{R}^N\ |\ |y|/\varepsilon-1\le|x|\le|y|/\varepsilon+1\})}\le C\left(\int_{B(y/\varepsilon,2)}(w_\varepsilon)^{p+1}\,dx\right)^{1/(p+1)}$$

for some positive constant C, independent of $\varepsilon>0$. Using this and (41) we obtain (38).

Proof of (39): From (8), the definition of χ_ε and the fact that $0\notin A^{4\delta}$, it follows that there is a constant $r_0>0$ such that

$$\int_{B(0,2r_0/\varepsilon)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)dx\le C\varepsilon^{(N-1)+3(p+1)/(p-1)} \tag{42}$$

for small $\varepsilon > 0$ and for some $C > 0$. By (42) and Lemma 4, we have

$$\int_{B(0,2r_0/\varepsilon)} (\alpha_\varepsilon)^{(p+1)/(p-1)} K(\varepsilon x) Q\left(\bar{u}_\varepsilon, \bar{v}_\varepsilon\right) dx \le C\varepsilon^{3(p+1)/(p-1)} \tag{43}$$

for small $\varepsilon > 0$ and for some $C > 0$. As (Q_2) and (43) are satisfied, we see that

$$\int_{B(0,2r_0/\varepsilon)} (w_\varepsilon)^{p+1}\, dx \le C\varepsilon^{3(p+1)/(p-1)} \tag{44}$$

for small $\varepsilon > 0$ and for some $C > 0$. Then using Theorem 9.20 in [26] and (44) we conclude that

$$||w_\varepsilon||_{L^\infty(B(0,r_0/\varepsilon))} \le C\varepsilon^{N/(p+1)}\varepsilon^{3/(p-1)}$$

for some positive constant C, independent of $\varepsilon > 0$. This shows (39). From (38) and (39) the lemma follows. □

In the next lemma, we also will use the arguments developed by Byeon and Wang in [18] adapted to our case.

Lemma 6. $\liminf_{\varepsilon\to 0} \varepsilon^{-2}\alpha_\varepsilon > 0$.

Proof. On the contrary, we assume for a subsequence, still denoted by ε, that $\varepsilon^{-2}\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Let ϕ be a cut-off function such that $\phi(x) = 1$ for $x \in A_\varepsilon^{4\delta}$, $\phi(x) = 0$ for $x \notin A_\varepsilon^{5\delta}$, $0 \le \phi \le 1$ and $|\nabla\phi| \le c\varepsilon$, $c > 0$. Then, it follows that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \left[|\nabla(\phi\bar{u}_\varepsilon)|^2 + |\nabla(\phi\bar{v}_\varepsilon)|^2\right] dx = 0. \tag{45}$$

In fact, since $V(\varepsilon x) \ge \delta_0 > 0$ for all $x \in \text{supp}(|\nabla\phi|)$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[|\nabla(\phi\bar{u}_\varepsilon)|^2 + |\nabla(\phi\bar{v}_\varepsilon)|^2\right] dx \\
&\le 2(c\varepsilon)^2\delta_0^{-1}\int_{\text{supp}(|\nabla\phi|)} [V_1(\varepsilon x)(\bar{u}_\varepsilon)^2 + V_2(\varepsilon x)(\bar{v}_\varepsilon)^2]dx \\
&\quad + 2\int_{\mathbb{R}^N} [|\nabla\bar{u}_\varepsilon|^2 + |\nabla\bar{v}_\varepsilon|^2]dx \\
&\le C\int_{\mathbb{R}^N} [|\nabla\bar{u}_\varepsilon|^2 + |\nabla\bar{v}_\varepsilon|^2 + V_1(\varepsilon x)(\bar{u}_\varepsilon)^2 + V_2(\varepsilon x)(\bar{v}_\varepsilon)^2]dx = C\left\|(\bar{u}_\varepsilon, \bar{v}_\varepsilon)\right\|_\varepsilon^2
\end{aligned}$$

for some positive constant C, independent of small $\varepsilon > 0$. This and Lemma 3 imply

$$\int_{\mathbb{R}^N} \left[|\nabla(\phi\bar{u}_\varepsilon)|^2 + |\nabla(\phi\bar{v}_\varepsilon)|^2\right] dx \le C_1\alpha_\varepsilon \le C_1\varepsilon^{-2}\alpha_\varepsilon$$

for some $C_1 > 0$ independent of small $\varepsilon > 0$. Hence, we get (45).

Now, using change of variables, (Q_2), the Hölder inequality and Sobolev imbedding results, we see that

$$\int_{A_\varepsilon^{4\delta}} K(\varepsilon x) Q(\bar{u}_\varepsilon(x), \bar{v}_\varepsilon(x)) dx$$

$$\leq C\left|A^{4\delta}\right|^{(2^*-(p+1))/2^*}\varepsilon^{-N}\Big(\int_{A^{4\delta}}\left(\phi\left(y/\varepsilon\right)\bar{u}_\varepsilon\left(y/\varepsilon\right)\right)^{2^*}dy\Big)^{(p+1)/2^*}$$
$$+C\left|A^{4\delta}\right|^{(2^*-(p+1))/2^*}\varepsilon^{-N}\Big(\int_{A^{4\delta}}\left(\phi\left(y/\varepsilon\right)\bar{v}_\varepsilon\left(y/\varepsilon\right)\right)^{2^*}dy\Big)^{(p+1)/2^*}$$
$$\leq C\left|A^{4\delta}\right|^{(2^*-(p+1))/2}\ \varepsilon^{N(p-1)/2}\Big(\int_{\mathbb{R}^N}\left|\nabla(\phi\bar{u}_\varepsilon)\right|^2dx\Big)^{(p+1)/2}$$
$$+C\left|A^{4\delta}\right|^{(2^*-(p+1))/2}\ \varepsilon^{N(p-1)/2}\Big(\int_{\mathbb{R}^N}\left|\nabla(\phi\bar{v}_\varepsilon)\right|^2dx\Big)^{(p+1)/2}$$

for some $C>0$ independent of ε. From (45) it follows that

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon^{4\delta}}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)dx=0. \tag{46}$$

From (8) and the definition of χ_ε, we conclude that

$$\lim_{\varepsilon\to 0}\int_{B(0,R_0/\varepsilon)\backslash A_\varepsilon^{4\delta}}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)dx=0 \tag{47}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\backslash B(0,R_0/\varepsilon)}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)dx=0. \tag{48}$$

As a consequence, from (46), (47) and (48) we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}K(\varepsilon x)Q(\bar{u}_\varepsilon,\bar{v}_\varepsilon)dx=0.$$

But this is a contradiction with (7). The proof of lemma is complete. □

Completion of the proof for Theorem 1. To complete the proof of Theorem 1, we use arguments developed in [11], [16], [17] and [18]. We define $U_\varepsilon\equiv(C\alpha_\varepsilon)^{1/(p-1)}\left(\bar{u}_\varepsilon+\bar{v}_\varepsilon\right)$, where $C>0$ was obtained in (37). Note that

$$-\Delta U_\varepsilon+V(\varepsilon x)U_\varepsilon\leq K(\varepsilon x)(U_\varepsilon)^p \text{ in } \mathbb{R}^N. \tag{49}$$

By Lemma 5,

$$\lim_{\varepsilon\to 0}\|U_\varepsilon\|_{L^\infty(\mathbb{R}^N)}=0. \tag{50}$$

Let

$$2c=\inf\left\{V(x)\mid x\in B(0,3R_0)\backslash Z^\delta\right\}>0.$$

So, we conclude that

$$U_\varepsilon(x)\leq\exp(-cd(x,\partial(B(0,3R_0/\varepsilon)\backslash Z_\varepsilon^\delta))) \tag{51}$$

for all $x\in B(0,3R_0/\varepsilon)\backslash Z_\varepsilon^\delta$ and for some $c>0$. Indeed, from (49) and (50) it follows that

$$\Delta U_\varepsilon-cU_\varepsilon\geq 0 \text{ in } B(0,3R_0/\varepsilon)\backslash Z_\varepsilon^\delta \tag{52}$$

for small $\varepsilon > 0$. Putting $F_\varepsilon(x) = \exp(-\sqrt{c}d(x, \partial(B(0, 3R_0/\varepsilon)\backslash Z_\varepsilon^\delta)))$ we deduce that

$$\Delta F_\varepsilon - cF_\varepsilon < 0 \text{ in } B(0, 3R_0/\varepsilon)\backslash Z_\varepsilon^\delta. \tag{53}$$

By (50), (52), (53) and the comparison principle we obtain (51). Using (51) we have that

$$U_\varepsilon(x) \le \exp(-c\delta/\varepsilon) \text{ in } Z_\varepsilon^{3\delta}\backslash Z_\varepsilon^{2\delta} \tag{54}$$

for some constant $c > 0$.

For a connected component Λ of $\mathrm{int}(Z^{4\delta}\backslash A^{4\delta})$, we consider the first eigenvalue problem on Λ,

$$\begin{cases} -\Delta\phi = \lambda_1\phi & \text{in} \quad \Lambda, \\ \phi = 0 & \text{on} \quad \partial\Lambda. \end{cases} \tag{55}$$

We can assume that $\max_{x\in\Lambda\cap\partial Z^{3\delta}} \phi(x) \ge 1$. Now, we claim that

$$U_\varepsilon(x) \le C\exp(-c\delta/\varepsilon) \text{ in } \Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}, \tag{56}$$

where $\Lambda_\varepsilon = \left\{x \in \mathbb{R}^N \mid \varepsilon x \in \Lambda\right\}$. To justify the assertion (56), we define $\phi_\varepsilon(x) \equiv \exp(-c\delta/\varepsilon)\phi(\varepsilon x)$. Using (Q_2) and (8), we see that

$$\int_{\mathbb{R}^N} \chi_\varepsilon(U_\varepsilon)^{p+1}dx \le C\,(\alpha_\varepsilon)^{(p+1)/(p-1)} \tag{57}$$

for some positive constant C. Now, let $z \in Z_\varepsilon^{3\delta}\backslash A_\varepsilon^{3\delta}$. Then, by Lemma 4, (57) and the definition of χ_ε, we conclude that

$$\int_{B(z,\delta/\varepsilon)} (U_\varepsilon)^{p+1}dx \le C\varepsilon^{3(p+1)/(p-1)} \tag{58}$$

for small $\varepsilon > 0$. From Theorem 9.20 in [26] and (58) it follows that

$$\sup_{B(z,\delta/2\varepsilon)} U_\varepsilon \le C\varepsilon^{3/(p-1)}$$

for small $\varepsilon > 0$. Thus,

$$U_\varepsilon \le C\varepsilon^{3/(p-1)} \text{ in } Z_\varepsilon^{3\delta}\backslash A_\varepsilon^{3\delta}. \tag{59}$$

From (49) and (59) we have

$$\Delta U_\varepsilon + C_1\varepsilon^3 U_\varepsilon \ge 0 \text{ in } \Lambda_\varepsilon \cap Z_\varepsilon^{3\delta} \tag{60}$$

for some positive constant C_1. Since ϕ satisfies (55), we deduce that, for small $\varepsilon > 0$,

$$\Delta\phi_\varepsilon + C_1\varepsilon^3\phi_\varepsilon \le 0 \text{ in } \Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}. \tag{61}$$

From (54) and the fact that $\phi(x) \ge 1$ for $x \in \Lambda \cap \partial Z^{3\delta}$, we conclude that $(U_\varepsilon - \phi_\varepsilon)_+ = 0$ on $\Lambda_\varepsilon \cap (Z_\varepsilon^{3\delta}\backslash Z_\varepsilon^{2\delta})$. From (60) and (61) we see that

$$-\Delta\left(U_\varepsilon - \phi_\varepsilon\right) \le C_1\varepsilon^3\left(U_\varepsilon - \phi_\varepsilon\right) \text{ in } \Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}. \tag{62}$$

As in [11], using $(U_\varepsilon - \phi_\varepsilon)_+$ as a test function in (62) and the Poincaré inequality, we obtain

$$
\begin{aligned}
(63)\qquad &\int_{\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}} \left|\nabla (U_\varepsilon - \phi_\varepsilon)_+\right|^2 dx \\
&\le C_1\varepsilon^3 \int_{\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}} \left((U_\varepsilon - \phi_\varepsilon)_+\right)^2 dx \\
&\le C_1\varepsilon^3 (\left|\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}\right| / \omega_N)^{2/N} \int_{\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}} \left|\nabla (U_\varepsilon - \phi_\varepsilon)_+\right|^2 dx \\
&\le C\varepsilon \int_{\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}} \left|\nabla (U_\varepsilon - \phi_\varepsilon)_+\right|^2 dx
\end{aligned}
$$

for some $C > 0$. From (63) it follows that $(U_\varepsilon - \phi_\varepsilon)_+ = 0$ in $\Lambda_\varepsilon \cap Z_\varepsilon^{3\delta}$ for small $\varepsilon > 0$. This shows (56). From (51) and (56), we deduce that for some C, $c > 0$,

$$
(64)\qquad \|U_\varepsilon\|_{L^\infty(B(0,3R_0/\varepsilon - \delta/\varepsilon)\setminus A_\varepsilon^{4\delta})} \le C \exp(-c\delta/\varepsilon).
$$

Our next goal is to prove that

$$
(65)\qquad U_\varepsilon(x) \le C\left(\varepsilon/|x|\right)^{\gamma/(p+1)}
$$

for all $x \in \mathbb{R}^N \setminus B(0, 2R_0/\varepsilon)$, where $C > 0$ is a constant independent of y.

Let $y \in \mathbb{R}^N \setminus B(0, 2R_0/\varepsilon)$. From (8), the definition of χ_ε and of the fact that $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ are radially symmetric functions, we have

$$
\begin{aligned}
&\int_{B(y,2)} (\alpha_\varepsilon)^{(p+1)/(p-1)} Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx \\
&\le \frac{C}{|y|^{N-1}} \int_{\{x \in \mathbb{R}^N \,|\, |y|-2 \le |x| \le |y|+2\}} (\alpha_\varepsilon)^{(p+1)/(p-1)} Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx \\
(66)\qquad &\le C\left(\varepsilon/R_0\right)^{N-1} 2^\gamma \left(\varepsilon/|y|\right)^\gamma (\alpha_\varepsilon)^{(p+1)/(p-1)}
\end{aligned}
$$

for some constant $C > 0$. Thus, from (Q_2), (66) and Lemma 4 it follows that

$$
(67)\qquad \int_{B(y,2)} (U_\varepsilon)^{p+1} dx \le C\left(\varepsilon/|y|\right)^\gamma
$$

for sufficiently small $\varepsilon > 0$ and for some positive constant C. Then, from (67) and Theorem 9.20 in [26], we have that

$$
\sup_{B(y,1)} U_\varepsilon \le C_0\left(\varepsilon/|y|\right)^{\gamma/(p+1)} \le C_1\left(\varepsilon/|x|\right)^{\gamma/(p+1)}
$$

for some constants C_0, $C_1 > 0$, for small $\varepsilon > 0$ and for any $x \in B(y, 1)$. Hence, (65) follows. We define

$$
\omega_\varepsilon \equiv -\frac{(N-2) + \sqrt{(N-2)^2 + 4\lambda/\varepsilon^2}}{2}.
$$

Consequently, $(\omega_\varepsilon)^2 + (N-2)\omega_\varepsilon - \frac{\lambda}{\varepsilon^2} = 0$. Then, setting $\Psi_\varepsilon(x) = |x|^{\omega_\varepsilon}$, we conclude from condition (V) that

$$\begin{aligned} -\Delta\Psi_\varepsilon(x) + V(\varepsilon x)\Psi_\varepsilon(x) &\geq (2\lambda/\varepsilon^2 - (\omega_\varepsilon)^2 - (N-2)\omega_\varepsilon)r^{\omega_\varepsilon - 2} \\ &= \frac{\lambda}{\varepsilon^2 |x|^2}\Psi_\varepsilon(x) \text{ for } |x| \geq R_0/\varepsilon. \end{aligned} \tag{68}$$

Using (65), (68) and the fact that $\gamma(p-1)/(p+1) > 2$, we have

$$-\Delta\Psi_\varepsilon + V(\varepsilon x)\Psi_\varepsilon \geq K(\varepsilon x)\,(U_\varepsilon)^{p-1}\,\Psi_\varepsilon \tag{69}$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$ and small $\varepsilon > 0$. From (64), we deduce that for some C, $c > 0$,

$$U_\varepsilon \leq C\exp(-c/\varepsilon) \text{ in } \partial B(0, 2R_0/\varepsilon). \tag{70}$$

Let $\tilde{\Psi}_\varepsilon(x) = C\exp(-c/\varepsilon)(\frac{2R_0}{\varepsilon})^{-\omega_\varepsilon}\Psi_\varepsilon(x)$. We claim that

$$U_\varepsilon(x) \leq C\exp(-c/\varepsilon)(2R_0/\varepsilon)^{-\omega_\varepsilon}\Psi_\varepsilon(x) \tag{71}$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$ and some constants C, $c > 0$. In fact, as a consequence of (70), $(U_\varepsilon - \tilde{\Psi}_\varepsilon)_+ = 0$ on $\partial B(0, 2R_0/\varepsilon)$. From (69), we note that

$$-\Delta\tilde{\Psi}_\varepsilon + V(\varepsilon x)\tilde{\Psi}_\varepsilon \geq K(\varepsilon x)\,(U_\varepsilon)^{p-1}\,\tilde{\Psi}_\varepsilon \tag{72}$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$. As in [11], using (49) and (72) we see that

$$-\Delta(U_\varepsilon - \tilde{\Psi}_\varepsilon) + V(\varepsilon x)(U_\varepsilon - \tilde{\Psi}_\varepsilon) \leq (U_\varepsilon)^{p-1}\,K(\varepsilon x)(U_\varepsilon - \tilde{\Psi}_\varepsilon) \tag{73}$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$. Multiplying both sides of (73) by $(U_\varepsilon - \tilde{\Psi}_\varepsilon)_+$ and integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N \backslash B(0,2R_0/\varepsilon)} [|\nabla(U_\varepsilon - \tilde{\Psi}_\varepsilon)_+|^2 + V(\varepsilon x)((U_\varepsilon - \tilde{\Psi}_\varepsilon)_+)^2]dx \\ \leq &\int_{\mathbb{R}^N \backslash B(0,2R_0/\varepsilon)} (U_\varepsilon)^{p-1}\,K(\varepsilon x)((U_\varepsilon - \tilde{\Psi}_\varepsilon)_+)^2 dx. \end{aligned} \tag{74}$$

Using (V), (65) and the fact that $\gamma(p-1)/(p+1) - 2 > 0$, we deduce that, for some constants C_0, $C_1 > 0$,

$$\begin{aligned} K(\varepsilon x)(U_\varepsilon(x))^{p-1} &\leq C_0\varepsilon^{\gamma(p-1)/(p+1)}\frac{1}{|x|^{\gamma(p-1)/(p+1)-2}}\frac{4\lambda}{|x|^2} \\ &\leq C_1\varepsilon^{\gamma(p-1)/(p+1)}V(\varepsilon x) \end{aligned} \tag{75}$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$ and small $\varepsilon > 0$. (74) and (75) imply

$$\begin{aligned} &\int_{\mathbb{R}^N \backslash B(0,2R_0/\varepsilon)} [|\nabla(U_\varepsilon - \tilde{\Psi}_\varepsilon)_+|^2 + V(\varepsilon x)((U_\varepsilon - \tilde{\Psi}_\varepsilon)_+)^2]dx \\ \leq\ & C_1\varepsilon^{\gamma(p-1)/(p+1)}\int_{\mathbb{R}^N \backslash B(0,2R_0/\varepsilon)} V(\varepsilon x)((U_\varepsilon - \tilde{\Psi}_\varepsilon)_+)^2 dx. \end{aligned}$$

This implies that, for sufficiently small $\varepsilon > 0$, $(U_\varepsilon - \tilde{\Psi}_\varepsilon)_+ = 0$ in $\mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$ and the proof of (71) is over.

Verification of (25). Indeed, from (Q_2), (64) and Lemma 6, we infer that

$$
\begin{aligned}
Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) &\le 2\eta_2(\bar{u}_\varepsilon + \bar{v}_\varepsilon)^{p+1} \\
&= C_0(\alpha_\varepsilon)^{-(p+1)/(p-1)}(U_\varepsilon)^{p+1} \\
&\le C\varepsilon^{-2(p+1)/(p-1)} \exp(-c_1/\varepsilon) \text{ in } B(0, 3R_0/\varepsilon - \delta/\varepsilon)\backslash A_\varepsilon^{4\delta} \qquad (76)
\end{aligned}
$$

for small $\varepsilon > 0$ and for some constants c_1, C_0, $C > 0$. Thus, using the definition of χ_ε and (76) we see that

$$
\int_{B(0,R_0/\varepsilon)\backslash A_\varepsilon^{4\delta}} \chi_\varepsilon(x) Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx \le C_3 \varepsilon^{-(2N-1)-5(p+1)/(p-1)} \exp(-c_1/\varepsilon) \qquad (77)
$$

and

$$
\int_{B(0,2R_0/\varepsilon)\backslash B(0,R_0/\varepsilon)} \chi_\varepsilon(x) Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx \le C_4 \varepsilon^{-(2(p+1)/(p-1)+2\gamma+N)} \exp(-c_1/\varepsilon) \qquad (78)
$$

for some constants C_3, $C_4 > 0$, independent of ε. Moreover, from (Q_2), Lemma 6 and (71) it follows that, for some constants C_5, $c_2 > 0$,

$$
Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \le C_5 \varepsilon^{-2(p+1)/(p-1)} \exp(-c_2/\varepsilon)(2R_0)^{-(p+1)\omega_\varepsilon} \varepsilon^{(p+1)\omega_\varepsilon} |x|^{(p+1)\omega_\varepsilon} \qquad (79)
$$

for all $x \in \mathbb{R}^N \backslash B(0, 2R_0/\varepsilon)$ and small $\varepsilon > 0$. Then, combining (79) with the definition of χ_ε, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N\backslash B(0,2R_0/\varepsilon)} \chi_\varepsilon(x) Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx \\
&\le C_5 \exp(-c_2/\varepsilon)(2R_0)^{-(p+1)\omega_\varepsilon} \\
&\quad \times \varepsilon^{-2(p+1)/(p-1)+(p+1)\omega_\varepsilon-\gamma} \int_{\mathbb{R}^N\backslash B(0,2R_0/\varepsilon)} |x|^{\gamma+(p+1)\omega_\varepsilon} dx \\
&= C_6 \frac{1}{-\gamma - (p+1)\omega_\varepsilon - N} (2R_0)^{\gamma+N} \varepsilon^{-2(p+1)/(p-1)-2\gamma-N} \exp(-c_2/\varepsilon) \qquad (80)
\end{aligned}
$$

for some constant $C_6 > 0$. From (77), (78), (80) and of the fact that $\chi_\varepsilon \equiv 0$ in $A_\varepsilon^{4\delta}$, we deduce that

$$
\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \chi_\varepsilon(x) Q(\bar{u}_\varepsilon, \bar{v}_\varepsilon) dx = 0.
$$

This proves (25).

As a consequence of (25) we have $\beta_\varepsilon = 0$. Using (7), the homogeneity of Q, Lemma 3 and $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ as test function in $(S_{\alpha_\varepsilon,\,0})$, we obtain $\frac{M_\varepsilon}{p+1} = \alpha_\varepsilon$. This implies that $(u_\varepsilon, v_\varepsilon)$, where $u_\varepsilon(x) = (\alpha_\varepsilon)^{1/(p-1)} \bar{u}_\varepsilon(\varepsilon^{-1}x)$ and $v_\varepsilon(x) = (\alpha_\varepsilon)^{1/(p-1)} \bar{v}_\varepsilon(\varepsilon^{-1}x)$, satisfies (S). Note that of (64) and (71), we have

$$
u_\varepsilon(x), v_\varepsilon(x) \le C \exp(-c\delta/\varepsilon) \; \forall x \in B(0, 2R_0)\backslash A^{4\delta} \qquad (81)
$$

and

$$u_\varepsilon(x), v_\varepsilon(x) \le C\exp(-c/\varepsilon)(|x|/2R_0)^{\omega_\varepsilon}\ \forall x \in \mathbb{R}^N \backslash B(0,2R_0). \tag{82}$$

The property (1) is proved in Lemma 5. We now show the property (2), i.e.,

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2/(p-1)} \|u_\varepsilon + v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0.$$

We define $W_\varepsilon \equiv \varepsilon^{-2/(p-1)}(u_\varepsilon + v_\varepsilon)$. Then, it suffices to show that

$$\liminf_{\varepsilon\to 0} \|W_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0.$$

From (Q_1) and of the fact that $(u_\varepsilon, v_\varepsilon)$ is solution of (S) we see that, for some $C > 0$,

$$-\Delta W_\varepsilon + \frac{1}{\varepsilon^2}V(x)W_\varepsilon \le CK(x)(W_\varepsilon)^p \text{ in } \mathbb{R}^N. \tag{83}$$

Multiplying both sides of (83) by W_ε and integrating by parts, we obtain

(84)

$$\int_{\mathbb{R}^N}[|\nabla W_\varepsilon|^2 + \frac{1}{\varepsilon^2}V(x)(W_\varepsilon)^2]dx \le C\int_{\mathbb{R}^N}(W_\varepsilon)^{p+1}dx$$
$$\le C\|W_\varepsilon\|^{p-1}_{L^\infty(\mathbb{R}^N)}\int_{A^{5\delta}}(W_\varepsilon)^2dx + C\|W_\varepsilon\|^{(p-1)/2}_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N\backslash A^{5\delta}}(W_\varepsilon)^{(p-1)/2}(W_\varepsilon)^2dx.$$

Now, to conclude our proof once more, we will use the arguments developed by Byeon-Wang in [17] and [18]. We take $\varphi \in C_0^\infty(\text{int}(A^{5\delta}))$ such that $\varphi(x) = 1$ for $x \in A^{4\delta}$. As $\inf_{x\in \text{supp}(\varphi)\backslash A^{4\delta}} V(x) > 0$ and $\inf_{x\in A^{5\delta}\backslash A^{4\delta}} V(x) > 0$, it follows, by definition of φ and by the Poincaré inequality, that

$$\begin{aligned}\int_{A^{5\delta}}(W_\varepsilon)^2dx &\le 2\int_{A^{5\delta}}[(\varphi W_\varepsilon)^2 + (1-\varphi)^2(W_\varepsilon)^2]dx\\ &\le C_0\int_{\text{supp}(\varphi)}|\nabla(\varphi W_\varepsilon)|^2dx + C_1\frac{1}{\varepsilon^2}\int_{A^{5\delta}\backslash A^{4\delta}}V(x)(W_\varepsilon)^2dx\\ &\le C_2\frac{1}{\varepsilon^2}\int_{\text{supp}(\varphi)\backslash A^{4\delta}}V(x)(W_\varepsilon)^2dx + 2C_0\int_{\mathbb{R}^N}|\nabla W_\varepsilon|^2dx\\ &\quad + C_1\frac{1}{\varepsilon^2}\int_{A^{5\delta}\backslash A^{4\delta}}V(x)(W_\varepsilon)^2dx\\ &\le C\int_{\mathbb{R}^N}[|\nabla W_\varepsilon|^2 + \frac{1}{\varepsilon^2}V(x)(W_\varepsilon)^2]dx \end{aligned} \tag{85}$$

for some positive constants C_0, C_1, C_2 and C, independent of small $\varepsilon > 0$. On the other hand, using the Hölder inequality and Sobolev embedding results, we get

(86)

$$\int_{\mathbb{R}^N\backslash A^{5\delta}}(W_\varepsilon)^{(p-1)/2}(W_\varepsilon)^2dx \le C(\int_{\mathbb{R}^N\backslash A^{5\delta}}(W_\varepsilon)^{N(p-1)/4}dx)^{2/N}\int_{\mathbb{R}^N}|\nabla W_\varepsilon|^2dx.$$

In view of (81) and (82) we deduce that, for some constants C_1, $C_2 > 0$,

$$W_\varepsilon(x) \le C_1 \varepsilon^{-2/(p-1)} \exp(-c\delta/\varepsilon) \ \forall x \in B(0, 2R_0) \backslash A^{4\delta} \tag{87}$$

and

$$W_\varepsilon(x) \le C_2 \varepsilon^{-2/(p-1)} \exp(-c/\varepsilon)(2R_0)^{-\omega_\varepsilon} |x|^{\omega_\varepsilon} \ \ \forall x \in \mathbb{R}^N \backslash B(0, 2R_0). \tag{88}$$

From (87) and (88) we have, for some constants c_1, c_2, C_3, C_4, $C_5 > 0$, that

$$\begin{aligned} &\int_{\mathbb{R}^N \backslash A^{5\delta}} (W_\varepsilon)^{N(p-1)/4} dx \\ &\le C_3 \varepsilon^{-N/2} \exp(-c_1/\varepsilon) + C_4 \varepsilon^{-N/2} \exp(-c_2/\varepsilon)(2R_0)^{-N(p-1)\omega_\varepsilon/4} \\ &\quad \times \int_{\mathbb{R}^N \backslash B(0,2R_0)} |x|^{N(p-1)\omega_\varepsilon/4} \, dx \\ &= C_3 \varepsilon^{-N/2} \exp(-c_1/\varepsilon) + C_5 \varepsilon^{-N/2} \exp(-c_2/\varepsilon)(2R_0)^N \frac{1}{-N(p-1)\omega_\varepsilon/4 - N}. \end{aligned} \tag{89}$$

From (89), $\int_{\mathbb{R}^N \backslash A^{5\delta}} (W_\varepsilon)^{N(p-1)/4} dx \le 1$ for sufficiently small $\varepsilon > 0$. This and (86) imply

$$\int_{\mathbb{R}^N \backslash A^{5\delta}} (W_\varepsilon)^{(p-1)/2} (W_\varepsilon)^2 dx \le C \int_{\mathbb{R}^N} [|\nabla W_\varepsilon|^2 + \frac{1}{\varepsilon^2} V(x)(W_\varepsilon)^2] dx \tag{90}$$

for small $\varepsilon > 0$. From (84), (85) and (90) it follows that $\|W_\varepsilon\|^{p-1}_{L^\infty(\mathbb{R}^N)} + \|W_\varepsilon\|^{(p-1)/2}_{L^\infty(\mathbb{R}^N)} \ge C$ for some positive constant C. Then $\|W_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \ge C_1 > 0$, where C_1 is a constant independent of $\varepsilon > 0$. This completes the proof of property (2).

Now, we claim that $u_\varepsilon, v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$. In fact, from (81) and (82) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (u_\varepsilon)^2 dx \le& \int_{A^{4\delta}} (u_\varepsilon)^2 dx + C_0 \exp(-c_0/\varepsilon) \\ &+ C_1 \exp(-c_1/\varepsilon)(2R_0)^{-2\omega_\varepsilon} \int_{\mathbb{R}^N \backslash B(0,2R_0)} |x|^{2\omega_\varepsilon} \, dx \\ =& \int_{A^{4\delta}} (u_\varepsilon)^2 dx + C_0 \exp(-c_0/\varepsilon) \\ &+ C_2 \exp(-c_1/\varepsilon)(2R_0)^N \frac{1}{-2\omega_\varepsilon - N} \end{aligned} \tag{91}$$

for some constants C_0, C_1, C_2, c_0, $c_1 > 0$. Using the Lemma 3 and change of variables, we have

$$\begin{aligned} \varepsilon^2 \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^2 \, dx &= (\alpha_\varepsilon)^{2/(p-1)} \int_{\mathbb{R}^N} |\nabla \bar{u}_\varepsilon(x/\varepsilon)|^2 \, dx \\ &= \varepsilon^N \, (\alpha_\varepsilon)^{2/(p-1)} \int_{\mathbb{R}^N} |\nabla \bar{u}_\varepsilon(y)|^2 \, dy \end{aligned}$$

$$\le \varepsilon^N (\alpha_\varepsilon)^{2/(p-1)} \|(\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\varepsilon^2$$

$$= (p+1)(\alpha_\varepsilon)^{(p+1)/(p-1)} \varepsilon^N. \tag{92}$$

From (91) and (92) we conclude that $u_\varepsilon \in W^{1,2}(\mathbb{R}^N)$. Similarly, we obtain $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$. The decay property (3) follows from (81) and (82). □

Acknowledgement. The second author would like to thank the Department of Mathematic of UFMG for their hospitality during his visit.

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