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A NOTE ON THE GENERALIZED MYERS THEOREM FOR FINSLER MANIFOLDS

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Abstract. In this note we establish a generalized Myers theorem under line integral curvature bound for Finsler manifolds.

1. Introduction

The celebrated Myers theorem in global Riemannian geometry says that if a Riemannian manifold M satisfies $\mathbf{Ric}(v) \geq (n-1)a > 0$ for all unit vector v, then M is compact and

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{a}}.$$

There are many generalizations of Myers theorem (see e.g., [2, 3, 7]). In [7] the author proved the following result.

Theorem 1.1. *Let (M, g) be an n-dimensional complete Riemannian manifold. Then for any $\delta > 0, a > 0$, there exists $\epsilon = \epsilon(n, a, \delta)$ satisfying the following. If for any $p \in M$ and each minimal geodesic γ emanating from p, the Ricci curvature satisfies*

$$\int_\gamma \max\{(n-1)a - \mathbf{Ric}(\gamma'(t)), 0\} dt \leq \epsilon(n, a, \delta),$$

then M is compact with

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{a}} + \delta.$$

Myers theorem has also been generalized to Finsler manifolds [1]. In this note we shall prove the following result which generalizes Theorem 1.1.

Theorem 1.2. *Let (M, F) be an n-dimensional forward complete Finsler manifold. If there is $\Lambda > 0$ such that for any $p \in M$ and each minimal geodesic γ*

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emanating from p, the Ricci curvature satisfies

$$\int_\gamma \max\{(n-1)a - \mathbf{Ric}(\gamma'(t)), 0\}dt \le \Lambda,$$

then M is compact with

$$\mathrm{diam}(M) \le \frac{\pi}{\sqrt{a}} + \frac{\Lambda}{(n-1)a}.$$

2. Finsler geometry

In this section we briefly recall some fundamental materials of Finsler geometry, and for details one is referred to see [1, 4, 5, 6]. Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be local coordinates on TM, and $\pi : TM\backslash 0 \to M$ the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. The *fundamental tensor* g_{ij} is defined by

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}.$$

Let $R_j{}^i{}_{kl}$ be the *first Chern curvature tensor*, and $R_{ijkl} := g_{js}R_i{}^s{}_{kl}$. Write $\mathbf{g}_y = g_{ij}(x, y)dx^i \otimes dx^j, \mathbf{R}_y = R_{ijkl}(x, y)dx^i \otimes dx^j \otimes dx^k \otimes dx^l$. For a tangent plane $P \subset T_xM$, let

$$\mathbf{K}(P, y) = \mathbf{K}(y; u) := \frac{\mathbf{R}_y(y, u, u, y)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $y, u \in P$ are tangent vectors such that $P = \mathrm{span}\{y, u\}$. We call $\mathbf{K}(P, y)$ the *flag curvature of P with flag pole y*. Let

$$\mathbf{Ric}(y) = \sum_i \mathbf{K}(y; e_i),$$

where $\{e_1, \ldots, e_n\}$ is a $\mathbf{g}_y$-orthogonal basis for the corresponding tangent space. We call $\mathbf{Ric}(y)$ the *Ricci curvature of y*.

Let $V = v^i\partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric $\widetilde{g} = \mathbf{g}_V$ and a linear connection ∇^V (called *Chern connection*) on the tangent bundle over $\mathcal{U}$ as follows:

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x, v)\frac{\partial}{\partial x^k},$$

where $\Gamma^i_{jk}(x, v)$ are the Chern connection coefficients.

The *Legendre transformation* $l : TM \to T^*M$ is defined by

$$l(Y) = \begin{cases} \mathbf{g}_Y(Y, \cdot), & Y \ne 0 \\ 0, & Y = 0. \end{cases}$$

Now let $f : M \to \mathbb{R}$ be a smooth function on M. The *gradient* of f is defined by $\nabla f = l^{-1}(df)$. Thus we have

$$df(X) = \mathbf{g}_{\nabla f}(\nabla f, X), \quad X \in TM.$$

Let $\mathcal{U} = \{x \in M : \nabla f \mid_x \neq 0\}$. We define the *Hessian* $H(f)$ of f on $\mathcal{U}$ as follows:

$$H(f)(X,Y) := XY(f) - \nabla_X^{\nabla f} Y(f), \quad \forall X, Y \in TM \mid_{\mathcal{U}} .$$

It is known that $H(f)$ is symmetric, and it can be rewritten as (see [6])

$$H(f)(X,Y) = \mathbf{g}_{\nabla f}(\nabla_X^{\nabla f} \nabla f, Y).$$

It should be noted that the notion of Hessian defined here is different from that in [4]. In that case $H(f)$ is in fact defined by

$$H(f)(X,X) = X \cdot X \cdot (f) - \nabla_X^X X(f),$$

and there is no definition for $H(f)(X,Y)$ if $X \neq Y$. The advantage of our definition is that $H(f)$ is a symmetric bilinear form and we can treat it by using the theory of symmetric matrix. For any fixed $p \in M$ let $r = d_F(p,\cdot)$ be the distance function from p induced by Finsler metric F, and (r,θ) the polar coordinates on $M \backslash C(p)$, where $C(p)$ is the cut loci of p. The following lemma is crucial to prove Theorem 1.2.

Lemma 2.1. *Let* $h = h(r,\theta) = \mathrm{trace}_{\mathbf{g}_{\nabla r}} H(r)$. *Then* $\lim_{r\to+0} h = +\infty$, *and*

$$\frac{dh}{dr} \leq -\mathbf{Ric}(\nabla r) - \frac{h^2}{n-1}.$$

Proof. Let $E_1, \ldots, E_{n-1}, E_n = \nabla r$ be the local $\mathbf{g}_{\nabla r}$-orthonormal frame fields on $M \backslash C(p)$. We have the following equality where r is smooth (see (5.1) of [6]):

$$\frac{d}{dr}\mathrm{trace}_{\mathbf{g}_{\nabla r}} H(r) = -\mathbf{Ric}(\nabla r) - \sum_{i,j} (H(r)(E_i,E_j))^2.$$

Note that ∇r is a geodesic field, and thus $H(r)(\nabla r,\cdot) = 0$, which together with above equality and Schwartz inequality we clearly have the desired inequality. On the other hand, for sufficiently small ϵ let b be the upper bound of flag curvature on $B_p(\epsilon)$, then by Hessian comparison theorem [6] it follows that

$$h(r,\theta) \geq (n-1)\mathrm{ct}_b(r) = \begin{cases} (n-1)\sqrt{b}\cdot \mathrm{cotan}(\sqrt{b}r), & b > 0 \\ \frac{n-1}{r}, & b = 0 \\ (n-1)\sqrt{-b}\cdot \mathrm{cotanh}(\sqrt{-b}r), & b < 0 \end{cases}, \quad \forall r < \epsilon,$$

and consequently, $\lim_{r\to+0} h = +\infty$. □

3. Proof of Theorem 1.2

Now let us complete the proof of Theorem 1.2. For any fixed $p, q \in M$ let $\gamma : [0,L] \to M$ be the minimal unit-speeded geodesic from p to q with $L = r(q) = d_F(p,q)$. Let $h = h(r,\theta)$ be defined by Lemma 2.1, and consider $f = f(t) := h(\gamma(t))$, then f is smooth on $(0,L)$. By Lemma 2.1 one has

$$f'(t) \leq -\mathbf{Ric}(\gamma'(t)) - \frac{f(t)^2}{n-1},$$

and consequently,

$$
\begin{aligned}
\left(\operatorname{arccot}\left(\frac{f}{(n-1)\sqrt{a}}\right)\right)' &= \frac{-\frac{1}{(n-1)\sqrt{a}}f'}{1+\frac{f^2}{(n-1)^2a}} \\
&\geq \frac{\mathbf{Ric}(\gamma')+\frac{f^2}{n-1}}{\left(1+\frac{f^2}{(n-1)^2a}\right)(n-1)\sqrt{a}} \qquad (1)\\
&= \frac{\mathbf{Ric}(\gamma')-(n-1)a+(n-1)a\left(1+\frac{f^2}{(n-1)^2a}\right)}{\left(1+\frac{f^2}{(n-1)^2a}\right)(n-1)\sqrt{a}} \\
&\geq -\frac{1}{(n-1)\sqrt{a}}\max\{(n-1)a-\mathbf{Ric}(\gamma'),0\}+\sqrt{a}.
\end{aligned}
$$

For any small $\epsilon > 0$ integrating (1) on $(\epsilon, L-\epsilon)$ we get

$$
\begin{aligned}
&\pi - \operatorname{arccot}\left(\frac{f(\epsilon)}{(n-1)\sqrt{a}}\right) \\
(2)\quad &> \operatorname{arccot}\left(\frac{f(L-\epsilon)}{(n-1)\sqrt{a}}\right) - \operatorname{arccot}\left(\frac{f(\epsilon)}{(n-1)\sqrt{a}}\right) \\
&\geq -\frac{1}{(n-1)\sqrt{a}}\int_{\epsilon}^{L-\epsilon}\max\{(n-1)a-\mathbf{Ric}(\gamma'(t)),0\}dt+(L-2\epsilon)\sqrt{a}.
\end{aligned}
$$

On the other hand, $\lim_{t\to+0} f(t) = +\infty$ by Lemma 2.1, and thus

$$\lim_{t\to+0}\operatorname{arccot}\left(\frac{f(t)}{(n-1)\sqrt{a}}\right) = 0.$$

Now let $\epsilon \to +0$ in (2) it follows that

$$
\begin{aligned}
\pi &\geq -\frac{1}{(n-1)\sqrt{a}}\int_0^L \max\{(n-1)a-\mathbf{Ric}(\gamma'(t)),0\}dt + L\sqrt{a} \\
&\geq -\frac{\Lambda}{(n-1)\sqrt{a}} + L\sqrt{a},
\end{aligned}
$$

and consequently,

$$L \leq \frac{\pi}{\sqrt{a}} + \frac{\Lambda}{(n-1)a}.$$

So we complete the proof.

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