

REGULARITY OF SOLUTIONS OF 3D NAVIER-STOKES EQUATIONS IN A LIPSCHITZ DOMAIN FOR SMALL DATA

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ABSTRACT. We consider the global existence of strong solutions of the 3D incompressible Navier-Stokes equations in a bounded Lipschitz domain under Dirichlet boundary condition. We present by a very simple argument that a strong solution exists globally when the product of L^2 norms of the initial velocity and the gradient of the initial velocity and $L^{p,2}$, $p \geq 4$ norm of the forcing function are small enough. Our condition is scale invariant and implies many typical known global existence results for small initial data including the sharp dependence of the bound on the volume of the domain and viscosity. We also present a similar result in the whole domain with slightly stronger condition for the forcing.

1. Introduction

We consider the initial boundary value problem of the incompressible Navier-Stokes equations,

$$(1) \quad u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f,$$

$$(2) \quad \nabla \cdot u = 0,$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ or in \mathbb{R}^3 itself. Here u denotes the velocity of a homogeneous, viscous incompressible fluid, f is the density of force per unit volume, p denotes the pressure, and ν is the kinematic viscosity. We require that the forcing function f and the initial data u_0 satisfy

$$\nabla \cdot f = \nabla \cdot u_0 = 0.$$

As a boundary condition, we use

$$u|_{\partial\Omega} = 0$$

when Ω is a bounded domain.

By the classical results of Leray and Hopf ([12], [18]), there exists a global weak solution of the Navier-Stokes equations if the initial data is in L^2 and

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satisfies (2). It is also known that the solution becomes necessarily strong (regular) for all regular data in two dimensional domain. But for the case $\Omega \subset \mathbb{R}^3$, global strong solutions have only been guaranteed for small initial data and small forcing term (see [2], [7], [8], [9], [13], [14], [23], [24], [25], [27] and references therein). The same is true for solutions in a negative Besov space (see [5], [22] and references therein) and for different boundary conditions (see [6], [10], [11], [15], [16], [17], [19], [21] and reference therein).

We here provide with some simple argument for global existence for small data in a Lipschitz domain, which recovers many known such results. Our argument also holds true in the whole \mathbb{R}^3 . Concretely, we show the global existence of strong solutions if the initial data and the forcing function satisfy

$$\|u_0\|^2 \|\nabla u_0\|^2 + 4\nu^{(4-4p)/p} \lambda^{(4-3p)/p} \|f\|_{p,0}^4 \leq \frac{\nu^4}{K^4}, \quad 4 \leq p \leq \infty$$

for an interpolation constant K , which are independent of $|\Omega|$. Here, $|\Omega|$ denotes the volume of Ω , λ is the first eigenvalue of the Stokes operator, and

$$\|f\|_{p,0}^p = \int_0^\infty \|f\|_{L^2}^p(t) dt.$$

This condition is scale invariant under the Navier-Stokes scaling,

$$v \rightarrow bv(abt, ax), \quad \nu \rightarrow \frac{b}{a}\nu, \quad \lambda \rightarrow a^2\lambda, \quad f \rightarrow ab^2f(abt, ax).$$

It implies that the H^1 norm of the initial velocity need not to be small for the global regularity. It also recovers that if the L^2 norm of the gradient of the initial velocity is small enough compared with a sixth power of the volume of Ω , global existence is guaranteed when the forcing term vanishes.

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2. Regularity

We denote

$$H = \{u \in L^2(\Omega) \mid \nabla \cdot u = 0, u \cdot n(\partial\Omega, \cdot) = 0\}$$

when Ω is a bounded domain and

$$H = \{u \in L^2(\mathbb{R}^3) \mid \nabla \cdot u = 0\}$$

when $\Omega = \mathbb{R}^3$. Also, $V \equiv H \cap W_0^{1,2}(\Omega)$ when Ω is bounded and $V \equiv H \cap W^{1,2}(\Omega)$ when $\Omega = \mathbb{R}^3$. We remark that if Ω is a bounded Lipschitz domain, $\|\nabla u\|$ is an equivalent norm for V due to the Poincaré type inequality. For convenience's sake, we also denote $\|\cdot\|_p = \|\cdot\|_{L^p}$, $\|\cdot\|_2 = \|\cdot\|$. \mathbb{P} is the Leray projection

of $L^2(\Omega) = L^2(\Omega, \mathbb{R}^3)$ into H . Now, applying the Leray projection, we can reformulate (1) in the following evolutionary equation on the Hilbert space V ,

$$(3) \quad u_t + \nu Au + B(u, u) = \mathbb{P}f,$$

where $Au = -\mathbb{P}\Delta u$, and the bilinear form $B(u, v) = \mathbb{P}(u \cdot \nabla)v$. We will be interested in solutions of (3) with the initial data u_0 and $f = f(t)$ satisfying

$$(4) \quad u_0 \in V, \quad f(t) \in L^\infty([0, \infty), H).$$

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^3 . The Navier-Stokes evolutionary equation (3) has a solution*

$$u \in C^0([0, \infty), H) \cap L^\infty((0, \infty), V)$$

whenever u_0 and f satisfy

$$(5) \quad \|u_0\|^2 \|\nabla u_0\|^2 + 4\nu^{\frac{4-4p}{p}} \lambda^{\frac{4-3p}{p}} \|f\|_{p,0}^4 \leq \frac{\nu^4}{K^4}$$

for some $p \in [4, \infty]$. Here, K is a scale invariant embedding constant, λ is the first eigenvalue of the Stokes operator, and $\|\cdot\|_{p,0} = \|\cdot\|_{L^p((0,\infty),L^2)}$. Moreover, in this case

$$(6) \quad \|\nabla u\|^2(t) \leq \|\nabla u_0\|^2 + 4\nu^{\frac{2-2p}{p}} \lambda^{\frac{2-p}{p}} \|f\|_{p,0}^2$$

for all $t > 0$.

Proof. Since $u_0 \in H^1$ and $f \in L^2(0, T; L^2)$ for all $T > 0$, there exists a local in time unique strong solution, u corresponding to u_0 and f . Suppose that this solution blows up in a finite time. This means that there exists a maximal time of existence, $T^* > 0$ at which the solution quits to belong H^1 . Now, let us restrict ourself $t < T^*$. By taking the scalar product of (3) with u and using the fact that $\langle B(u, u), u \rangle = 0$, we find that

$$(7) \quad \frac{d}{dt} \|u\|^2 + 2\nu \|\nabla u\|^2 \leq 2\|f\| \|u\|.$$

Now, we borrow the following inequality for a bounded Lipschitz domain from [3] and [1]

$$(8) \quad \|u\|_\infty \leq K \|\nabla u\|^{1/2} \|Au\|^{1/2}$$

to get

$$\begin{aligned} |\langle B(u, u), Au \rangle| &= \left| \int u \cdot \nabla u \cdot Au \right| \\ &\leq \|u\|_{L^\infty} \|\nabla u\| \|Au\| \\ &\leq K \|\nabla u\|^{3/2} \|Au\|^{3/2} \\ &\leq K \|u\|^{1/2} \|\nabla u\|^{1/2} \|Au\|^2. \end{aligned}$$

Here, in the last line, we use the inequality

$$(9) \quad \|\nabla u\|^2 = - \int u \cdot Au \leq \|u\| \|Au\|.$$

Note that K is a scaling invariant constant which may depend on the Lipschitz norm of $\partial\Omega$. Taking the scalar product of (3) with Au and using the above estimate, we obtain

$$(10) \quad \frac{d}{dt} \|\nabla u\|^2 + 2\nu \|Au\|^2 \leq 2\|f\| \|Au\| + K(\|u\| \|\nabla u\|)^{1/2} \|Au\|^2.$$

Next, we multiply (7) by $\|\nabla u\|^2$ and (10) by $\|u\|^2$ and adding them to have

$$(11) \quad \begin{aligned} & \frac{d}{dt} (\|u\|^2 \|\nabla u\|^2) + 2\nu \|\nabla u\|^4 + 2\nu \|u\|^2 \|Au\|^2 \\ & \leq 2\|f\| \|u\| (\|\nabla u\|^2 + \|u\| \|Au\|) + K(\|u\| \|\nabla u\|)^{1/2} \|u\|^2 \|Au\|^2. \end{aligned}$$

By the Hölder inequality and (9), we have

$$\begin{aligned} 2\|f\| \|u\| (\|\nabla u\|^2 + \|u\| \|Au\|) & \leq 4\|f\| \|u\|^2 \|Au\| \\ & \leq \nu \|u\|^2 \|Au\|^2 + \nu \|\nabla u\|^4 + \frac{4}{\nu^3 \lambda^2} \|f\|^4. \end{aligned}$$

Denoting $G = \|u\|^2 \|\nabla u\|^2$, we thus arrive at the differential inequality

$$\frac{d}{dt} G + \nu \lambda G \leq \left[K G^{1/4} - \nu \right] \|u\|^2 \|Au\|^2 + \frac{4}{\nu^3 \lambda^2} \|f\|^4.$$

By (5), $K G(0)^{1/4} - \nu \leq 0$. And, since $G(t)$ is continuous for $t < T^*$, there exists a maximal $T_1 \leq T^*$ such that $K G^{1/4} - \nu \leq 0$ on $[0, T_1)$. Then we have by the Grönwall inequality

$$\begin{aligned} G(t) & \leq G(0) e^{-\nu \lambda t} + \frac{4}{\nu^3 \lambda^2} \int_0^t \|f\|^4 e^{\nu \lambda (s-t)} ds \\ & < G(0) + 4\nu^{(4-4p)/p} \lambda^{(4-3p)/p} \|f\|_{p,0}^4, \quad 4 \leq p \leq \infty \end{aligned}$$

for $t < T_1$ (Equality cannot hold in the last line). This implies that $K G(t)^{1/4} - \nu < 0$ for all $t < T_1$ by (5) and thus $T_1 = T^*$. Now, using $G(t) \leq \frac{\nu^4}{K^4}$ for $t < T^*$, we have from (10)

$$\frac{d}{dt} \|\nabla u\|^2 + \frac{\nu}{2} \|Au\|^2 \leq \frac{2}{\nu} \|f\|^2, \quad t < T^*.$$

By the Grönwall inequality and the inequality

$$\lambda \|\nabla u\|^2 \leq \|Au\|^2,$$

we have finally

$$\begin{aligned} \|\nabla u\|^2(t) & \leq \|\nabla u\|^2(0) e^{-\frac{\nu \lambda}{2} t} + \frac{2}{\nu} \int_0^t \|f\|^2 e^{\frac{\nu \lambda}{2} (s-t)} ds \\ & \leq \|\nabla u\|^2(0) + 4\nu^{\frac{2-2p}{p}} \lambda^{\frac{2-p}{p}} \|f\|_{p,0}^2 \end{aligned}$$

for all $t < T^*$. This leads a contradiction and we conclude $T^* = +\infty$. The above inequality shows (6) then for all $t > 0$. □

Remark. 1. The condition (5) is in a sense a condition of smallness of the initial data and external force. However, this condition allows for initial data with large H^1 norm provided that the L^2 norm of the initial data is small enough. In particular, when $f = 0$, the above theorem tells that even for initial data with large H^1 norm, there exists a globally regular solution if $\|u_0\|$ is small enough compared with $\nu^2 \|\nabla u_0\|^{-1}$.

2. Although we show the above theorem under Dirichlet boundary condition for a bounded domain, it still holds for a cylindrical domain under mixed no-slip and periodic boundary condition.

3. The condition $p \geq 4$ could be relaxed to $p \geq 2$ if we just want to obtain the global regularity of the solution. However, we may lose the simple scale invariant bound of H^1 norm of the solution in this approach.

The above theorem is also in consistent with the previously known global regularity result for small initial data. For example, the case $p = 2$ and Ω being C^2 is reduced to Theorems 9.3 and 9.4 in [7]. It also gives the dependence of the bound on the volume of the domain. The following corollary reveals this, which is a consequence of a simple embedding theorem.

Corollary 2.2. *Given initial data $u_0 \in H^1 \cap H$ and $4 \leq p \leq \infty$. There exists a scaling invariant constant C such that if*

$$\|\nabla u_0\| \leq C\nu|\Omega|^{-1/6}, \quad \|f\|_{p,0} \leq C\nu^{(2p-1)/p}|\Omega|^{-(3p-4)/6p},$$

there exists a globally regular solution of (3).

Proof. Since $\lambda\|u_0\|^2 \leq \|\nabla u_0\|^2$, the condition (5) is satisfied if

$$\|\nabla u_0\|^4 \leq \frac{\nu^4}{5K^4}\lambda, \quad \|f\|_{p,0}^4 \leq \frac{\nu^{(8p-4)/p}\lambda^{(3p-4)/p}}{5K^4}.$$

Meanwhile, since

$$\lambda = \inf_{v \in D(A^{1/2})} \frac{\|\nabla v\|^2}{\|v\|^2},$$

the scaling $x \rightarrow |\Omega|^{1/3}x$ gives $\lambda(\Omega) = |\Omega|^{-2/3}\lambda(\Omega_0)$. Here, Ω_0 is a scaling of Ω with unit volume. Further, $\lambda(\Omega_0) \geq \lambda^D(\Omega_0)$, the first Dirichlet eigenvalue by the above definition. Thus, denoting the ball of unit volume by B , $\lambda(\Omega_0) \geq \lambda^D(B)$ by symmetrization. Putting these together and defining C suitably, we finish the proof. \square

The above corollary in particular tells that when $p = 2$, both of the bounds grows like $|\Omega|^{-1/6}$.

Under slightly stronger condition on the external force, an analogy of the above theorem holds on the whole of \mathbb{R}^3 .

Theorem 2.3. *When $\Omega = \mathbb{R}^3$, (3) has a solution*

$$u \in C^0([0, \infty), H) \cap L^\infty((0, \infty), V)$$

whenever u_0 and f satisfy

$$(12) \quad \|u_0\|^2 \|\nabla u_0\|^2 + \frac{2}{\nu} \left(\|u_0\| + \int_0^\infty \|f\| \right)^2 \int_0^\infty \|f\|^2 \leq \frac{\nu^4}{K}.$$

Further,

$$(13) \quad \|\nabla u\|^2(t) \leq \|\nabla u_0\|^2 + \frac{1}{\nu} \int_0^t \|f\|^2$$

for all $t > 0$.

Proof. Exactly by the same argument as in the previous theorem, (7) and (10) holds in this case, too. In particular, the inequality (8) reduces to the Gagliardo-Nirenberg inequality in this case and is well-known. Dividing by $2\|u\|$, we reduce (7) further to

$$\frac{d}{dt} \|u\| \leq \|f\|.$$

Integrating the above inequality, we have

$$(14) \quad \|u\|(t) \leq \|u_0\| + \int_0^t \|f\|.$$

Meanwhile, from (7) and (10), we again have (11). To manipulate (11) further, we estimate the forcing term as follows.

$$2\|f\|\|u\|(\|\nabla u\|^2 + \|u\|\|Au\|) \leq \nu\|\nabla u\|^4 + \nu\|u\|^2\|Au\|^2 + \frac{2}{\nu}\|f\|^2\|u\|^2.$$

Then, (11) is reduced to

$$\frac{d}{dt} G \leq \left[KG^{1/4} - \nu \right] \|u\|^2\|Au\|^2 + \frac{2}{\nu}\|f\|^2\|u\|^2.$$

Here, $G = \|u\|^2\|\nabla u\|^2$ as before. Again by the continuation argument and the Grönwall inequality, $G(t) \leq G(0)$ if

$$G(0) + \frac{2}{\nu} \int_0^\infty \|f\|^2\|u\|^2 \leq \frac{\nu^4}{K^4}.$$

By (14), the above inequality is satisfied under (12). Finally, once $G(t) \leq G(0) \leq (\nu/K)^4$, from (10),

$$\begin{aligned} \frac{d}{dt} \|\nabla u\|^2 + \nu\|Au\|^2 &\leq \|f\|\|Au\| \\ &\leq \frac{1}{\nu}\|f\|^2 + \nu\|Au\|^2. \end{aligned}$$

Integrating the above inequality, we recover (13) and finish the proof. \square

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