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ON DISCRETENESS OF MÖBIUS GROUPS

Xi Fu

Abstract. It's known that one could use a fixed loxodromic or parabolic element in $M(\overline{\mathbb{R}}^n)$ as a test map to test the discreteness of a non-elementary Möbius group G. In this paper, we discuss the discreteness of G by using a fixed elliptic element.

1. Introduction

The discreteness of Möbius groups is an old and interesting problem which has been studied by many authors. For instance, in [4], Jørgensen obtained a useful necessary condition for two-generator Möbius groups of $PSL(2,\mathbb{C})$, which is known as Jørgensen's inequality. As an application, he established the following discreteness criterion in [5].

Theorem J. *A non-elementary subgroup G of $PSL(2,\mathbb{C})$ is discrete if and only if every two-generator subgroup in G is discrete.*

This implies that the discreteness of a non-elementary Möbius group G depends on the discreteness of its two-generator subgroups. There are many further discussions in this direction (see [6], [8], [11]). In 2001, Wang and Yang [12] generalized Theorem J to the case of $M(\overline{\mathbb{R}}^n)$ and proved the following.

Theorem WY. *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary. Then G is discrete if and only if $WY(G)$ is discrete and each non-elementary subgroup generated by two loxodromic elements in G is discrete.*

Here,

$$WY(G)=\{g \in G : g \text{ fixes every fixed point of each loxodromic element of } G\}.$$

Obviously, if $G \subset PSL(2,\mathbb{C})$ is non-elementary, then $WY(G) = \{I\}$. According to [12], we know that the condition "$WY(G)$ is discrete" in Theorem WY is necessary when $n \geq 3$.

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In [10], Wang, Li and Cao obtained further generalizations of Theorem WY. By using a fixed loxodromic or parabolic element in G, they proved the following theorems.

Theorem $\mathbf{W_1}$ ([10, Theorem 3.1]). *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary. Then G is discrete if and only if $WY(G)$ is discrete and each non-elementary subgroup generated by two elements of G_f is discrete, where $f \in G$ is loxodromic.*

Theorem $\mathbf{W_2}$ ([10, Theorem 3.2]). *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary containing parabolic elements. Then G is discrete if and only if $WY(G)$ is discrete and each non-elementary subgroup generated by two elements of G_f is discrete, where $f \in G$ is parabolic.*

Here G_f in Theorems W_1 and W_2 are defined as follows:

$$G_f = \{g \in G : g \text{ is conjugate to } f \text{ and } \langle f, g\rangle \text{ is non-elementary}\} \cup \{f\}.$$

The novelty of Theorems W_1 and W_2 is that the discreteness of G is totally determined by a loxodromic (resp. parabolic) element of G.

In [7], Li and the author showed that the assumption "$f \in G$ " in Theorems W_1 and W_2 was unnecessary and obtained the following theorems.

Theorem $\mathbf{LF_1}$ ([7, Theorem 1.1]). *Let $G \subset M(\overline{\mathbb{R}}^n)$ be a non-elementary group and $f \in M(\overline{\mathbb{R}}^n)$ loxodromic. If $WY(G)$ is discrete and each non-elementary group $\langle f, gfg^{-1}\rangle$ is discrete, where $g \in G$, then G is discrete.*

Theorem $\mathbf{LF_2}$ ([7, Theorem 1.2]). *Let $G \subset M(\overline{\mathbb{R}}^n)$ be a non-elementary group and $f \in M(\overline{\mathbb{R}}^n)$ parabolic. If $WY(G)$ is discrete and each non-elementary group $\langle f, gfg^{-1}\rangle$ is discrete, where $g \in G$, then G is discrete.*

Naturally, we asked the following question.

Conjecture LF ([7]). *Let $G \subset M(\overline{\mathbb{R}}^n)$ be a non-elementary group and $f \in M(\overline{\mathbb{R}}^n)$ elliptic. If $WY(G)$ is discrete, and each non-elementary group*

$$\langle f, gfg^{-1}\rangle$$

is discrete, where $g \in G$, then G is discrete.

We constructed an example in [7] which showed that if $f|_{M(G)} = I$, then Conjecture LF may not be true.

Example LF ([7]). Let $G_0 \subset M(\overline{\mathbb{R}}^2)$ be a non-elementary and non-discrete group containing no elliptic elements, and let G be the Poincaré extension of G_0 in $\overline{\mathbb{R}}^4$. Let f be a rotation around $\overline{\mathbb{R}}^2$ of order p ($p \geq 3$). Then f is an elliptic element acting on $\overline{\mathbb{R}}^4$ whose fixed point set is $\overline{\mathbb{R}}^2$. Obviously, $f \notin G$, $WY(G) = \langle I\rangle$ is finite but there exists no non-elementary group generated by f and gfg^{-1} for $g \in G$.

In this paper, we discuss Conjecture LF further and some new discreteness criteria of Möbius groups are obtained.

2. Preliminaries

For $n \geq 2$, we denote by $\overline{\mathbb{R}}^n$ the one-point compactification of $\mathbb{R}^n$ obtained by adding ∞. The group of orientation-preserving Möbius transformations of $\overline{\mathbb{R}}^n$ is denoted by $M(\overline{\mathbb{R}}^n)$. We regard $\overline{\mathbb{R}}^n$ as the boundary at infinity of the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ and let $\overline{\mathbb{H}}^{n+1} = \mathbb{H}^{n+1} \cup \overline{\mathbb{R}}^n$. It's known that every Möbius transformation f in $\overline{\mathbb{R}}^n$ can be extended to an isometry $\widetilde{f}$ (Poincaré extension) in $\mathbb{H}^{n+1}$.

For a non-trivial element $f \in M(\overline{\mathbb{R}}^n)$, we let

$$fix(f) = \{x \in \overline{\mathbb{H}}^{n+1} : f(x) = x\}$$

f is called *loxodromic* if it has two fixed points in $\overline{\mathbb{R}}^n$ and none in $\mathbb{H}^{n+1}$, *parabolic* if it has only one fixed point in $\overline{\mathbb{R}}^n$ and none in $\mathbb{H}^{n+1}$, and *elliptic* if it has a fixed point in $\mathbb{H}^{n+1}$.

Let G be a subgroup of $M(\overline{\mathbb{R}}^n)$. For a point $z \in \overline{\mathbb{H}}^{n+1}$, the set $G(z) = \{g(z) : \ g \in G\}$ is called G-orbit of z. The *limit set* $L(G)$ of G is defined as follows:

$$L(G) = \overline{G(z)} \cap \overline{\mathbb{R}}^n.$$

We call G *elementary* if $L(G)$ contains fewer than three points. Otherwise, it is called *non-elementary.*

Proposition 2.1 ([10])**.** *Let $G \subset M(\overline{\mathbb{R}}^n)$. Then we have the following*

(1) *if G contains a loxodromic element, then G is elementary if and only if it fixes a point in $\overline{\mathbb{R}}^n$ or a point-pair $\{x, y\} \subset \overline{\mathbb{R}}^n$;*

(2) *if G contains a parabolic element but no loxodromic element, then G is elementary if and only if it fixes a point in $\overline{\mathbb{R}}^n$;*

(3) *if G is purely elliptic, then G fixes a point in $\overline{\mathbb{H}}^{n+1}$.*

Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary. We denote $M(G)$ the smallest G-invariant hyperbolic subspace of $\mathbb{H}^{n+1}$, $\phi(g)$ the restriction of g to $M(G)$ for all $g \in G$, that is

$$\phi(g) = g|_{M(G)}, \ \phi(G) = \{g|_{M(G)} : g \in G\}.$$

Obviously,

$$WY(G) = \{g \in G : \phi(g) = I\}.$$

If there exists a sequence of distinct elements in G converging to the identity, then we say that G is not *discrete.* Otherwise, we say that G is *discrete.*

Proposition 2.2 ([9])**.** *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary. Then G is discrete if and only if both groups $WY(G)$ and $\phi(G)$ are discrete.*

For $f_r = \left(\begin{smallmatrix} a_r & b_r \\ c_r & d_r \end{smallmatrix}\right) \in M(\overline{\mathbb{R}}^n)$ $(r = 1, 2)$, we define

$$\|f_1 - f_2\| = (|a_1 - a_2|^2 + |b_1 - b_2|^2 + |c_1 - c_2|^2 + |d_1 - d_2|^2)^{\frac{1}{2}}.$$

The following lemma is crucial for our investigation.

Lemma 2.1 ([13]). *Let f, $g \in M(\overline{\mathbb{R}}^n)$. If $\langle f, g\rangle$ is a discrete and non-elementary group, then*

$$\|f - I\| \cdot \|g - I\| > \frac{1}{32}.$$

In the following, we give an example which shows that in some special case, Conjecture LF may be true.

Proposition 2.3. *Let $G \subset M(\overline{\mathbb{R}}^5)$ be non-elementary with $M(G) = \mathbb{H}^6$, and let f be an elliptic element of $M(\overline{\mathbb{R}}^5)$ such that f^2 has only one fixed point in $\mathbb{H}^6$ and none in $\overline{\mathbb{R}}^5$. If each non-elementary group $\langle f, gfg^{-1}\rangle$ is discrete, where $g \in G$, then G is discrete.*

Proof. Suppose not. Since $G \subset M(\overline{\mathbb{R}}^5)$ is non-elementary with $M(G) = \mathbb{H}^6$, by [3], we know that G is dense in $M(\overline{\mathbb{R}}^5)$. It follows that there exists a sequence $\{f_i\} \subset G$ such that for each i, f_i is loxodromic and

$$f_i \to I \text{ as } i \to \infty.$$

By our assumptions and Lemma 2.1, it's easy to see that for large enough i, $\langle f^2, f_i f^2 f_i^{-1}\rangle$ are elementary. It deduces that

$$f_i(fix(f^2)) = fix(f^2).$$

It's the desired contradiction since f^2 has only one fixed point in $\mathbb{H}^6$. $\square$

Motivated by Example LF and Proposition 2.3, we obtain the following main results.

Theorem 2.1. *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary and $f \in M(\overline{\mathbb{R}}^n)$ be elliptic with $Card[fix(f^2)] = 1$. If $WY(G)$ is discrete, and each non-elementary group $\langle f, gfg^{-1}\rangle$ is discrete, where $g \in G$, then G is discrete.*

Theorem 2.2. *Let $G \subset M(\overline{\mathbb{R}}^n)$ be non-elementary and $f \in M(\overline{\mathbb{R}}^n)$ be elliptic with $f^2|_{M(G)} \neq I$. If $WY(G)$ is discrete, and each non-elementary group $\langle f, g\rangle$ is discrete, where $g \in G$, then G is discrete.*

Remark 2.1. Following [3], if $f \in M(\overline{\mathbb{R}}^n)$ be elliptic with $Card[fix(f)] = 1$, then n must be odd.

3. Proofs of main results

3.1. Proof of Theorem 2.1

Suppose not. Then there exists a sequence $\{f_i\} \subset G$ such that

$$f_i \to I \text{ as } i \to \infty.$$

Since $Card[fix(f^2)] = 1$, we can see that f^2 has no fixed point in $\overline{\mathbb{R}}^n$ (that is $\widetilde{f^2}$ has only one fixed point in $\mathbb{H}^{n+1}$). Without loss of generality, we assume that $fix(f^2) = \{x\}$, where $x \in \mathbb{H}^{n+1}$. Then for large enough i, we know that the subgroups $\langle f^2, f_i f^2 f_i^{-1}\rangle$ are elementary since

$$\|f^2 - I\| \cdot \|f^{-2} f_i f^2 f_i^{-1} - I\| < \frac{1}{32}.$$

By Proposition 2.1, we know that for large enough i, $x \in fix(f_i)$. Since G is non-elementary, we can find finitely many loxodromic elements $g_1, g_2, \ldots, g_t$ in G such that the set $S = \{A_{fix(g_1)}, A_{fix(g_2)}, \ldots, A_{fix(g_t)}\}$ can spans $M(G)$, where $A_{fix(g)}$ denote the attractive fixed point of a loxodromic element g. For each k, let $U_{A_{fix(g_k)}}$ be a small neighborhood of $A_{fix(g_k)}$ in $\overline{\mathbb{H}}^{n+1}$, where ($k = 1, 2, \ldots, t$) (cf. [7]). Then we can find an integer N such that for each k, $g_k^N(x) \in U_{A_{fix(g_k)}}$. Now, let's consider the subgroups $\langle g_t^N f^2 g_t^{-N}, f_i g_t^N f^2 g_t^{-N} f_i^{-1}\rangle$. Since

$$\langle g_t^N f^2 g_t^{-N}, f_i g_t^N f^2 g_t^{-N} f_i^{-1}\rangle = g_t^N \langle f^2, g_t^{-N} f_i g_t^N f^2 g_t^{-N} f_i^{-1} g_t^N\rangle g_t^{-N}$$

and

$$\langle f^2, g_t^{-N} f_i g_t^N f^2 g_t^{-N} f_i^{-1} g_t^N\rangle = \langle f^2, g_t^{-N} f_i g_t^N f^2 g_t^{-N} f_i^{-1} g_t^N f^{-2}\rangle,$$

by the assumptions and Lemma 2.1, we know that the groups

$$\langle g_t^N f^2 g_t^{-N}, f_i g_t^N f^2 g_t^{-N} f_i^{-1}\rangle$$

are elementary for large enough i. It easily follows that for each k and large enough i, f_i has a fixed point in $U_{A_{fix(g_k)}}$. This means that for large enough i, $f_i \in WY(G)$. It's a contradiction.

3.2. Proof of Theorem 2.2

Suppose that G is not discrete. Then there exists a sequence $\{f_i\} \subset G$ such that for each i,

$$f_i \to I \quad \text{as} \quad i \to \infty.$$

It follows a discussion similar to that in the proof of Theorem 2.1, we can find finitely many loxodromic elements $g_1, g_2, \ldots, g_t$ in G such that the set $S = \{A_{fix(g_1)}, A_{fix(g_2)}, \ldots, A_{fix(g_t)}\}$ can span $M(G)$ and an integer N such that for each k, $g_k^N(fix(f)) \in U_{A_{fix(g_k)}}$ ($k = 1, 2, \ldots, t$). Since

$$\langle g_k^N f^2 g_k^{-N}, f_i\rangle = g_k^N \langle f^2, g_k^{-N} f_i g_k^N\rangle g_k^{-N}$$

and

$$\|g_k^{-N} f_i g_k^N - I\| \cdot \|f^2 - I\| < \frac{1}{32}$$

for large enough i, we can see that the subgroups $\langle g_k^N f^2 g_k^{-N}, f_i\rangle$ are elementary. By Proposition 2.1, we know that for each k ($k = 1, 2, \ldots, t$),

$$fix(f_i) \cap U_{A_{fix(g_k)}} \neq \emptyset.$$

It follows that for sufficiently large i,

$$f_i \in WY(G).$$

It's a contradiction.

4. A discreteness criterion for isometric subgroups of PU(n,1)

It's similar to Möbius groups, in [6], Li obtained the following discreteness criteria for subgroups of $PU(n,1)$.

Theorem L ([6, Theorem 1.3])**.** *Let $G \subset PU(n,1)$ be non-elementary and $M(G) = \mathbb{H}^n_{\mathbb{C}}$. Suppose that $f \in G$ is elliptic with order at least 3. Then G is discrete if and only if each non-elementary subgroup generated by f and an elliptic of G is discrete.*

By [3], we know that if $M(G) = \mathbb{H}^n_{\mathbb{C}}$, then G is either discrete or dense. Since $\dim(M(G))$ is even, it follows from a discussion similar to that in the proof of [2, Theorem 1.2], we have:

Theorem 4.1. *Let $G \subset PU(n,1)$ be non-elementary and $M(G) = \mathbb{H}^n_{\mathbb{C}}$. Then G is discrete if and only if each group generated by an elliptic element of G is discrete.*

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Department of Mathematics
Shaoxing University
Shaoxing, Zhejiang 312000, P. R. China
E-mail address: fuxi1000@yahoo.com.cn