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ENTROPY AND PERIODIC ORBITS FOR GENERIC DIFFEOMORPHISMS

Ruixin Chen

Abstract. We prove an inequality between topological entropy and asymptotical growth of periodic orbits for C^1 generic diffeomorphisms.

Let M be a compact Riemannian manifold. The distance function on M is d. For a diffeomorphism f, one can define another metric d_n by $d_n(x,y) = \max_{0\leq i\leq n-1} d(f^i(x), f^i(y))$ for any $x, y \in M$. Let $N_n(\varepsilon)$ be the minimal number of ε-balls in the metric d_n that can cover M. Since M is compact, $N_n(\varepsilon)$ is finite. The topological entropy $h(f)$ of f is defined to be

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\ln N_n(\varepsilon)}{n}.$$

Let $P_n(f) = \{x \in M : f^n(x) = x\}$ be the set of periodic points with period n. There are some relationships between topological entropy and periodic orbits. For instance, Katok [4, Corollary 4.4] proved for any $C^{1+\alpha}$ ($\alpha > 0$) surface diffeomorphism f, one has $\limsup_{n\to\infty} \ln \#P_n(f)/n \geq h(f)$, where $\#P_n(f)$ is the cardinal number of $P_n(f)$. To my knowledge, there is no generalization of this result for higher-dimensional diffeomorphisms. As in [4, Page 141]: "Indeed it might be true generically, i.e., for any f from some dense G_δ set in the space $\mathrm{Diff}^r(M)$ of all C^r diffeomorphisms of M with C^r topology ($r \geq 1$). Note that even in the two-dimensional case the answer is not known for $r = 1$." In this paper we can give a positive answer of the above question for the C^1 topology.

Main Theorem. *There is a dense G_δ set $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$,*

$$\limsup_{n\to\infty} \frac{\ln \#P_n(f)}{n} \geq h(f).$$

We need some preparation for the proof. $f \in \mathrm{Diff}^1(M)$ is called *star* if there is a C^1 neighborhood $\mathcal{U}$ of f such that any periodic orbit of any $g \in \mathcal{U}$ is hyperbolic. f is called *Axiom A* if $\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{\mathrm{Per}(f)}$.

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Theorem 1 (Mañé [6], Hayashi [3]). *Every star diffeomorphism is Axiom A.*

Theorem 2 (Bowen [2]). *For every Axiom A diffeomorphism f, one has*

$$\lim_{n\to\infty} \frac{\ln \# P_n(f)}{n} = h(f).$$

f is called *Kupka-Smale* if every periodic orbit of f is hyperbolic and the stable manifold of one hyperbolic periodic orbit intersects the unstable manifold of another one transversely. [5, 7] proved that the set of Kupka-Smale diffeomorphisms is a dense G_δ in $\mathrm{Diff}^r(M)$ for any $r \geq 1$. Note that Kupka-Smale diffeomorphisms may not be star diffeomorphisms: Shub constructed robust non-Axiom A diffeomorphisms; Kupka-Smale diffeomorphisms are dense among robust non-Axiom A diffeomorphisms. So Theorem 1 implies the claim. Let $\mathcal{G}_{KS}$ be the set of Kupka-Smale diffeomorphisms. Thus, one has:

Lemma 1. *For any $f \in \mathcal{G}_{KS}$, if there is a sequence of diffeomorphisms $\{f_n\}_{n\in\mathbb{N}}$ satisfying*

- $\lim_{n\to\infty} f_n = f$,
- *each f_n contains a non-hyperbolic periodic orbit Q_n,*

then $\lim_{n\to\infty} \pi(Q_n) = \infty$, where $\pi(Q_n)$ denotes the period of Q_n.

Proof. If not, by taking a subsequence if necessary, one can assume that $\pi(Q_n)$ is a constant for all n, say N. Take $q_n \in Q_n$ for each n. By taking a subsequence again, one can assume that $\{q_n\}$ converges and its limit is q. By the continuity of f, one has $f^N(q) = q$. Since f is Kupka-Smale, $\mathrm{Orb}(q)$ is hyperbolic. By the robustness of hyperbolicity, we have that Q_n is a hyperbolic periodic orbit of f_n. This gives a contradiction. □

Let $HP_n(f) = \{x : f^n(x) = x,\ x \text{ is hyperbolic}\}$.

Proposition 1. *There is a dense G_δ set $\mathcal{G}$ in $\mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$, one has either f is Axiom A, or*

$$\limsup_{n\to\infty} \frac{\ln \# HP_n(f)}{n} = \infty.$$

Proof. Let

$$H_k = \{f : \sup_{n\in\mathbb{N}} \frac{\ln \# HP_n(f)}{n} > k\},$$

$$N_k = \{f : \exists\ C^1 \text{ neighborhood } \mathcal{U} \text{ of } f \text{ s.t. } \forall g \in \mathcal{U},\ \sup_{n\in\mathbb{N}} \frac{\ln \# HP_n(g)}{n} \leq k\}.$$

By the definition, $H_k \cup N_k$ is dense in $\mathrm{Diff}^1(M)$ and N_k is open. By the property of hyperbolic periodic orbits, one has H_k is open. Let

$$\mathcal{G} = (\bigcap_{k\in\mathbb{N}} (H_k \cup N_k)) \cap \mathcal{G}_{KS},$$

where $\mathcal{G}_{KS}$ is the set of Kupka-Smale diffeomorphisms. If $f \in \mathcal{G}$ is star, by Theorem 1, f is Axiom A. For any non-star diffeomorphism $f \in \mathcal{G}$, there is a sequence of diffeomorphisms $\{f_n\}$ such that $\lim_{n\to\infty} f_n = f$ and each f_n has a non-hyperbolic periodic orbit Q_n. $\lim_{n\to\infty} \pi(Q_n) = \infty$ by Lemma 1. By [1, Corollary 2.18], one can assume that the eigenvalues of $Df^{\pi(Q_n)}|_{T_{Q_n}M}$ have only one eigenvalue with modulus 1, where $q_n \in Q_n$. The eigenvalue may be 1 or -1. Since $f \in \mathcal{G}$, $f \in H_k \cup N_k$ for any k. We assume that there is $m \in \mathbb{N}$ such that $f \in N_m$. Take f_n C^1-close to f such that $f_n \in N_m$. Thus there is a C^1 neighborhood $\mathcal{U}_n$ of f_n such that any $g \in \mathcal{U}_n$ is in N_m. Thus, for $\pi(Q_n)$, one has for any $g \in \mathcal{U}_n$,

$$\frac{\ln \#HP_{\pi(Q_n)}(g)}{\pi(Q_n)} \le m \text{ and } \frac{\ln \#HP_{2\pi(Q_n)}(g)}{2\pi(Q_n)} \le m.$$

For f_n, since Q_n is a non-hyperbolic periodic orbit with only one eigenvalues with modulus 1, there is g_n which can be arbitrarily C^1-close to f_n such that g_n can have $10\exp(m\pi(Q_n))$ hyperbolic periodic orbits with period $\pi(Q_n)$ or $10\exp(m2\pi(Q_n))$ hyperbolic periodic orbits with period $2\pi(Q_n)$. This contradicts to $f_n \in N_m$. Here we need to consider $2\pi(Q_n)$ because if the eigenvalue is -1, after a perturbation we may have a periodic orbit with period $2\pi(Q_n)$. Thus for non-star diffeomorphism f, one has $f \in H_k$ for any $k \in \mathbb{N}$. Thus

$$\sup_{n\in\mathbb{N}} \frac{\ln \#HP_n(f)}{n} = \infty.$$

In fact, if $\#HP_n(f) < \infty$ for all $n \in \mathbb{N}$, then the equality follows from $\sup_{n\in\mathbb{N}} \frac{\ln \#HP_n(f)}{n} = \infty$. And if $\#HP_n(f) = \infty$ for some $n \in \mathbb{N}$, then $\#HP_{kn}(f) = \infty$ for all $k \in \mathbb{N}$ and we have the desired equality. □

Proof of Main Theorem. Let $\mathcal{G}$ be the dense G_δ set as in Proposition 1. For $f \in \mathcal{G}$,

- either f is Axiom A, then by Theorem 2, one gets the conclusion,
- or,

$$\limsup_{n\to\infty} \frac{\ln \#HP_n(f)}{n} = \infty,$$

Since $h(f)$ is finite and $HP_n(f) \subset P_n(f)$, one has

$$\limsup_{n\to\infty} \frac{\ln \#P_n(f)}{n} > h(f).$$

□

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College of Mathematics and Physics
Qingdao University of Science and Technology
Qingdao 266042, P. R. China
E-mail address: mathcrx@126.com