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WALLMAN COVERS AND QUASI-F COVERS

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Abstract. Observing that for any space X, there is a Wallman sublattice $\mathscr{A}_X$ and that QFX is homeomorphic to a subspace X_q of the Wallman cover $\mathscr{L}(\mathscr{A}_X)$ of $\mathscr{A}_X$, we show that βQFX and $\mathscr{L}(\mathscr{A}_X)$ are homeomorphic.

1. Introduction

All spaces in this paper are assumed to be Tychonoff spaces and $(\beta X, \beta_X)$ denotes the Stone-Čech compactification of a space X.

Iliadis constructed the absolute of a Hausdorff, which is the minimal extremally disconnected cover and they turn out to be the perfect onto projective covers $\big([6]\big)$. To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various authors $\big([1], [3], [4], [5], [7], [8], [9]\big)$.

In particular, Henriksen and Gillman introduced the concept of quasi-F spaces which is a generalization of F-spaces, in which every cozero-set is C^*-embedded. Each space X has the minimal quasi-F cover (QFX, Φ_X) $\big([3]\big)$.

Henriksen, Vermeer and Woods introduced the concept of Wallman covers and proved that if X is a compact space, then QFX is given by the Wallman cover $\mathscr{L}(Z(X)^{\#})$ of the Wallman sublattice $Z(X)^{\#}$ and $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$ $\big(\alpha \in \mathscr{L}(Z(X)^{\#})\big)$ $\big([4]\big)$.

In this paper, we introduce a new Wallman sublattice $\mathscr{A}_X$ for a space X and show that QFX is homeomorphic to the subspace X_q of a Wallman cover $\mathscr{L}(\mathscr{A}_X)$. Using this, we will show that βQFX and $\mathscr{L}(\mathscr{A}_X)$ are homeomorphic.

For the terminology, we refer to [2] and [9].

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2. QUASI-F COVERS AND WALLMAN COVERS

The set $\mathcal{R}(X)$ of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows : for any $A \in \mathcal{R}(X)$ and any

$\{A_i \mid i \in I\} \subseteq \mathcal{R}(X)$,
$\vee\{A_i \mid i \in I\} = cl_X(\cup\{A_i \mid i \in I\})$,
$\wedge\{A_i \mid i \in I\} = cl_X\big(int_X(\cap\{A_i \mid i \in I\})\big)$, and
$A' = cl_X(X - A)$

and a sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and meets.

Recall that a map $f : Y \longrightarrow X$ is called *a covering map* if it is a continuous, onto, perfect, and irreducible map.

Lemma 2.1 ([7]). (1) *Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : \mathcal{R}(Y) \longrightarrow \mathcal{R}(X)$, defined by $\psi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by $\psi^{-1}(B) = cl_Y\big(f^{-1}(int_X(B))\big) = cl_Y\big(int_Y(f^{-1}(B))\big)$.*
(2) *Let X be a dense subspace of a space K. Then the map $\phi : \mathcal{R}(K) \longrightarrow \mathcal{R}(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_K(B)$.*

Definition 2.2. A space X is called *a quasi-F space* if for any zero-sets A, B in X, $cl_X\big(int_X(A \cap B)\big) = cl_X\big(int_X(A)\big) \cap cl_X\big(int_X(B)\big)$, equivalently, every dense cozero-set in X is C^*-embedded in X.

It is well-known that a space X is a quasi-F space if and only if βX is a quasi-F space.

Definition 2.3. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : X \longrightarrow Y$ is a covering map,
(2) *a quasi-F cover of* X if (Y, f) is a cover of X and Y is a quasi-F space, and
(3) *a minimal quasi-F cover of* X if (Y, f) is a quasi-F cover of X and for any quasi-F cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ such that $f \circ h = g$.

For any space X, there is a minimal quasi-F cover (QFX, Φ_X) $\big([4]\big)$

Definition 2.4 ([4]). Let X be a space and $\mathscr{A}$ a sublattice of $\mathcal{R}(X)$. Then $\mathscr{A}$ is said to be *a Wallman sublattice* if

(1) for any $A \in \mathscr{A}$ and $x \in X - A$, there is a $B \in \mathscr{A}$ such that $x \in int_X(B)$ and $A \wedge B = \emptyset$, and

(2) for any $A, B \in \mathscr{A}$ with $A \wedge B = \emptyset$, there are C, D in $\mathscr{A}$ such that $A \wedge C = B \wedge D = \emptyset$ and $C \vee D = X$.

Let X be a compact space and $\mathscr{A}$ a Wallman sublattice of $\mathcal{R}(X)$ such that $\mathscr{A}$ is a base for closed sets in X. Let $\mathscr{L}(\mathscr{A}) = \{\alpha \mid \alpha$ is an $\mathscr{A}$-ultrafilter $\}$ and for any $A \in \mathscr{A}$, $\sum_{\mathscr{A}}^{A} = \{\alpha \in \mathscr{L}(\mathscr{A}) \mid A \in \alpha\}$. Let $\mathscr{L}(\mathscr{A})$ be the space, equipped with the topology for which $\{\sum_{\mathscr{A}}^{A} \mid A \in \mathscr{A}\}$ is a base for closed sets and define a map $\Phi_{\mathscr{A}} : \mathscr{L}(\mathscr{A}) \longrightarrow X$ by $\Phi_{\mathscr{A}}(\alpha) = \cap\{A \mid A \in \alpha\}$. Then the pair $(\mathscr{L}(\mathscr{A}), \Phi_{\mathscr{A}})$ is called *a Wallman cover of X with respect to $\mathscr{A}$* $\big([4]\big)$.

For any space X, $Z(X)^{\#}$ is a Wallman sublattice of $\mathcal{R}(X)$ $\big([4]\big)$.

Lemma 2.5 ([4])**.** *Let X be a compact space. Then we have the following :*

(1) *for any $A, B \in \mathscr{A}$, $\sum_{\mathscr{A}}^{A} \wedge \sum_{\mathscr{A}}^{B} = \sum_{\mathscr{A}}^{A} \cap \sum_{\mathscr{A}}^{B}$,*

(2) *$\Phi_{\mathscr{A}} : \mathscr{L}(\mathscr{A}) \longrightarrow X$ is a covering map, and*

(3) *$\big(\mathscr{L}(Z(X)^{\#}), \Phi_{Z(X)^{\#}}\big)$ is a minimal quasi-F cover of X.*

Let X be a space and $\mathscr{A}$ a Wallman sublattice of $\mathcal{R}(X)$. Let $\mathscr{L}(\mathscr{A}) = \{\alpha \mid \alpha$ is a $\mathscr{A}$-ultrafilter$\}$ be the space which is equipped with the topology for which $\{\sum_{\mathscr{A}}^{A} \mid A \in \mathscr{A}\}$ is a base for closed sets. By Lemma 2.1, $Z(X)^{\#}$ and $Z(\beta X)^{\#}$ are isomorphic, $\mathscr{L}(Z(\beta X)^{\#})$ is homeomorphic to the space $\mathscr{L}(Z(X)^{\#})$.

For any space X, let $\mathscr{A}_X = \{cl_X\big(int_X(\Phi_X(A))\big) \mid A \in Z(QFX)^{\#}\}$. By Lemma 2.1, $\mathscr{A}_X$ is a Wallman sublattice of $\mathcal{R}(X)$.

For any space X, let $(QF(\beta X), \Phi_\beta)$ be the minimal quasi-F cover of βX.

Recall that a $\mathscr{A}$-filter α is called fixed if $\cap\{A \mid A \in \alpha\} \neq \emptyset$.

Proposition 2.6. *Let X be a space. Then we have the following :*

(1) *$\mathscr{A}_X$ is a Wallman sublattice of $\mathcal{R}(X)$ with $Z(X)^{\#} \subseteq \mathscr{A}_X$, and*

(2) *for any fixed $\mathscr{A}_X$-ultrafilter α, $\alpha_q = \{A \in Z(QFX)^{\#} \mid \Phi_X(A) \in \alpha\}$ is a fixed $Z(QFX)^{\#}$-ultrafilter and $|\cap\{B \mid B \in \alpha_q\}| = 1$.*

Proof. (1) Since $Z(QFX)^{\#}$ is a Wallman sublattice of $\mathcal{R}(QFX)$ and $\Phi_X : QFX \longrightarrow X$ is a covering map, $\mathscr{A}_X$ is a Wallman sublattice of $\mathcal{R}(X)$. Let $A \in Z(X)^{\#}$. By Lemma 2.1, $\Phi_X\big(cl_{QFX}(\Phi_X^{-1}(int_X(A)))\big) = A$ and $cl_{QFX}\big(\Phi_X^{-1}(int_X(A))\big) \in Z(QFX)^{\#}$. Hence $A \in \mathscr{A}_X$ and so $Z(X)^{\#} \subseteq \mathscr{A}_X$.

(2) Clearly, α_q is a $Z(QFX)^{\#}$-filter. Suppose that $A \in Z(QFX)^{\#} - \alpha_q$. By the definition of α_q, $\Phi_X(A) \notin \alpha$. Since α is a $Z(X)^{\#}$-ultrafilter, there is a $B \in \alpha$

such that $B \wedge \Phi_X(A) = \emptyset$. By Lemma 2.1, $cl_{QFX}\big(\Phi_X^{-1}(int_X(B))\big) \wedge A = \emptyset$ and $cl_{QFX}\big(\Phi_X^{-1}(int_X(B))\big) \in Z(QFX)^{\#}$. Since $\Phi_X\Big(cl_{QFX}\big(\Phi_X^{-1}(int_X(B))\big)\Big) = B \in \alpha$, $cl_{QFX}\big(\Phi_X^{-1}(int_X(B))\big) \in \alpha_q$. Hence α_q is a $Z(QFX)^{\#}$-ultrafilter.

Let $x \in \cap\{A \mid A \in \alpha\}$. Since α_q is a $Z(QFX)^{\#}$-ultrafilter, $\{B \cap \Phi_X^{-1}(x) \mid B \in \alpha_q\}$ has the finite intersection property. Since $\Phi_X^{-1}(x)$ is compact, $\cap\{B \cap \Phi_X^{-1}(x) \mid B \in \alpha_q\} \neq \emptyset$ and $\cap\{B \mid B \in \alpha_q\} \neq \emptyset$. This implies that α_q is fixed. Since $Z(QFX)^{\#}$ is a base for closed sets in QFX, $|\cap \{B \mid B \in \alpha_q\}| = 1$. □

Let X be a space and $X_q = \{\alpha \in \mathscr{L}(\mathscr{A}_X) \mid \alpha \text{ is fixed}\}$ the subspace of $\mathscr{L}(\mathscr{A}_X)$. Define a map $g_X : X_q \longrightarrow QFX$ by $g_X(\alpha) = \cap\{B \mid B \in \alpha_q\}$.

In the following, for any $A \in \mathscr{A}_X$, let $\sum_A = \sum_{\mathscr{A}_X}^A$ and $S_A = \sum_A \cap X_q$.

Theorem 2.7. *Let X be a space. Then $g_X : X_q \longrightarrow QFX$ is a homeomorphism.*

Proof. First, we will show that g_X is one-to-one and onto. Let $\alpha \neq \delta$ in X_q. Then there are $C, D \in \mathscr{A}_X$ such that $C \in \alpha$, $D \in \delta$ and $C \wedge D = \emptyset$. Since $C, D \in \mathscr{A}_X$, there are $A, B \in Z(QFX)^{\#}$ such that $\Phi_X(A) = C$ and $\Phi_X(B) = D$. Then $\Phi_X(A) \wedge \Phi_X(B) = \Phi_X(A \wedge B) = \emptyset$ and by Lemma 2.1, $A \wedge B = \emptyset$. Since QFX is a quasi-F space, $A \cap B = \emptyset$. Since $\Phi_X(A) = C \in \alpha$, $A \in \alpha_q$ and $g_X(\alpha) \in A$. Similarly, $g_X(\delta) \in B$ and $g_X(\alpha) \neq g_X(\delta)$. Thus g_X is one-to-one.

Let $y \in QFX$ and $\gamma = \{\Phi_X(C) \mid y \in C \in Z(QFX)^{\#}\}$. Then clearly, γ is a $\mathscr{A}_X$-filter. Let $D \in \mathscr{A}_X - \gamma$. Then there is an $E \in \mathscr{A}_X$ such that $\Phi_X(E) = D$ and $y \notin E$. Since $Z(QFX)^{\#}$ is a Wallman sublattice, there is an $F \in Z(QFX)^{\#}$ such that $y \in int_{QF}(F)$ and $F \wedge E = \emptyset$. By Lemma 2.1, $\Phi_X(F) \wedge \Phi_X(E) = \Phi_X(F) \wedge D = \emptyset$ and since $\Phi_X(F) \in \gamma$, γ is an $\mathscr{A}_X$-ultrafilter. Since $h_X(\gamma_q) = y$, h_X is onto.

Let $E \in Z(QFX)^{\#}$ and $\mu \in X_q - g_X^{-1}(E)$. Then $g_X(\mu) = \cap\{H \mid H \in \mu_q\} \notin E$ and $E \notin \mu_q$. Hence $\Phi_X(E) \notin \mu$ and so $\mu \notin \sum_{\Phi_X(E)} \cap X_q = S_{\Phi_X(E)}$. Thus $S_{\Phi_X(E)} \subseteq g_X^{-1}(E)$. Suppose that $\xi \in g_X^{-1}(E)$. Then $g_X(\xi) \in E$ and for any $K \in \xi_q$, $K \wedge E \neq \emptyset$. Since ξ_q is a $Z(QFX)^{\#}$-ultrafilter, $E \in \xi_q$ and $\Phi_X(E) \in \xi$. Hence $\xi \in S_{\Phi_X(E)}$ and $g_X^{-1}(E) \subseteq S_{\Phi_X(E)}$. Thus $g_X^{-1}(E) = S_{\Phi_X(E)}$.

Since $Z(QFX)^{\#}$ is a base for closed sets and g_X is one-to-one and onto, g_X is a homeomorphism. □

For any space X, let $q_X = \Phi_X \circ g_X$.

Corollary 2.8. *Let X be a space. Then (QFX, Φ_X) and (X_q, q_X) are equivlaent cover of X, that is, there is a homeomorphism $g_X : X_q \longrightarrow QFX$ such that $q_X = \Phi_X \circ g_X$.*

Let X be a space. It is well-known that $QF(\beta X) = \mathscr{L}(Z(\beta X)^{\#})$. Since $Z(X)^{\#}$ and $Z(\beta X)^{\#}$ are Boolean isomorphic and $Z(X)^{\#} \subseteq \mathscr{A}_X$, there is a covering map $l_X : \mathscr{L}(\mathscr{A}_X) \longrightarrow QF\beta X$ such that $\Phi_\beta \circ l_X = \Phi_{\mathscr{A}_X}$ ([3]). In fact, $l_X(\alpha)$ is the $Z(X)^{\#}$-ultrafilter such that $\alpha \cap Z(X)^{\#} \subseteq l_X(\alpha)$.

Theorem 2.9. *Let X be a space. Then there is a homeomorphism $h_X : \beta QFX \longrightarrow \mathscr{L}(\mathscr{A}_X)$ such that $h_X \circ \beta_{QFX} = j \circ g_X^{-1}$, where $j : X_q \longrightarrow \mathscr{L}(\mathscr{A}_X)$ is the inclusion map.*

Proof. By Theorem 2.7, βX_q and βQFX are homeomorphic. Since $j \circ g_X^{-1} : QFX \longrightarrow \mathscr{L}(\mathscr{A}_X)$ is a dense embedding, there is a continuous map $h_X : \beta QFX \longrightarrow \mathscr{L}(\mathscr{A}_X)$ such that $h_X \circ \beta_{QFX} = j \circ g_X^{-1}$. Since βQFX and $\mathscr{L}(\mathscr{A}_X)$ are compact spaces and β_{QFX} and j are dense embeddings, h_X is a covering map.

Let A, B be disjoint zero-sets in X_q. Then there are C, D in $Z(X_q)^{\#}$ such $C \cap D = \emptyset$, $A \subseteq C$ and $B \subseteq D$. Since $g_X : X_q \longrightarrow QFX$ is a homeomorphism, $g_X(C)$ and $g_X(D)$ belong to $Z(QFX)^{\#}$. Hence $\Phi_X(g_X(C)) = q_X(C) \in \mathscr{A}_X$ and $\Phi_X(g_X(D)) = q_X(D) \in \mathscr{A}_X$.

In the proof of Theorem 2.7, $g_X\big(S_{\Phi_X(g_X(C))}\big) = q_X(C)$ and $g_X\big(S_{\Phi_X(g_X(D))}\big) = q_X(D)$. Note that $S_{\Phi_X(g_X(C))} = S_{q_X(C)} = \sum_{q_X(C)} \wedge X_q$ and $S_{\Phi_X(g_X(D))} = S_{q_X(D)} = \sum_{q_X(D)} \wedge X_q$. Since $C \cap D = \emptyset$, $\sum_{q_X(C)} \wedge \sum_{q_X(D)} \wedge X_q = \emptyset$ and since $q_X(C) \in \mathscr{A}_X$ and $q_X(D) \in \mathscr{A}_X$, $\sum_{q_X(C)} \cap \sum_{q_X(D)} = \emptyset$. Since $\Phi_{\mathscr{A}_X}(\sum_{q_X(C)}) = q_X(C) = \Phi_{\mathscr{A}_X}(C)$, $C \subseteq \sum_{q_X(C)}$ and similarly, $D \subseteq \sum_{q_X(D)}$. Hence $cl_{\mathscr{L}(\mathscr{A}_X)}(C) \cap cl_{\mathscr{L}(\mathscr{A}_X)}(D) = \emptyset$ and by the Urysohn's extension theorem, h_X is a homeomorphism. □

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