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Toward Students' Full Understanding of Trigonometric Ratios[1]

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Trigonometric ratios are difficult concepts to teach and learn in middle school. One of the reasons is that the mathematical terms (sine, cosine, tangent) don't convey the idea literally. This paper deals with the understanding of a concept from the learner's standpoint, and searches the orientation of teaching that make students to have full understanding of trigonometric ratios. Such full understanding contains at least five constructs as follows: skill-algorithm, property-proof, use-application, representation-metaphor, history-culture understanding [Usiskin, Z. (2012). What does it mean to understand some mathematics? In: *Proceedings of ICME12, COEX, Seoul Korea; July 8–15,2012* (pp. 502–521). Seoul, Korea: ICME-12]. Despite multi-aspects of understanding, especially, the history-culture aspect is not yet a part of the mathematics class on the trigonometric ratios. In this respect this study investigated the effect of history approach on students' understanding when the history approach focused on the mathematical terms is used to teach the concept of trigonometric ratios in Grade 9 mathematics class. As results, the experimental group obtained help in more full understanding on the trigonometric ratios through such teaching than the control group. This implies that the historical derivation of mathematical terms as well as the context of mathematical concepts should be dealt in the math class for the more full understanding of some mathematical concepts.

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INTRODUCTION

Trigonometric ratios are one of difficult concepts to learn in middle school. Teaching is consist of simply presenting and regulating the definition that explains, the constant ratios of pairs of side in a right triangle is the same as sine, cosine, tangent for an internal angle of a right triangle. After such teaching, many students often ask as follows; "Why is the ratios of pairs of sides in a right triangle called sine, cosine, tangent?", "Why isn't there the hypotenuse in tangent?" and so on. Some students used to answer the questions; "It is because mathematicians determined it so." Many Students seem to regard the definition of mathematical concepts or mathematical terms as a promise of mathematicians. Although they try to just accept it, they still have much difficulty to understand it. There are many mathematical terms which concepts aren't promptly come up to students. According to Ko (1998), the mathematical terms that doesn't convey the idea literally would happen the difficulty of mathematical learning, and such an example is the terms 'sin, cos, tan'.

Cavanagh (2008) says that a learning-teaching of mathematical terms needs to be formed as mathematical tools that organize mathematical concepts while also noting that mathematical terms should be included in the students' mathematical learning. Usiskin (2012) states that mathematical concepts involve at least five aspects of understanding including the procedural and relational understanding which Skemp (1976, p. 516) suggested, and by connecting the various aspects of understanding, one can take any of these and turn into a concept. Therefore, a teaching of mathematical concepts is needed to be oriented towards full understanding of it, considering the students' difficulty related to the mathematical terms.

When students manage the mathematical concepts, they cannot help but think of the meaning of terms as a container of concept. Park (2003) states that there are some mathematical terms which read Chinese characters phonetically in Hangul cannot easily evoke its concepts from students, and to solve such problems, it needs to suggest the semantic investigation of mathematical terms. The mathematical terms in English which is not being translated from daily words also would increase many students' difficulty as well as the trigonometric ratios 'sin, cos, tan'. Therefore, the students' understanding related to mathematical terms is needed to be analyzed. There are some studies[2] (Freudenthal, 1978; Kim, 2009; Kim & Lee, 2003; Choi, 2011) related to the mathematical terms. They analyzed levels using it but there are some mathematical concepts that include the low and high level understanding separately[3].It implies that the analysis related to the mathematical terms could be done through the multi-aspects of understanding and no levels of understanding. Hence, this study searches an orientation of instruction towards a full understanding of trigonometric ratios, especially students' understanding of the mathematical terms which is treated little in the current teaching.

Aims of This Study

The aim of the study can be described as follows;

- To explore the understanding of a concept from the learner's standpoint
- To search an orientation of teaching that make students have full understanding of trigonometric ratios

Problem of the Study

What is the effect of the history approach focused on the terms (sine, cosine, tangent) over traditional instruction on middle school students' understanding of trigonometric ratios?

THEORETICAL BACKGROUND

Trigonometric Ratios in School Mathematics:

Textbook is a mirror of current school mathematics, since teachers usually depend on textbooks. Textbooks of mathematics in Korea and other countries[4] are investigated. The focus of analysis is about the definition of trigonometric ratios, the trigonometric ratios

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2 Freudenthal (1978, pp. 233–242) stated that there are four levels of the mathematical language; the ostensive language level, the relative language level, the conventional variable level, and the functional level. Kim (2009) and Kim & Lee (2003) analyzed students' mathematical language by Freudenthal's theory. And Choi (2011) investigates students' concept image through mathematical term when background knowledge related to mathematical terms is given.

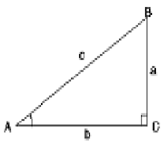
3 Many level theories assume and show that hierarchy exists between the levels. Yet there are a few students who say or write the definition of the concept without having a concrete example of a concept.

4 Textbooks analyzed in this study are as follows; Grade-9 (middle school 3th grade) mathematics (Rhew et al., 2011); UCSMP, 2002, quoted in Song (2008); UCSMP, 1992; New Syllabus Additional Mathematics 8th edition (Shing Lee, Singapore), quoted in Lee (2010).

regarding an acute angle and the method developed it. The content of each textbook is similar as follows:

One identifies the ratios of pairs of sides in right triangles with one equal angle are the same by using the property of similar triangles and then 'is called' the three ratios as sine, cosine, tangent (Table 1).

Table 1. Definition of Trigonometric Ratios

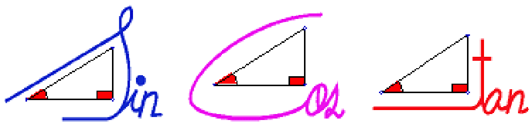


In a right angled triangle

$$\sin A = \frac{a}{c} = \frac{height}{hypotenuse} \quad \cos A = \frac{b}{c} = \frac{base}{hypotenuse} \quad \tan A = \frac{a}{b} = \frac{height}{base}$$

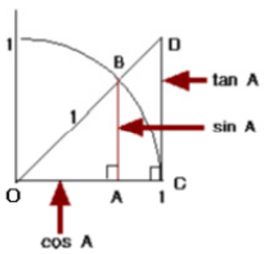
Also as a supplementary explanation, images shaped with s, c, and t (such as SOH, CAH, TOA) are dealt with as memory device(Table 2).

Table 2. Memory Device



Next, by using quarter-circle(Table 3), without explaining the necessity of it, it is dealt with trigonometric ratio regarding an acute angle ranging from 0° to 90°.

Table 3. Trigonometric ratios as length



In a quadrant with its radius 1

$$\sin A = \frac{\overline{AB}}{\overline{OB}} = \frac{\overline{AB}}{1} = \overline{AB}$$

$$\cos A = \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OA}}{1} = \overline{OA}$$

$$\tan A = \frac{\overline{CD}}{\overline{OC}} = \frac{\overline{CD}}{1} = \overline{CD}$$

The curriculum content achievement standards (Ministry of Education, Science and Technology, 2011) on the trigonometric ratios is 'to know the meaning of and to be able to find the values of trigonometric ratios.'

However, as the meaning and terms on trigonometric ratios isn't dealt through the connection of the right triangle and quadrant (circle), students' earlier questions is still left unsolvable and students' understanding is not yet fulfilled. Weber (2006) mentioned the goal of teaching and learning mathematics as follows:

> There is a broad consensus among mathematics education researchers that the goal of mathematics courses is not only for students to memorize procedures and acquire reliable methods for producing correct solutions on paper-and-pencil exercises; rather methodsstudents should learn mathematics with understanding (Weber, 2006, p. 92).

Since trigonometric terms explained in textbooks are only considered by students as an arbitrary correspondence to the value of the ratio and quadrant is addressed suddenly only for the trigonometric ratios regarding an acute angle, they cannot be movements toward the achievement of the goal of learning-teaching on the trigonometric ratios.

Understandingof a concept from students' standpoint

There are many studies (Pirie & Kieren, 1989; 1995; Martin, 2008; Conzales Astudillo, Codes, Delgado & Monterrubio, 2012) related to the students' understanding of mathematical concepts. These studies show that the growth of understanding is not linear, but they have a basic assumption that exists in the levels (or layers) of understanding concepts, that is, the higher level of understanding needs the lower level of understanding. However, there are students who solve an application problem or say the definition of a mathematical concept without saying its concrete example or representing its image.

Usiskin (2012) suggests five dimensions to the understanding of a concept in mathematics from the learners' standpoint, against Skemp's dichotomy about the understanding, that is, the instrumental understanding and relational understanding essentially meaning procedural understanding and conceptual understanding. According to his study, those are all different, more than two aspects of understanding the same subject and each aspect can be mastered to be relatively independent of the others, therefore, these aspects is called the dimensions of understanding. This is for full understanding of some mathematics.

Skill -algorithm dimension:

This is related to get the right answer using algorithms or procedures. Because that is often done automatically, one often views applying the procedures as the opposite or a lower level of understanding. However, there is much more to procedural understanding than merely applying an algorithm.

Property-proof dimension:

This is related to identify the mathematical properties or theories that underlie why the way of obtaining the answer worked.

Use-application (modelling) dimension:

This is related in knowing when or where the mathematics could be. This involves not a higher order of thinking than skill but a different kind of thinking, because some application is harder than some skill, but some skill is harder than some application.

Representation-metaphor dimension:

This is connected with representing the mathematical concept in some way (using concrete objects, pictorial representations, metaphor and so on).

History-culture dimension:

This is connected with knowing the answers of the following questions: "How and why did a certain bit of mathematics arise? How has it developed over time? How is it treated in different cultures, and so on?

Usiskin (2012, p. 515) states that the last dimension of understanding is not usually found in school mathematics. This suggests that the history-culture aspect of mathematics should be dealt with in the mathematics teaching for full understanding of a concept.

History of Trigonometric Terms

Hipparchus (180BC–125BC), was one of the greatest astronomers of antiquity, he created mathematics, which eventually became what we know as modern day trigonometry. In his work he dealt with triangles that were inscribed in circles. Because he was often dealing with triangles in the heavenly sphere, he developed spherical trigonometry at the same time that he was developing plane trigonometry. A basic problem was to evaluate the three angles and three sides of the inscribed triangle (Crossfield, Shepherd, Stein & Williams, 2009, p. 18).

The following is an interesting etymological evolution that would finally lead to our modern word "sine."

Early astronomers regarded the calculation of the length of the chord as very important, the Greek astronomer Claudius Ptolemy (c. 90AD –c. 168AD) tabulated lengths of chords corresponding to central angles in a circle. Aryabhata of India (c. 476AD–c. 550AD) streamlines astronomical calculations by creating tables of half chords.

Aryabhata uses the word *ardha-jya* for the half-chord, which he sometimes turns around to *jya-ardha*(chord half); in due time he shortens it to *jya* or *jiva*. When the Arabs learned trigonometry from the Indians, they retained the word *jiva* without translating its meaning from this word. In Arabic, however, vowels are generally omitted in writing. Thus, all that was written were the two consonants *jb*. The pronunciation of the missing vowels was understood through common usage. Thus jiva could also be pronounced as *jiba* or *jaib*. When Latin translators translated the Arabic works, they took this word to be a different Arabic word, namely *jaib*, which means fold, bay, or inlet. Thus, they translated it by the comparable Latin word *sinus*, form which come our word "sine." In Figure1, the length of the half-chord is to a modern word.

The cosine function, which we regard today as equal in importance to the sine, first arose from the need to compute the sine of the complementary angle. That is, the word "cosine" is simply short for "sine of the complement of an angle", or, simply "sine complement."

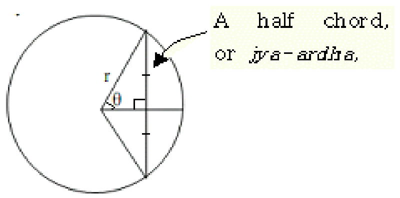


Figure 1. Sine

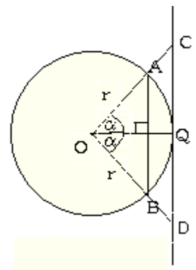


Figure 2. Tangent

The word "tangent" comes from the Latin *tangere*, to touch; its association with the tangent function may have come from the following observation: in a circle with center O and radius *r* (Fig.2), let AB be the chord subtended by the central angle 2α, and OQ the bisector of this angle. Draw a line parallel to AB and tangent to the circle at Q, and extend OA and OB until they meet this line at C and D, respectively, we have

$$AB = 2r \sin \alpha, \quad CD = 2r \tan \alpha$$

showing that the tangent function is related to the tangent line in the same way as the sine function is to the chord (Maor, 1998, pp. 35–38) .

METHODOLOGY

This study lasts 3 weeks and focuses on the "Trigonometric ratios" unit of mathematics course. The experimental study was carried out throughout September and October in the 2012 academic year.

Participants

The sample of this study consists of 198 third year middle school students. These students are in six different classes and are taught by two teachers. Classes are randomly selected as experimental and control group. In the experimental group there were 66 individuals and in the control group there were 67 individuals. In this group the teacher was informed of the experiment. There were another 65 individuals in a blind control group with the teacher unaware in order to be independent of the teaching variable. The informed teacher was told that they were participants of a research study but they were not told whether they were in the experimental or control group. The participants had very close grade point averages, and they all received similar lessons. Therefore, they have similar academic characteristics.

Following the experimental teaching, the one who was in the achievement mid-level experimental group and the other who was also in the mid-level control group were asked their understanding. They were chosen because their answer sheets were to be noticed in the perspective of this study.

Research Method

This research is an experimental design research. Experimental and control grouped students are considered as equal academic success at the beginning of the study since they have close grade points. It is compared to the effect of the experimental teaching by post-test.

Another research method is case study. School mathematics is not included historic dimension, so it is not able to question through post-test. It needs to examine this dimension by interview. It is asked what trigonometric ratios are, and how to know or solve it with student's test paper.

Data Gathering Instruments

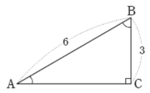
The class is analyzed by video-recording and written down. An achievement test is developed by the researcher to measure the effect of history oriented teaching of trigonometric ratios. Then the students' test papers are graded. The interview is also analyzed by video-recording and written down.

Questions of the Test

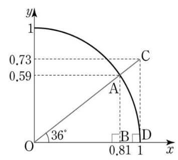
The aim of test is to determine attainment level and to analyze understanding of students after history oriented teaching process. While developing this test, the researcher considers the related attainment targets, checks previous final, midterm exams, and resources related to trigonometric ratios unit. The developed test consists of classical problems that also aim to assess skill-algorithm (Q1), property-proof (Q2), use-application (modelling) (Q3), representation-metaphor (Q4) understanding of students. The problems that are prepared by the researcher are checked about corresponding dimension by a professor, a teacher, and two PhD students. Q1 is expected for students to show skills, procedures, or strategies to be approach their concept. Q2 is expected for students to show properties, relations, or justifications to solve. Q3 is expected for students to show available concept in application context. Q4 is expected for students to show their representations as expression. The questions are presented in table 4 below.

Table 4. Questions

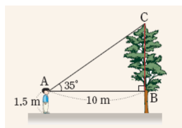
Q1. If in a right-angled triangle ABC, AB = 6, BC = 3 and $\angle C = 90^\circ$. Find the values of all the trigonometric ratios of angle A and B. Explain your answer.



Q2. Evaluate $\sin 36^\circ + \tan 36^\circ + \sin 54^\circ - \cos 54^\circ$ using the following figure. Explain your answer.

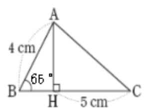


Q3. Write the expression as trigonometric ratios to find the length of the height of tree.



Q4. Find the length of given segments, $\overline{AH}$ and $\overline{BH}$ using the given table. Explain your answer.

angle	sin	cos	tan
65°	0.9	0.4	2.1



Analysis of Data

In order to compare to groups, it is calculated percentages of correct answer quantitatively. In order to examine students' understanding dimensions, it is analyzed students' strategies qualitatively.

Teaching Strategy that is Followed in Experimental Group

The unit is dealing with the historic origin of term, at first, the calculation necessary in astronomy and the atmosphere is introduced, the origins of the trigonometric terms are explained, and the ratio of the length of edges in the similar right triangle is addressed.By presenting the contexts in the history of mathematics, it is expected for students in this group to understandof usefulness in terms of the indirect measuring tools and then to connectfrom the trigonometric ratio towards non standard angle. The students in experimental group firstly study the meaning of the mathematical terms based on history origins. After this, they were followed to textbook practices.

Teaching Strategy that is Followed in Control Group

In this group the unit is taught by employing traditional teaching strategies (teacher centered; teaching by telling, question-answer-exercises). The same textbook is used in this group also (Rhew et al., 2011). The mathematics course was run by two instructors with the same printout content during the study.

RESULTS AND INTERPRETATION

In this section comparison of data gathered from experimental(E1,2) and control group(C1,2 and Blind Control; BC1,2) will be presented. Table5 is shown the results and the blank is the number individuals.

Table 5. Percentage of correct answer in experimental and control group

Question number	E1 (32)	E2 (34)	C1 (34)	C2 (33)	BC1 (34)	BC2 (31)
1	62.5 (20)	70.6 (24)	70.6 (24)	69.7 (23)	88.2 (30)	67.7 (21)
2	40.6 (13)	23.5 (8)	35.3 (12)	33.3 (11)	35.3 (12)	38.7 (12)
3	50.0 (16)	44.1 (15)	47.1 (16)	66.7 (22)	58.8 (20)	58.1 (18)
4-1	53.1 (17)	62.5 (16)	61.8 (21)	63.6 (21)	58.8 (20)	58.1 (18)
4-2	46.9 (15)	47.1 (15)	64.7 (22)	60.6 (20)	58.8 (20)	58.1 (18)

There hasn't been any recognizable change or effect upon experimental groups.

It is examined by interview in the case of drawing attention perspective to full understanding of trigonometric ratios concepts. This found students' strategies and errors in answer sheets.

Q1 had investigated that students could resolve the trigonometric ratios according to its procedures. The result of Skill-algorithm understanding is that students don't have any trouble very much to finding the values although they regard concepts as to be unfamiliar. It shows that most of the students have implemented the memory strategy (Fig.4) and picture rotation strategy (Fig.5).

Q2 had investigated whether the students were aware of the property or the theory on the basis of the procedures or not. As a result of identifying of Property-proof understanding, the percentage of correct answer was lower due to more the half failures. As for the successful cases, it had been high frequency of substitute values rather than being able to notice its relations. Fig.6 is the result of explaining the relations based on the procedures. She(S1) used her own examples. She said that cosine is the sine of complement angle. She could see the inverse relation of tangent between complement angles.

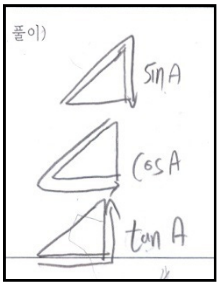


Figure 4. Memory strategy

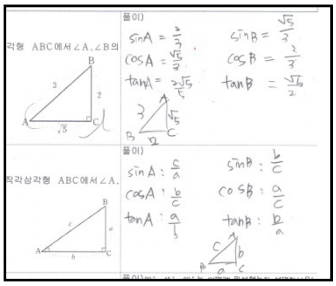


Figure 5. Picture rotation strategy

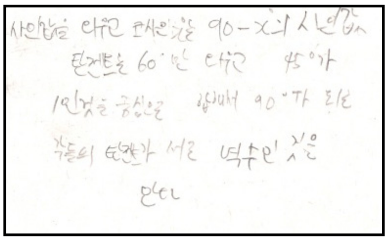


Figure 6. Solution of S1

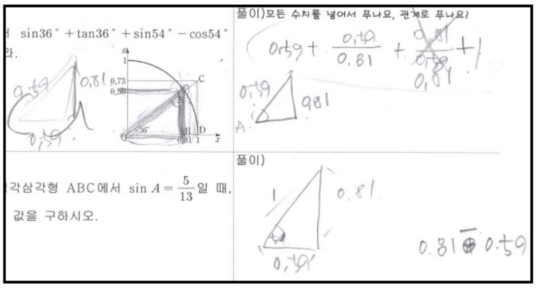


Figure 7. Solution of S2

There was noticeable solution in control grouped student (S2);

$$\tan O = \frac{59}{81}$$

(Fig.7).

The researcher(R) asked for each student (S1 in experimental group and S2 in control group) what is the tangent and how did you solve it. S1 connected the meaning of tangent as trigonometric ratios to the tangent line. Whereas S2 was not recognized the adjoining relation in tangent.

Interview S1

R: (with Fig.7) What is the tangent?

S1: The segment DC

R: Do you know that reason in why it is related to its meaning?

S1: It is adjoining, so DC.

Interview S2

R: What is the tangent?

S2: Height over base.

R: Do you know why tangent became the term to this ratio?

S2: Is that reason? Wasn't it determined through mathematicians?

...

R: Where did your answer $\frac{59}{81}$ come from?

S2: I didn't see outer triangle. I usually practice by myself. I usually corrected before although I didn't consider it.

R: Then why did you answer $\sin 54°$ was $\frac{81}{59}$ and $\cos 54°$ was 1?

S2: (with pointing out her upper drawing) I rotated drawing.

R: (with the drawing) Is this rotation right?

S2: Yes.

R: What do you see the length of this segment (with AB) is? Is it 0.59?

S2: Ah. (With correcting her wrong answer to another drawing in bottom) 1.

R: Can you correct your answer?

S2: I see (She was completing it perfectly).

This interview revealed if students know the history of tangent then they reduce confusion. So they can obtain richer understanding of their concept.

Q3 is about that the students could figure out the situational issues. As a result of use-application understanding, there were not many difficulties. Fig. 8 is a sample of correct notation of using the trigonometric ratios.

Unexpectedly, some students had a question of what it was regarding as an error on account of the standard angle, some of them argued that it was not possible as long as it had been an angle $30°$. The angles of the textbook problem are located to the bottom on the left-hand. Most of the questions in textbook are about the special angles. The students did not get used to applying it into the real life situations and are stuck with using the special angles. This response revealed some trouble in Korean textbook, although it is not the topic of this study.

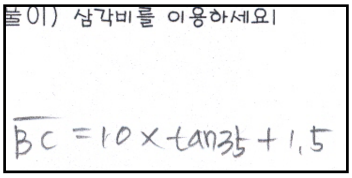


Figure 8. A sample of correct notation

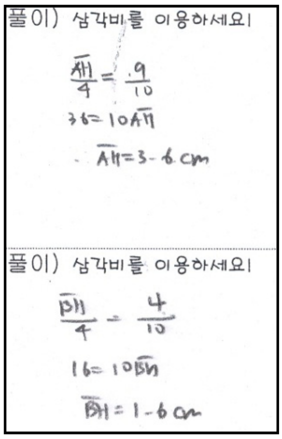


Figure 9. A sample without using 'sin', 'cos'

Q4 has been researched that students could exploit and figure out the trigonometric ratios related figuration. Following the identifying of understanding Representation-metaphor, there were no issues on resolving the values; however, it had the high frequency of using the uniqueness of the value of ratios not using the sign of its (Fig. 9). It may examine not of using sign in the representation, but of revealing to have the trigonometric ratios' concept.

Following the class and test, the interview carried out confirming their solution on purpose. Then, it was verified the effect of history oriented instruction in regards to obtaining students' full understanding. It has chosen two students (S1 and S2) with attractive answer sheets. It turned out the difference between experimental and control group. The controlled class student (S2) was aware of its definition, while the experimental one (S1) showed us huge understanding including historical meaning. It is assumed that the origin of the history class led multi-dimensional entity for the fully understanding.

CONCLUSIONAND DISCUSSION

Trigonometry is an important course in the secondary school curriculum. Unfortunately, even the initial stages of learning about trigonometry made it much more difficult for students. This study explored the students' understanding on trigonometric ratios regarding the effect of the history approach focused on trigonometric terms, sine, cosine, tangent over traditional teaching. The results indicated that the history approach effect students' full understanding on the trigonometric ratios. That is, in the case of the trigonometric ratios, it needs to be dealt with the historical derivation of mathematical terms as well as the context of mathematical concepts. Because the mathematical terms that include the link hook about the concepts is helpful (Choi, 2011) in making students come and go freely between the term and the concept.

History-culture aspect of understanding is little in school mathematics. So, complementary measures in the side of the aspect may increase students' understanding fully. Katz (1997) stated "The teaching of mathematics is difficult and it is not necessarily easier when one uses the history of mathematics as part of this teaching. But our aim is to discover the ways in which the use of the history of mathematics makes learning better for the student." According to him, a history approach may be one teaching method for students' better understanding. There are many aspects of understanding a same subject (Usiskin, 2012).It needs to merge them, especially a history-culture aspect which is rarely taught in a teaching of some mathematics, into teaching of concepts. This study was to search an orientation of teaching that make students have full understanding of trigonometric ratios. We hope that the history approach of this study will be one teaching method for an achievement of the learning-teaching goal on the trigonometric ratios.

However, this study has limitations that did not identify an awareness of Grade 9 students about the regulation of the mathematical terms, and specify a teaching of concepts and a teaching of terms. More studies regarding such the limitations are required, and we expect that these studies can help the students' full understanding of concepts.

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