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Teaching of Division of Fractions through Mathematical Thinking[1]

CHENG, Chun Chor Litwin
Department of Mathematics and Information Technology, Hong Kong Institute of Education, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong; Email: cccheng@ied.edu.hk

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Division of fractions is always a difficult topic for primary school students. Most of the presentations in teaching the topic in textbooks are procedural, asking students to invert the second fraction and multiply it with the first one, that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Such procedural approach in teaching diminishes both the understanding of structure in mathematics and the interest in learning the subject. This paper discussed the formulation of teaching the division of fractions, which based on research lessons in some primary five classrooms. The formulated lessons started with an analogy to division of integers and working with division of fractions with equal denominators and then extended to division of fractions in general. It is found that the using of analogy helps students to invent their procedure in working the division problem. Some procedures found by students are discussed, with the focus on the development of their invention and mathematical thinking.

Keywords: mathematical thinking
MESC Classification: C32, F32
MSC2010 Classification: 97C50

INTRODUCTION

The teaching of fractions division by

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

is only a description of a procedure, or a conclusion of a procedure, or a representation of

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a schema. Knowing the formula of the procedure is different from conceptual understanding. Such “memory and procedural” approach involve a lot of “correct calculation” without sufficient depth in concept understanding of the structure of fraction division. For example, in some cases students inverted the wrong fraction and got

$$\frac{a}{b} \div \frac{c}{d} = \frac{b}{a} \times \frac{c}{d},$$

and some even inverted the fraction in multiplication and gave

$$\frac{a}{b} \times \frac{c}{d} = \frac{b}{a} \times \frac{d}{c}.$$

Behr, et al (1983) proposed a four-part model for teaching fractions, in which the subconstruct of fraction as ratio is important in understanding the operation of fraction division. The research by Charalambous, et al (2007) indicated that the concept of equivalence fractions helps students to work with the subconstructs of fractions and manage the operation of fractions more easily. Using the concept of ratio and table to teach division of fraction is shared by many researchers (Sharp 1998; Sharp 2002; Tirosh, 2000). Sun (2011) described how the Chinese textbooks used the principle of variation problems and compared the Chinese and American textbooks in this issue.

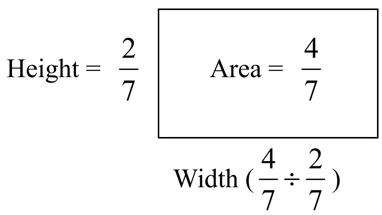
Yim (2009) worked with a tactic in knowing how children deal with fractions division by asking them to find the length of a rectangle given the area and width. Yim found that students could formulate three strategies by themselves, namely making the width equal to 1, or making the area equal to 1, or changing both area and width to natural numbers. For example, students are asked to find the width of a rectangle with

$$\text{area } \frac{4}{7} \text{ and height } \frac{2}{7}.$$

Through drawing a diagram and consider the expression

$$\frac{4}{7} \div \frac{2}{7},$$

students obtained the answer of the division of fraction as 2.



Textbooks in Hong Kong usually did not explain much on the procedure of “inverted fraction multiplication” and hence some pedagogical content knowledge is used to explain the inverted multiplication procedure. The explanation involves the “principle of unchanged quotient in ratio”. Based on $b \div a = (b \times p) \div (a \times q)$, students are taught to apply the rule to the division of fractions to obtain

$$\frac{11}{12} \div \frac{5}{8} = (\frac{11}{12} \times \frac{8}{5}) \div (\frac{5}{8} \times \frac{8}{5}).$$

The multiplication of

$$\frac{8}{5}$$

was meant to change the value of the divisor to 1, and obtain

$$\frac{11}{12} \div \frac{5}{8} = \frac{11}{12} \times \frac{8}{5}.$$

However, such direct explanation is a bit abstract and students did not know why they should do that at the start. Some embodiment of similar concepts is needed.

THE FRAMEWORK OF THE STUDY

Tall (2007) suggested a view of presentation of mathematics in three level. The first level is the multiple embodiments to understand, formulate, and solve the problem. The second level is the using of symbolism in presenting the problem and connects their knowledge to solve the problem, and finally the formal mathematics level, using mathematics structure to solve the problem. The following is the description of the framework of the three levels.

Table 1. Using the framework of embodiment, symbolism and formal mathematics

Example: A metal bar with $\frac{3}{4}$ metre in length weights $\frac{5}{6}$ kg. What will be the weight of the metal bar if it is one metre in length?		
Multiple embodiment to represent the concept	Using of symbolism to present the problem	Using formal mathematics in solving the problem
Ratio Diagram analogy	A bar 4 metre long weights 8 kg, then each metre will weight 8÷4 =2 kg。 Symbolic expression $\frac{5}{6} \div \frac{3}{4}$	Procedural $\frac{5}{6} \div \frac{3}{4}$ $= (\frac{5}{6} \times \frac{4}{3}) \div (\frac{3}{4} \times \frac{4}{3})$ $= (\frac{5}{6} \times \frac{4}{3})$

Analogy is an important component of mathematical thinking and mathematics learning design is best to allow such thinking to happen (English, 2004). In this regards, the correspondence in using of problems in division of numbers and division of fractions are important.

Zhang (2008) proposed that learning mathematics based on basic skills, basic knowledge, and mathematical thinking. This is part of the "Four Basics model". With reference to this model, basic skills means the ability to convert equivalence fractions, and basic knowledge means connecting fraction division with subtraction of integers and ratio comparison. Basic mathematical thinking is to use analogy and transfer the knowledge of division and subtraction for integers to division of fractions. The process involves activities in "concept and structure correspondence" and "conjecturing with verification".

TEACHING DESIGN

Apart from "invert and multiply"; there are others ways in teaching fractions division. For example, using ratio table to compare fractions, and using equivalence fractions so that division of fraction become comparison of the numerators. This will be conducted in this study.

The teaching was in three parts:

1) Division of a fraction by a whole number, then
2) The division of two fractions with the same value of denominators, and
3) The division of fractions in general. In the course of teaching, oral presentation, posing problem with analogy of question and diagram are conducted with students.

The teaching was conducted in three primary schools in Hong Kong[2], either at the end of Primary 4 or first term of Primary 5[3]. Table 2 is the design of teaching with respect to concept formation in fractions division.

A. Teaching of division of the type $\frac{ma}{b} \div m = \frac{a}{b}$

Based on a very simple division problem "8 ÷ 2 = 4", students started to discuss the solution of the following question with

"$\frac{2}{5} \div 2$".

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2 The author would like to thank the following three schools for their involvement in the research. They are Tai Kok Tsui Catholic Primary School(Hoi Fan Road) , Baptist (Sha Tin Wai) Lui Ming Choi Primary School, Heep Yunn Primary school.

3 According to the Hong Kong Curriculum, division of fractions is taught in the second semester of the Primary Five. In this study, the teaching lesson was conducted in later part of Primary 4 or early Primary 5 to avoid students learning the same topic again.

Through using equivalent fractions, diagram and abstraction, expression such as

$$“\frac{6}{7} \div 4”$$

are considered.

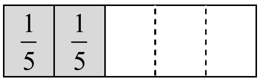
Question: A metal bar of length $\frac{2}{5}$ metre is divided into two equal parts, what is the length of each part? $\frac{2}{5}$  Find the answer again if the original length is $\frac{3}{5}$ metre.

Table 2. The design of teaching with respect to concept formation in fractions division

	Learning Tasks	Related tasks
A	Division of fraction by integers $\frac{ma}{b} \div m = \frac{a}{b}$ $\frac{b}{a} \div p = \frac{b \div p}{a}$	$\frac{6}{7} \div 2 = \frac{3}{7}$ $\frac{6}{7} \div 4 = \frac{12}{14} \div 4 = \frac{3}{14}$ 。
B	Division with fractions of equal (multiple) denominators $\frac{a}{b} \div \frac{c}{b} = a \div c$ $\frac{a}{b} \div \frac{c}{mb} = \frac{ma}{mb} \div \frac{c}{mb} = ma \div c$	$\frac{4}{7} \div \frac{2}{7}, = 2, \quad \frac{4}{7} \div \frac{3}{7} = \frac{4}{3}$ $\frac{4}{7} \div \frac{5}{14} = \frac{8}{14} \div \frac{5}{14} = \frac{8}{5}$
C	Division of fractions in general $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = ad \div bc$	$\frac{5}{6} \div \frac{3}{4} = \frac{5 \times 4}{6 \times 4} \div \frac{3 \times 6}{4 \times 6} = 20 \div 18$

From the diagram (in the question),

$$\frac{2}{5} \div 2 = \frac{1}{5},$$

and some students also employ fraction to get the solution:

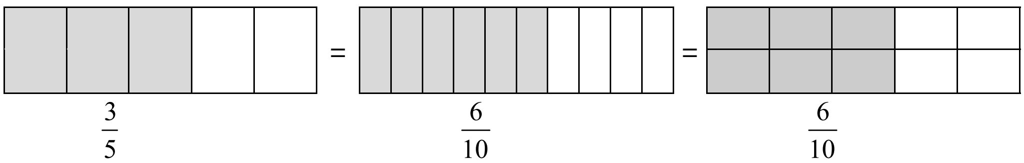
$$\frac{2}{5} \div 2 = \frac{2}{5} \div \frac{2}{1} = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}.$$

Using equivalent fractions as a tool to connect the concept, students work on

$$“\frac{3}{5} \div 2”.$$

The answer was also verified by using diagram

$$“\frac{3}{5} \div 2 = \frac{6}{10} \div 2 = \frac{3}{10}”$$



The exploration on their approaches also leads to the solution for

$$“\frac{6}{7} \div 4”.$$

Apart from using multiplication

$$“\frac{6}{7} \div 4 = \frac{6}{7} \times \frac{1}{4} = \frac{3}{14}”,$$

students arrived at the process

$$“\frac{6}{7} \div 4 = \frac{12}{14} \div 4 = \frac{3}{14}”.$$

Through the process of getting correct answer for

$$\frac{ma}{b} \div m = \frac{a}{b},$$

students aware and understand the relation of

$$\frac{b}{a} \div p = \frac{b \div p}{a}.$$

B. Teaching division of fractions with equal denominators, $\frac{a}{b} \div \frac{c}{b} = a \div c$

The teaching

$$\frac{a}{b} \div \frac{c}{b} = a \div c$$

is introduced by:

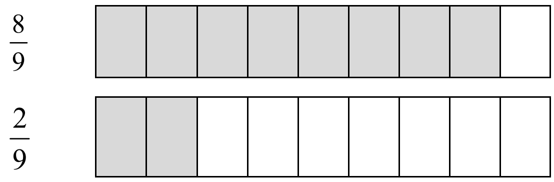
1) Concept of series of subtraction (with integer solution), and
2) Ratio through diagram.

The following example is used. The volume of a box of juice is $\frac{8}{9}$ litre and each time $\frac{2}{9}$ litre of the juice is poured. How many times could it be poured?

Through series of subtraction,

$$“\frac{8}{9}-\frac{2}{9}-\frac{2}{9}-\frac{2}{9}-\frac{2}{9}=0”,$$

the answer is 4 times. Students are then required to draw a diagram for the expression as follow:



From the diagram, students observed that

$$“\frac{8}{9}\div\frac{2}{9}=8\div 2=4”.$$

Similar diagrams help students to understand

$$\frac{5}{7}\div\frac{2}{7}=5\div 2$$

through ratio considerations. The using of two approaches (series of subtraction and ratio through diagram) allowed students to use ratio to understand division that did not give decimal solution, such as

$$\frac{5}{7}\div\frac{3}{7}=5\div 3.$$

Later on, students need to make sense of the expression

$$\frac{8}{9}\div\frac{2}{9}$$

by providing daily examples (for example,

a $\frac{8}{9}$ pizza was cut into a share of $\frac{2}{9}$ each time).

Students' discovery of the rule $\frac{a}{b}\div\frac{c}{d}=\frac{a\div c}{b\div d}$

A student worked on

$$\frac{6}{7}\div\frac{4}{7},$$

and changed the first fraction to an equivalence fraction so that both the numerator and denominator could be divided by the numerator and denominator of the second fraction and obtain

$$“\frac{6}{7}\div\frac{4}{7}=\frac{12}{14}\div\frac{4}{7}=\frac{12\div 4}{14\div 7}=\frac{3}{2}”.$$

Other students try similar working and obtain

$$“\frac{5}{7} \div \frac{3}{7} = \frac{15}{21} \div \frac{3}{7} = \frac{15 \div 3}{21 \div 7} = \frac{5}{3}”; “\frac{2}{7} \div \frac{3}{7} = \frac{12}{42} \div \frac{3}{7} = \frac{12 \div 3}{42 \div 7} = \frac{4}{6}” \text{ etc.}$$

They then guessed that there is a rule of

$$“\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}” \text{ as both } “\frac{2}{7} \div \frac{3}{7} = 2 \div 3” \text{ and } “\frac{2}{7} \div \frac{3}{7} = \frac{2 \div 3}{7 \div 7} = 2 \div 3”$$

are the same expression. The students could not explain the rule at this point and it will be addressed in part C when the general fraction division is considered.

C. Teaching of division of fractions $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = ad \div bc$

The problem given to students are fractions that the one denominator is a multiple of the other, and with integral solutions such as

$$“\frac{3}{4} \div \frac{1}{8} = 6”.$$

Using denominators in such relations aimed to reduce the cognitive load in learning. For example,

$$“\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} \div \frac{1}{8} = 6”.$$

From the diagrams drawn, students could reduce the mathematical expression as follow

$$“\frac{3}{4} \div \frac{1}{8} = (\frac{3}{4} \times 8) \div (\frac{1}{8} \times 8) = 6 \div 1 = 6”$$

To summaries, students solved the problem in two approaches.

1) By division	$\frac{3}{4} = \frac{6}{8}$, $\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} \div \frac{1}{8} = 6$.
2) By multiplication	$\frac{3}{4} \div \frac{1}{8} = (\frac{3}{4} \times 8) \div (\frac{1}{8} \times 8) = 6 \div 1 = 6$

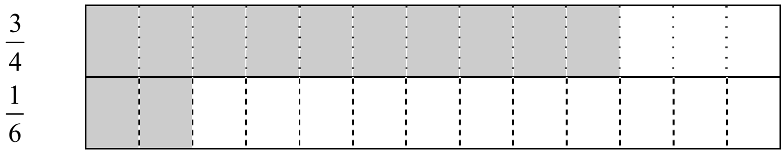
The next step is to solve fractions division with any numbers in the denominators. Using of a problem with daily background and use analogy of integral values.

Questions for analogy: Find the weight of the metal bar if it is one metre in length.

It is 4 metre in length and weights 8 kg. The answer is 8 ÷ 4 = 2.	It is $\frac{1}{6}$ metre in length and weights $\frac{3}{4}$ kg. Students arrived at similar expression $\frac{3}{4} \div \frac{1}{6}$

Then the using diagram to explain the calculation, that is,

$$\frac{3}{4} \div \frac{1}{6} = \frac{9}{12} \div \frac{2}{12} = \frac{9}{2}.$$



STUDENTS' INVENTION ON PROCEDURE

When the general divisions of fractions are considered, students could now transfer their skills and knowledge in solving fraction problem discussed previously.

> **Question:**
> A metal bar with $\frac{3}{4}$ metre in length weights $\frac{5}{6}$ kg. What will be the weight of the metal bar if it is one metre in length?

The answer mathematics expression is

$$\frac{5}{6} \div \frac{3}{4},$$

students then discussed how to obtain the answer through equivalence fractions. Most of them come with the results

$$\text{“}\frac{5}{6} \div \frac{3}{4} = \frac{5 \times 4}{6 \times 4} \div \frac{3 \times 6}{4 \times 6} = 20 \div 18\text{”}.$$

Also, with the following table, students tried to answer the question by ratio.

Weight	$\frac{5}{6}$	$\frac{10}{6}$	$\frac{15}{6}$	$\frac{20}{6}$	Kg
Length	$\frac{3}{4}$	$\frac{6}{4}$	$\frac{9}{4}$	$\frac{12}{4}$	m

As

$$\frac{12}{4} = 3,$$

a metal bar of 3 metre in length will weight

$$\frac{20}{6}\text{ kg.}$$

Which means each metre of the bar weight

$$\frac{20}{6} \div 3 = \frac{60}{18} \div 3 = \frac{20}{18}\text{ kg}.$$

The ratio table provides a chance for students to discuss the method on two numbers line proposed by teacher. Students obtained the weight of

$$\frac{1}{4}\text{ metre of the bar, which is } \frac{5}{6} \div 3.$$

And the weight of 1 metre of the bar is

$$\frac{5}{6} \div 3 \times 4.$$

And students know that the expression is the same as

$$\frac{5}{6} \div \frac{3}{4}.$$

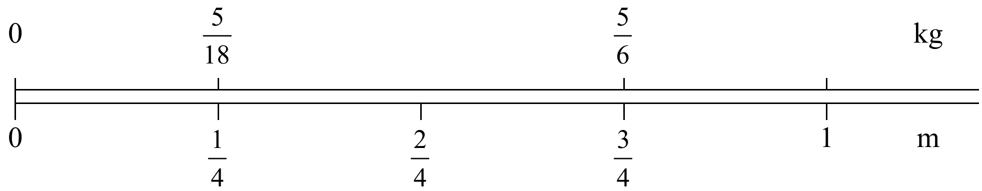
Hence

$$\frac{5}{6} \div \frac{3}{4} = \frac{5}{6} \div 3 \times 4.$$

As

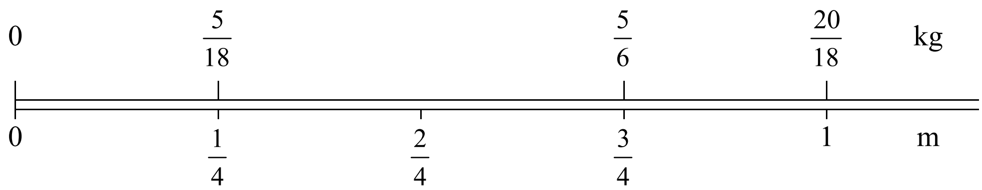
$$\frac{3}{4}\text{ m correspondence to } \frac{5}{6}\text{ kg, } \frac{1}{4}\text{ m correspondence to } \frac{5}{18}\text{ kg}$$

as below.



Hence the weight for 1 metre of the bar is

$$\frac{5}{18} \times 4 = \frac{20}{18}$$



When students worked on a similar problem

$$\text{“}\frac{6}{7} \div \frac{3}{4}\text{”},$$

they provide their following reasoning in working the solutions.

Response 1	$\frac{6}{7} \div \frac{3}{4} = \frac{24}{28} \div \frac{3}{4} = \frac{24 \div 3}{28 \div 4} = \frac{8}{7}$
Response 2	$\frac{6}{7} \div \frac{3}{4} = \frac{6 \times 4}{7 \times 4} \div \frac{3 \times 7}{4 \times 7} = \frac{24}{28} \div \frac{21}{28} = 24 \div 21 = \frac{8}{7}$
Response 3	$\frac{6}{7} \div \frac{3}{4} = \frac{6 \div 3}{7 \div 4} = \frac{2}{7/4} = \frac{2 \times 4}{7} = \frac{8}{7}$

Students' explanation of the rule $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$

As mentioned in part B, students obtain

$$“\frac{6}{7} \div \frac{4}{7} = \frac{12}{14} \div \frac{4}{7} = \frac{12 \div 4}{14 \div 7} = \frac{3}{2}”$$

and guess that

$$“\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}”.$$

In this example, they arrive at

$$“\frac{6}{7} \div \frac{3}{4} = \frac{24 \div 3}{28 \div 4}”$$

and know that it works according to the similarities. They provide the following discussion and verify that the guess rule is true. The arguments are as follow:

When divided by integers	$\frac{6}{7} \div 3 = \frac{6 \div 3}{7} = \frac{2}{7}$, $\frac{6}{7} \div 3 = \frac{6}{7} \div \frac{3}{1}$
When divided by unit fractions	$\frac{6}{7} \div \frac{1}{4} = \frac{24}{28} \div \frac{7}{28} = 24 \div 7 = \frac{24 \div 7}{28 \div 28}$.

And combine the two effects

$$“\frac{6}{7} \div 3 \text{ and } \frac{6}{7} \div \frac{1}{4}”,$$

the conclusion is

$$“\frac{6}{7} \div \frac{3}{4} = \frac{24 \div 3}{28 \div 4}”.$$

Then students use the results as a tool to work on all other problems.

Re-discover the procedural of inverted multiplication and its reasoning

The above process of division of fraction provides students with concept and making the concept into procedural operations. However, as most of the textbooks will finally come to teach the "inverted multiplication" procedure, the teaching needed to go on to lead students to acquire the reasoning of the "inverted multiplication" procedure. The following process allows this to happen.

Question 1: (unchanged quotient)	Question 2: (apply the rule in fractions)
$24 \div 5 = (24\times \square) \div (5\times \square)$	$\frac{6}{7} \div \frac{3}{4} = (\frac{6}{7} \times \square)\div(\frac{3}{4} \times \square)$.

Students are then asked to fill in a fraction (in the boxes) for the expression so as to simplify the division:

$$\frac{6}{7} \div \frac{3}{4} = (\frac{6}{7} \times \frac{\square}{\square})\div(\frac{3}{4} \times \frac{\square}{\square}).$$

The following are the response from students.

Response 1	$\frac{6}{7} \div \frac{3}{4} = (\frac{6}{7} \times \frac{28}{1})\div(\frac{3}{4} \times \frac{28}{1})$
Response 2	$\frac{6}{7} \div \frac{3}{4} = (\frac{6}{7} \times \frac{28}{3})\div(\frac{3}{4} \times \frac{28}{3})$
Response 3	$\frac{6}{7} \div \frac{3}{4} = (\frac{6}{7} \times \frac{4}{3})\div(\frac{3}{4} \times \frac{4}{3})$

All of these responses are correct and most importantly, they are provided by students themselves.

CONCLUSION

Accurate calculation without concept could not lead to understanding of mathematical structure. And without the understanding of structure, it is hard to extend the concept and invent new procedure. The teaching process in this study usually involves 3 hours for the whole process of discussion and investigation to happen. With students provide their own strategies in tackling division of fractions in general, it is safe to say that the concept is well delivered. When students are asked to draw a diagram, they have to decide the number of boxes they need in doing the comparison in ratio. For example, the working of

$$\frac{3}{4} \div \frac{1}{6}$$

may require students to use 12 boxes or 24 boxes. The using of two or more approaches in discussion of the same topic is important in terms of multiple embodiments, as it enhances the level of understanding concepts. The process of discussion allows students to use their skills (equivalence fractions) and knowledge (ratio and comparison), and invent their procedure under mathematical thinking, transferring the concept of division in integers to fractions division. This allow learning with less cognitive load (more deduction with analogy), and less memory load (using more deduction). Psychologically, the model of teaching helps to build the confidence of the students, since they have the opportunities to invent their procedures in obtaining their answers.

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