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# A NIELSEN TYPE NUMBER OF FIBRE PRESERVING MAPS

Seoung Ho Lee

Abstract. We introduce a Nielsen type number of a fibre preserving map, and show that it is a lower bound for the number of $n$-orbits in the homotopy class. Under suitable conditions we show that it is equal to the Nielsen type relative essential $n$-orbit number. We also give necessary and sufficient conditions for it and the essential $n$-orbit number to coincide.

## 1. Introduction

Nielsen fixed point theory has been extended to a Nielsen type theory of periodic orbits [7, III.3]. Much has been introduced about the Nielsen number of fibre map [2, 3, 7, 8, 12]. Heath [3] defined a Nielsen type number for fibre preserving maps which are essentially fibre uniform, and exhibited connections between it and ordinary Nielsen numbers. Our aim in this paper is similar. We define a Nielsen type number of fibre preserving maps which are essentially $n$-orbit fibre uniform, using the Reducing Lemma in [8]. We show that a relation between this new Nielsen type number and the Nielsen type relative essential orbit number.

We consider a fibre preserving map $f : E \to E$ of a Hurewicz fibration $P : E \to B$ of compact ANR's. It induces a map $\bar{f} : B \to B$ and the restriction $f\,|_b : F_b \to F_{\bar{f}(b)}$ on fibres. Denote by $EO^{(n)}(f)$ the number of essential $n$-orbit classes of $f$. It is a lower bound for the number of $n$-orbits in the homotopy class. We define a Nielsen type number $EO^{(n)}_{\mathcal{F}}(f,p)$ of $f$ to be $EO^{(n)}(\bar{f})EO^{(m)}(h_b)$, where $d$ is the depth of the essential $\bar{f}$-orbit class containing $b$, $m = n/d$, and $h_b : F_b \to F_b$ is some homotopy of $f^d\,|_{F_b}$. Under suitable conditions, we have

$$EO^{(n)}_{\mathcal{F}}(f,p) = EO^{(n)}(h; E, p^{-1}(\xi)),$$

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where $EO^{(n)}(h; E, p^{-1}(\xi))$ is the Nielsen type relative essential $n$-orbit number of $h$, $\xi$ is a set of essential representatives for $\bar{h}$, and

$$EO_{\mathcal{F}}^{(n)}(f, p) = EO^{(n)}(f).$$

The paper consists of two sections. In the first section we review those definitions and results from [5, 8, 9, 10, 11] which are needed here. In the second section we define a Nielsen type number and obtain its properties.

For the basics of Nielsen fixed point theory, the reader is referred to [1, 7].

## 2. Review

Let $X$ be a compact connected ANR. Let $f : X \to X$ be a map. We denote by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of $f$. Two fixed points $x, y \in \mathrm{Fix}(f)$ are Nielsen related if there is a path $\lambda$ from $x$ to $y$ such that $f(\lambda)$ is homotopic to $\lambda$ by a homotopy keeping the end points fixed. This relation divides $\mathrm{Fix}(f)$ into a finite number of fixed point classes of $f$. The set of fixed point classes will be denoted by $\mathcal{FP}(f)$.

Let $n > 0$ be a given integer. Then $f$ acts on the set $\mathcal{FP}(f^n)$ of $n$-periodic point classes of $f$ by $\mathbf{F}_{f^n} \mapsto f(\mathbf{F}_{f^n})$. In [8], the $f$-orbit of a class $\mathbf{F}_{f^n}$ is called an $n$-orbit class, denoted by $\mathbf{F}_f^{(n)}$. The set of $n$-orbit classes is denoted by $\mathcal{O}^{(n)}(f)$. The length of the orbit $\mathbf{F}_f^{(n)}$ is the smallest integer $\ell > 0$ such that $\mathbf{F}_{f^n} = f^\ell(\mathbf{F}_{f^n})$. Standard fixed point index theory provides an integer index $\mathrm{ind}(\mathbf{F}_{f^n})$ for each periodic point class $\mathbf{F}_{f^n}$. A periodic point class $\mathbf{F}_{f^n}$ is essential if its index is nonzero. We let $\mathcal{E}(f^n)$ be the set of essential periodic point classes of $f$. Then the Nielsen number $N(f^n)$ of $f^n$ is the cardinality of $\mathcal{E}(f^n)$. We denote the set of essential $n$-orbit classes by $\mathcal{EO}^{(n)}(f)$. The essential $n$-orbit number $EO^{(n)}(f)$ is the the cardinality of the set $\mathcal{EO}^{(n)}(f)$.

Let $x$ be the base point in $X$, and take a path $w$ from $x$ to $f(x)$ as the base path for $f$. The induced endomorphism $f_*^w : \pi_1(X, x) \to \pi_1(X, x)$ is defined by $f_*^w(\langle\gamma\rangle) = \langle wf(\gamma)w^{-1}\rangle$ for any loop $\gamma$ at $x$. If $w$ is the constant path, $f_*^w$ will be denoted by $f_*^x$. For $n > 1$, we have $(f^n)_*^{w_n} = (f_*^w)^n$ if the base path for $f^n$ is taken to be $w_n = wf(w)\cdots f^{n-1}(w)$. For the sake of convenience, denote the induced endomorphism $f_*^w$ by $\varphi$, and $\pi_1(X, x) := \pi_X$.

Given $\varphi : \pi_X \to \pi_X$, we have the Reidemeister left action of $\pi_X$ on $\pi_X$, given by $\beta \cdot \alpha = \beta\alpha\varphi(\beta^{-1})$. The Reidemeister classes are the orbits of this action, and the set of Reidemeister classes is denoted by $\mathcal{R}(\varphi)$.

Let $n > 0$ be a given integer. Then $\varphi$ acts on the Reidemeister set $\mathcal{R}(\varphi^n)$ by $[\alpha]_{\varphi^n} \overset{\varphi}{\mapsto} [\varphi(\alpha)]_{\varphi^n}$. In [8], the $\varphi$-orbit of a Reidemeister class $[\alpha]_{\varphi^n}$ is called the Reidemeister $n$-orbit of $\varphi$, and denoted by $[\alpha]_\varphi^{(n)}$. The Reidemeister $n$-orbit set of $\varphi$ is the set of all such $\varphi$-orbits, denoted by $\mathcal{RO}^{(n)}(\varphi)$. The length of the orbit $[\alpha]_\varphi^{(n)}$ is the smallest integer $\ell > 0$ such that $[\alpha]_{\varphi^n} = [\varphi^\ell(\alpha)]_{\varphi^n}$.

For $m \mid n$, we have a commutative diagram of pointed sets:

$$\begin{array}{ccc} \mathcal{R}(\varphi^m) & \stackrel{\iota_{m,n}}{\longrightarrow} & \mathcal{R}(\varphi^n) \\ \downarrow & & \downarrow \\ \mathcal{RO}^{(m)}(\varphi) & \stackrel{\iota_{m,n}}{\longrightarrow} & \mathcal{RO}^{(n)}(\varphi), \end{array}$$

where the vertical maps are projections, and the horizontal maps are induced by the level-change function $\iota_{m,n} : \pi_X \to \pi_X$ defined by

$$\iota_{m,n}(\beta) := \beta\varphi^m(\beta)\varphi^{2m}(\beta)\cdots\varphi^{n-m}(\beta).$$

Recall that an $\varphi$-orbit $[\alpha]^{(n)}_\varphi \in \mathcal{RO}^{(n)}(\varphi)$ is reducible to level $h$ if there exists a $[\beta]^{(h)}_\varphi \in \mathcal{RO}^{(h)}(\varphi)$ such that $\iota_{h,n}([\beta]^{(h)}_\varphi) = [\alpha]^{(n)}_\varphi$. The lowest level $d = d([\alpha]^{(n)}_\varphi)$ to which $[\alpha]^{(n)}_\varphi$ reduces is its depth.

It is well known that every fixed point class of $f$ is assigned a Reidemeister class in $\mathcal{R}(\varphi)$, called its coordinate. We get an injection $\rho : \mathcal{FP}(f) \hookrightarrow \mathcal{R}(\varphi)$, defined by $\rho(\mathbf{A}_f) := [cf(c^{-1})w^{-1}]_\varphi$ for any path $c$ from $x_0$ to a point $x$ in $\mathbf{A}_f$. Thus we also get an injection $\rho : \mathcal{O}^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(\varphi)$, defined by $\rho(\mathbf{A}^{(n)}_f) := [cf^n(c^{-1})w_n^{-1}]^{(n)}_\varphi$ for any path $c$ from $x_0$ to a point $x$ in $\mathbf{A}^{(n)}_f$. If $\ell \mid n$ and an $\ell$-orbit class $\mathbf{B}^{(\ell)}_f$ lies inside an $n$-orbit class $\mathbf{A}^{(n)}_f$, then their coordinates are related by $\rho(\mathbf{A}^{(n)}_f) = \iota_{\ell,n}(\rho(\mathbf{B}^{(\ell)}_f))$, hence $\rho(\mathbf{A}^{(n)}_f)$ is reducible to level $\ell$. The *depth* of an $n$-orbit class $\mathbf{A}^{(n)}_f$ is defined to be the depth of its coordinate $\rho(\mathbf{A}^{(n)}_f)$.

The following lemma in [8] is very important to this paper.

**Reducing Lemma.** *Suppose $X$ is a compact connected ANR, and $f : X \to X$ is a map. Suppose $x \in Fix(f^n)$ lies in an $n$-orbit class $\mathbf{A}^{(n)}_f$ of depth $d$. Then there exists a homotopy $H = \{h_t : X \to X\}_{0\le t\le 1}$ connecting $f = h_0$ and $g = h_1$, such that*

(1) *$x \in Fix(g^d)$,*

(2) *The loop $H^n(x) = \{h^n_t(x)\}_{0\le t\le 1}$ is contractible in $X$,*

(3) *$H$ equals $f$ outside of an arbitrarily given neighborhood of the point $f^{d-1}(x)$.*

Note that in (2) the notation $H^n = \{h^n_t : X \to X\}$ stands for the homotopy from $f^n = h^n_0$ to $g^n = h^n_1$ consisting of $h^n_t$, the $n$-time iterate of $h_t : X \to X$. The loop $H^n(x) = \{h^n_t(x)\}_{0\le t\le 1}$ is the trace of the point $x$ under this homotopy.

Recall the relative Nielsen theory in [10]. If $f : (X, A) \to (X, A)$ is a selfmap of compact ANR's with $X$ path connected. We shall write $f_A : A \to A$ for the restriction of $f$. An essential fixed point class of $f : X \to X$ is called an essential common fixed point class of $f$ and $f_A$ if it contains an essential fixed point class of $f_A$. We let $N(f, f_A)$ be the number of essential common fixed point classes of $f$ and $f_A$. The relative Nielsen number $N(f; X, A)$ of

$f : (X, A) \to (X, A)$ is defined by

$$N(f; X, A) = N(f_A) + N(f) - N(f, f_A).$$

Nielsen type numbers for periodic points on pairs of spaces was introduced in [6]. Since $f$-images of essential common fixed point classes of $f^n$ and $f_A^n$ are essential common, the cardinality of the set $\mathcal{E}O^{(n)}(f, f_A)$ of essential common $n$-orbits of $f$ and $f_A$ is denoted by $EO^{(n)}(f, f_A)$ [9]. The Nielsen type relative essential $n$-orbit number of $f$ on the pair $(X, A)$ is defined by

$$EO^{(n)}(f; X, A) = EO^{(n)}(f_A) + EO^{(n)}(f) - EO^{(n)}(f, f_A),$$

where $EO^{(n)}(f_A) = \sum_{[j]} EO^{(m_j)}(f_k^{c(j)})$, the restriction $f_k^{c(j)}$ is a selfmap of a path component $A_k$ of $A$, and $c(j)$ is the length of $A_k$. The summation runs over all equivalence classes $[j] \in C(f_A)$, and $m_j := n/c(j)$ is an integer (see [5, 3.3]).

In this paper we will assume that all of our fibrations $F \hookrightarrow E \to B$ (with projection $p : E \to B$) are Hurewicz fibrations with typical fibre $E$ and $B$ path-connected (see [11]). We say that $f : E \to E$ is a fibre preserving map provided there is a well-defined map $\bar{f} : B \to B$ with $pf = \bar{f}p$. When such a map exists it is unique, and when $B$ is a path connected locally path connected space it is enough that for all $b \in B$ the restriction of $f$ takes the fibre $F_b := p^{-1}(b)$ to another fibre. We will refer to the pair $(f, \bar{f})$ as a fibre preserving map. For any $b \in \mathrm{Fix}(\bar{f}^n)$, we will denote the restricted map on $F_b$ by $f_b^n$. For $x \in E$ let $j : F_{p(x)} \to E$ be the inclusion. Similarly, a fibre preserving homotopy $(H, \bar{H}) : (f, \bar{f}) \simeq (g, \bar{g})$ is a pair of homotopies $H : E \times I \to E$ and $\bar{H} : B \times I \to B$, satisfying $pH = \bar{H}(p \times 1)$.

## 3. The Nielsen type number of a fibre preserving map

We recall from the definition of [3, 3.1] that a fibre preserving map $f$ is essentially fibre uniform if $N(f_b)$ is independent of any $b$ in any essential class of $\bar{f}$. We see from [7, IV.2.5] that $N(f_b)$ is independent of $b$ within a single fixed point class of $\bar{f}$. Then we can generalize the definition of essentially fibre uniform as follows. Applying the Reducing Lemma for the base map $\bar{f} : B \to B$, we have:

**Proposition 3.1.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map. If $d$ is the depth of the essential $n$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$ of $\bar{f}$ containing $b$, $1 \le i < \ell$, $\ell$ is the length of $\mathbf{F}_{\bar{f}}^{(n)}$, and $m = n/d$. If $g$ is the fibre preserving map from the Reducing Lemma, then we have*

$$EO^{(m)}(g_b^d) = EO^{(m)}(g^d_{\bar{f}^i(b)}).$$

*Proof.* Since the depth is a multiple of the length of the $n$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$. By the third property of the Reducing Lemma, we have $\bar{g}^i(b) = \bar{f}^i(b)$ and

$\bar{g}^{d-i}(\bar{f}^i(b)) = \bar{g}^d(b) = b$. Consider the following fibre maps

$$g_b^i : F_b \to F_{\bar{f}^i(b)} \quad \text{and} \quad g_{\bar{f}^i(b)}^{d-i} : F_{\bar{f}^i(b)} \to F_b.$$

Then the desired equality follows from the commutativity of the Nielsen type number [7, III.4.10]. □

We call a subset $\xi_n \subset \mathrm{Fix}(\bar{f}^n)$ a set of *essential $n$-orbit representatives* for $\bar{f}$ if $\xi_n$ contains exactly one point from each essential $n$-orbit class of $\mathcal{E}O^{(n)}(\bar{f})$. Proposition 3.1 shows that the number $EO^{(m)}(g_b^d)$ is independent of the $\bar{f}$-image of the $n$-periodic point class, which is in the essential $n$-orbit class containing $b$.

For each $b \in \xi_n$, let $d$ be the depth of the essential $\bar{f}$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$ containing $b$. Now $b$ and $\bar{f}^d(b)$ are in the same fixed point class of $\bar{f}^n$ (because the depth is always a multiple of the length, of the $\bar{f}$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$), but not necessarily $\bar{f}^d(b) = b$. By the Reducing Lemma, there exists a homotopy $\bar{H} = \{\bar{h}_t : B \to B\}_{t\in I}$ connecting $\bar{f} = \bar{h}_0$ to some $\bar{g} = \bar{h}_1$ such that $b \in \mathrm{Fix}(\bar{g}^d)$, and the $n$-orbit class of $\bar{f}$ containing $b$ corresponds to the $n$-orbit class of $\bar{g}$ containing $b$ because the trace $H^n(x)$ is a contractible loop. We can do this for all $b \in \xi$ simultaneously, because the $\bar{H}$ above only changes $\bar{f}$ in a small neighborhood of the $\bar{f}$-orbit of $b$. By the homotopy lifting property of the fibration $p$, the homotopy $\bar{H}$ in $B$ lifts to a fibre preserving homotopy $H = \{h_t : E \to E\}_{t\in I}$ connecting $f = h_0$ to some $g = h_1$ (see p. 153 of [8]).

**Definition.** Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map. Let $\xi_n = \{b_1, \ldots, b_k\}$ be a set of essential $n$-orbit representatives for $\bar{f}$. We say that $f$ is *essentially $n$-orbit fibre uniform* if $EO^{(m_i)}(g_{b_i}^{d_i})$ is independent of $b_i \in \xi_n$, where $g$ is the fibre preserving map from the Reducing Lemma, $d_i$ is the depth of the $n$-orbit class of $\bar{f}$ containing $b_i$, and $m_i = n/d_i$.

**Definition.** Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map which is essentially $n$-orbit fibre uniform. Then the Nielsen type number $EO_{\mathcal{F}}^{(n)}(f,p)$ of $f$ is defined by

$$EO_{\mathcal{F}}^{(n)}(f,p) = EO^{(n)}(\bar{f})EO^{(m)}(g_b^d),$$

where $g$ is the fibre preserving map from the Reducing Lemma, $d$ is the depth of the essential $n$-orbit class of $\bar{f}$ containing $b$, and $m = n/d$.

Let $(H, \bar{H}) : (f_0, \bar{f}_0) \simeq (f_1, \bar{f}_1)$ be a fibre preserving homotopy, and let $\xi_n$ be a set of essential $n$-orbit representatives for $\bar{f}_0$. Then the homotopy $\bar{H}^n : \bar{f}_0^n \simeq \bar{f}_1^n$ induces a bijection between the essential classes of $\bar{f}_0^n$ and $\bar{f}_1^n$. Suppose $b \in \xi_n$ lies in the essential $n$-periodic point class $\mathbf{F}_{\bar{f}_0^n}$, which in turn is in the essential $n$-orbit class $\mathbf{F}_{\bar{f}_0}^{(n)}$. Let $\mathbf{F}_{\bar{f}_1^n}$ be the essential $n$-periodic point class corresponding to $\mathbf{F}_{\bar{f}_0^n}$ via $\bar{H}^n$ [7, I.4.5]. For every $b_i \in \xi_n$, choose an

$n$-periodic point $b_i'$ in $\mathbf{F}_{\bar{f}_1^n}$, then $\eta_n = \{b_i' \mid b_i \in \xi_n\}$ is a set of essential $n$-orbit representatives for $\bar{f}_1$.

**Lemma 3.2.** *Let $f_0$ be a fibre preserving map, and let $(f_0, \bar{f}_0) \simeq (f_1, \bar{f}_1)$ be a fibre preserving homotopy with $\xi_n$ and $\eta_n$ as above. Let $g_0$ and $g_1$ be the fibre preserving maps from the Reducing Lemma for $f_0$ and $f_1$, respectively. Then*

$$EO^{(m_i)}((g_0)_{b_i}^{d_i}) = EO^{(m_i)}((g_1)_{b_i'}^{d_i})$$

*for all $(b_i, b_i') \in \xi_n \times \eta_n$, where $d_i$ is the depth of the $n$-orbit class of $\bar{f}_0$ containing $b_i$, and $m_i = n/d_i$.*

*Proof.* If $g_0$ is the fibre preserving map from the Reducing Lemma for $f_0$, then from the above remark of Theorem 2.4 in [8], $(f_0, \bar{f}_0)$ is fibre preserving homotopic to $(g_0, \bar{g}_0)$. Thus we have three fibre preserving homotopies $(f_0^n, \bar{f}_0^n) \simeq (f_1^n, \bar{f}_1^n)$, $(f_0^n, \bar{f}_0^n) \simeq (g_0^n, \bar{g}_0^n)$, and $(f_1^n, \bar{f}_1^n) \simeq (g_1^n, \bar{g}_1^n)$. Suppose $(H^n, \bar{H}^n) : (g_0^n, \bar{g}_0^n) \simeq (g_1^n, \bar{g}_1^n)$ is a fibre preserving homotopy. The homotopy $\bar{H}^n$ induces a bijection between the essential $n$-orbit classes of $\bar{g}_0$ and $\bar{g}_1$. Furthermore, this bijection preserves the depth of the essential $n$-orbit class of $\bar{g}_0$ [7, III.3.4]. Note that by the Reducing Lemma, $\xi_n$ is also a set of essential $n$-orbit representatives for $\bar{g}_0$. Let $d_i$ be the depth of the essential class $\mathbf{F}_{\bar{g}_0}^{(n)}$ containing $b_i \in \xi_n$. Then $\mathbf{F}_{\bar{g}_0}^{(n)}$ is corresponding to the essential $n$-orbit class $\mathbf{F}_{\bar{g}_1}^{(n)}$ containing $b_i' \in \eta_n$, so the depth of $\mathbf{F}_{\bar{g}_1}^{(n)}$ is $d_i$. We can assume that $\mathbf{F}_{\bar{g}_0^n}$ containing $b_i$ is corresponding to $\mathbf{F}_{\bar{g}_1^n}$ containing $b_i'$ via $\bar{H}^n$. In the proof of [3, 4.6], $N((g_0^n)_{b_i}) = N((g_1^n)_{b_i'})$. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{E}(((g_0)_{b_i}^{d_i})^{m_i}) & \xrightarrow{\bar{H}_*^n} & \mathcal{E}(((g_1)_{b_i'}^{d_i})^{m_i}) \\ \downarrow & & \downarrow \\ \mathcal{E}O^{(m_i)}((g_0)_{b_i}^{d_i}) & \xrightarrow{\bar{H}_*^n} & \mathcal{E}O^{(m_i)}((g_1)_{b_i'}^{d_i}), \end{array}$$

where the vertical maps are projections. This completes the proof of the lemma. □

**Corollary 3.3** (Homotopy Invariance)**.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres. If $f_0$ is a fibre preserving map which is essentially $n$-orbit fibre uniform, and $(f_0, \bar{f}_0) \simeq (f_1, \bar{f}_1)$ is a fibre preserving homotopy, then $EO_{\mathcal{F}}^{(n)}(f_0, p) = EO_{\mathcal{F}}^{(n)}(f_1, p)$.*

*Proof.* This is trivial from the lemma, definitions, and the homotopy invariance of the Nielsen type number. □

Consider a subset $\xi = \{b_i, \bar{g}(b_i), \dots, \bar{g}^{d_i-1}(b_i) \mid b_i \in \xi_n\} \subset \mathrm{Fix}(\bar{g}^n)$, where $d_i$ is the depth of the $n$-orbit class of $\bar{f}$ containing $b_i$. By the Reducing Lemma, $d_i$ is the length of $b_i$ with respect to $g$ for every $b_i \in \xi_n$. We call the subset $\xi$ the *reduced $\bar{g}$-invariant set* with respect to $\xi_n$ [9]. Since $g$ is a fibre preserving map, then $p^{-1}(\xi)$ is a $g$-invariant subspace of $E$. We write $g_\xi = g\,|_{p^{-1}(\xi)}\colon p^{-1}(\xi) \to p^{-1}(\xi)$ for the restriction of $g$ to $p^{-1}(\xi)$.

**Proposition 3.4.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map. If $g$ is a fibre preserving map from the Reducing Lemma, and $\xi_n$ is a set of essential $n$-orbit representatives for $\bar{f}$, then*

$$EO^{(n)}(g_\xi) = \sum_{b_i \in \xi_n} EO^{(m_i)}(g_{b_i}^{d_i}),$$

*where $\xi$ is the reduced $\bar{g}$-invariant set with respect to $\xi_n$.*

*Proof.* If $\xi_n$ is a set of essential $n$-orbit representatives for $\bar{f}$, then by the Reducing Lemma, for every $b_i \in \xi_n$, the set

$$\{p^{-1}(b_i), p^{-1}(\bar{g}(b_i)), \ldots, p^{-1}(\bar{g}^{d_i-1}(b_i))\}$$

is a $g_\xi$-cycle in $p^{-1}(\xi)$ (see [5, 3.3]). By the definition of the Nielsen type relative essential $n$-orbit number of $g$ on the pair $(E, p^{-1}(\xi))$, we have the assertion. □

**Theorem 3.5.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map which is essentially $n$-orbit fibre uniform. Let $\xi_n$ be a set of essential $n$-orbit representatives for $\bar{f}$. Then $EO_{\mathcal{F}}^{(n)}(f,p)$ is a lower bound for the number of $n$-orbits in the homotopy class. Furthermore,*

$$EO_{\mathcal{F}}^{(n)}(f,p) = EO^{(n)}(g_\xi) \le EO^{(n)}(g; E, p^{-1}(\xi)) \le EO^{(n)}(g),$$

*where $g$ is a fibre preserving map from the Reducing Lemma, and $\xi$ is the reduced $\bar{g}$-invariant set with respect to $\xi_n$.*

*Proof.* When $f$ is essentially $n$-orbit fibre uniform, by Definitions and Proposition 3.4, we have the desired equality. The inequalities follow from the definition of the Nielsen type relative essential $n$-orbit number of $f$ on the pair $(E, p^{-1}(\xi))$. □

**Theorem 3.6.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map which is essentially $n$-orbit fibre uniform. Let $\xi_n$ be a set of essential $n$-orbit representatives for $\bar{f}$. If $N(f^n) \neq 0$, then*

$$EO_{\mathcal{F}}^{(n)}(f,p) = EO^{(n)}(g; E, p^{-1}(\xi)),$$

*where $g$ is a fibre preserving map from the Reducing Lemma, and $\xi$ is the reduced $\bar{g}$-invariant set with respect to $\xi_n$.*

*Proof.* When $N(g^n) = N(f^n) \neq 0$, by [3, 4.4.(ii)] and [12, 4.1], any essential fixed point class of $g^n$ is an essential common fixed point class of $g^n$ and $g_\xi^n$. Since $g$-images of essential common fixed point classes of $g^n$ and $g_\xi^n$ are essential common, and so

$$EO^{(n)}(g; E, p^{-1}(\xi)) = EO^{(n)}(g_\xi).$$

By Theorem 3.5, we have the conclusion. □

**Theorem 3.7.** *Let $p : E \to B$ be a fibration of compact connected ANR's with path connected fibres, and let $f$ be a fibre preserving map which is essentially $n$-orbit fibre uniform. Let $\xi_n$ be a set of essential $n$-orbit representatives for $\bar{f}$. If the homomorphism $j_* : \pi_1(F_{b_i}, x) \to \pi_1(E, x)$ induced by the inclusion of the fibre is injective, and if $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$ for every $b_i \in \xi_n$, then*

$$EO_{\mathcal{F}}^{(n)}(f, p) = EO^{(n)}(f).$$

*Proof.* The kernel of the homomorphism $j_* : \pi_1(F_{b_i}, x) \to \pi_1(E, x)$ induced by the inclusion of the fibre is trivial. When $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$, by [8, 2.4] we have

$$EO^{(n)}(f) = \sum_{b_i \in \xi_n} EO^{(m_i)}(g_{b_i}^{d_i}),$$

where $g$ is the fibre preserving map from the Reducing Lemma, $d_i$ is the depth of the $n$-orbit class of $\bar{f}$ containing $b_i$, and $m_i = n/d_i$. Since $f$ is essentially $n$-orbit fibre uniform, we get the desired equality. □

The principal application is to fibrations over tori. It should be useful in calculations on nil and solvmanifolds.

**Corollary 3.8.** *Suppose $p : E \to B$ is a fibration over a torus. Let $f$ be a fibre preserving map which is essentially $n$-orbit fibre uniform. Let $\xi_n$ be a set of essential $n$-orbit representatives for $\bar{f}$. If the homomorphism $j_* : \pi_1(F_{b_i}, x) \to \pi_1(E, x)$ induced by the inclusion of the fibre is injective for every $b_i \in \xi_n$, then*

$$EO_{\mathcal{F}}^{(n)}(f, p) = EO^{(n)}(f).$$

*Proof.* See [8, 2.6]. □

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Department of mathematics
Mokwon University
Daejeon 302-729, Korea
*E-mail address*: seoungho@mokwon.ac.kr