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A HYBRID METHOD FOR A COUNTABLE FAMILY OF LIPSCHITZ GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS AND AN EQUILIBRIUM PROBLEM

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the fixed points set of a countable family of uniformly Lipschitzian generalized asymptotically quasi-nonexpansive mappings and the solutions set of equilibrium problems. Some strong convergence theorems of the proposed iterative scheme are established by using the concept of W-mappings of a countable family of uniformly Lipschitzian generalized asymptotically quasi-nonexpansive mappings.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E and let $T : C \to C$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. Then T is said to be

(i) *nonexpansive if* $\|Tx - Ty\| \leq \|x - y\|$ *for all* $x, y \in C$;

(ii) *asymptotically nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$;

(iii) *asymptotically quasi-nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in C$, $p \in F(T)$ and $n \in \mathbb{N}$;

(iv) *generalized asymptotically nonexpansive* [24] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + c_n$$

for all $x, y \in C$ and $n \in \mathbb{N}$;

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(v) *generalized asymptotically quasi-nonexpansive* [24] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| + c_n$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$;

(vi) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

It is clear that a class of generalized asymptotically quasi-nonexpansive mappings includes as special cases the class of generalized asymptotically nonexpansive mappings, the class of asymptotically nonexpansive mappings, and the class of nonexpansive mappings. However, the converse of each of above statements may be not true (see [24]).

In 1953, Mann [13] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where the initial point x_1 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1].

However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see [1, 10, 22].

Question: How can we modify Mann's iteration process to obtain strong convergence theorems?

In 2003, Nakajo and Takahashi [18] introduced a modification of Mann iteration which is called *CQ method* for a single nonexpansive mapping T in a Hilbert space. They proved the following theorem:

Theorem 1.1. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following manner*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.2}$$

Then $\{x_n\}$ defined by (1.2) *converges strongly to $P_{F(T)} x_0$.*

In 2006, Kim and Xu [12] extended the result of Nakajo and Takahashi [18] from a single nonexpansive mapping to a single asymptotically nonexpansive mapping in a Hilbert space. They proved the following theorem:

Theorem 1.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following manner*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam}C)^2 \to 0$. Then $\{x_n\}$ defined by (1.3) converges strongly to $P_{F(T)}x_0$.

In 2008, Takahashi et al. [28] proved the following theorem by using a hybrid method which is different from Nakajo and Takahashi's hybrid method. Such a method is called *the shrinking projection method*.

Theorem 1.3 ([28])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows*:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Since 2003, to obtain strong convergence of nonlinear mappings, the CQ method and the shrinking projection method have been studied by many authors (see [12, 14, 15, 16, 17, 18, 19, 20, 23, 25, 26, 29, 31, 32]).

Let $F : C\times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for a bifunction F is to find a point $x^* \in C$ such that

$$F(x^*, y) \ge 0, \quad \forall y \in C. \tag{1.5}$$

We denote the solutions set of (1.5) by $EP(F)$.

Question. How can we construct iteration process to obtain strong convergence theorems for finding a common element of the solutions set of an equilibrium problem and the fixed points set?

Recently, Tada and Takahashi [26] proposed a new iteration for finding a common element of the solutions set of an equilibrium problem and the fixed points set of a nonexpansive mapping T in a Hilbert space H and then proved the following theorem:

Theorem 1.4 ([26]). *Let H be a Hilbert space and C a closed convex subset of H. Let $F : C \times C \to R$ be a bifunction and $T : C \to C$ a nonexpansive mapping such that $F(T) \cap EP(F) \neq \emptyset$. For an initial point $x_1 = x \in C$, let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tu_n, \\ C_n = \Big\{z \in C : \|y_n - z\| \leq \|x_n - z\|\Big\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{F(T)\cap EP(F)}x$.

In the recent years, the problem of finding a common element of the solutions set of equilibrium problems and the fixed points set in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors (see, for example, [2, 3, 5, 30, 4, 6, 7, 9, 11, 21, 27] and the references cited therein).

Let $\{T_n\}$ be a family of asymptotically nonexpansive mappings of C into itself and let $\alpha_{i,n} \in [0,1]$ for all $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$.

In 2008, Nakajo et al. [17] introduced a new concept of the W-mapping as follows:

$$(1.6) \qquad \begin{cases} U_{n,n} = \alpha_{n,n}T_n^n + (1-\alpha_{n,n})I \\ U_{n-1,n} = \alpha_{n-1,n}T_{n-1}^n U_{n,n} + (1-\alpha_{n-1,n})I \\ \vdots \\ U_{k,n} = \alpha_{k,n}T_k^n U_{k+1,n} + (1-\alpha_{k,n})I \\ \vdots \\ W_n = U_{1,n} = \alpha_{1,n}T_1^n U_{2,n} + (1-\alpha_{1,n})I. \end{cases}$$

Such a mapping W is called the *W-mapping* generated by $T_n, \ldots, T_2, T_1$ and $\alpha_{n,n}, \ldots, \alpha_{2,n}, \alpha_{1,n}$.

Motivated by Nakajo et al. [17], we introduce a new approximation scheme for finding a common element of the fixed points set of a countable family of Lipschitz generalized asymptotically quasi-nonexpansive mappings and the solutions set of an equilibrium problem. Using a shrinking projection technique, strong convergence theorems are also established.

2. Preliminaries

In this section, we give some useful lemmas and definitions for proving our main theorem.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C.

Lemma 2.1 ([14]). *Let C be a closed and convex subset of a real Hilbert space H and P_C the metric projection from H onto C. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if $\langle x - z, y - z \rangle \le 0$ for all $y \in C$.*

Lemma 2.2 ([18]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $P_C : H \to C$ the metric projection from H onto C. Then the following holds*:

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2, \quad \forall x \in H, \ \forall y \in C.$$

Lemma 2.3 ([14]). *Let H be a real Hilbert space. Then the followings hold*:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \ \forall x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \ \forall t \in [0,1]$ *and* $x, y \in H$.

Lemma 2.4 ([25]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \ \|y - v\|^2 \le \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

For solving the equilibrium problem, let us give the following assumptions for F and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x, y)$;

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.5 ([2]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(x, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.6 ([9]). *For $r > 0$, $x \in H$, defined the mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \Big\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C \Big\}.$$

Then the followings hold:

(1) *T_r is single-value*;

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle;$$

(3) $F(T_r) = EP(F)$;

(4) *$EP(F)$ is closed and convex.*

Lemma 2.7 ([29]). *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - \omega_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r(0) = \{x \in E : \|x\| \le r\}$ and $\lambda \in [0,1]$ where $\omega_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 2.8 ([8]). *Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a uniformly L-Lipschitzian. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Lemma 2.9 ([8]). *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mapping such that $F(T) \ne \emptyset$. Then $F(T)$ is a closed and convex subset of C.*

3. Results for families of mappings

In this section, we prove some lemmas concerning the W-mapping in a real Banach space.

Lemma 3.1. *Let C be a nonempty convex subset of a real Banach space E. Let $\{T_i\}_{i=1}^{\infty}$ be a uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive mapping of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ with $k_{i,n} \ge 1$ and $c_{i,n} \ge 0$ for all $i, n \in \mathbb{N}$ such that $\lim_{n\to\infty} k_{i,n} = 1$, $\lim_{n\to\infty} c_{i,n} = 0$ and let W_n be the W-mapping defined by (1.6). Then the followings hold:*

(i) *There exist $\{\gamma_{1,n}\} \subset [1,\infty)$ and $\{\varepsilon_{1,n}\} \subset [0,\infty)$ such that $\|W_n x - p\| \le \gamma_{1,n}\|x - p\| + \varepsilon_{1,n}$ for all $x \in C$, $p \in \bigcap_{i=1}^{\infty} F(T_i)$ and all $n \in \mathbb{N}$.*

(ii) *There exists a constant $L^* > 0$ such that*

$$\|W_n x - W_n y\| \le L^*\|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(iii) *If $0 < \ell \le \alpha_{i,n} \le k < 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$ for some ℓ, k with $0 < \ell < k < 1$ and $\lim_{n\to\infty}\gamma_{1,n} = 1$, $\lim_{n\to\infty}\varepsilon_{1,n} = 0$, then for each bounded sequence $\{x_n\}$ in C, we have*

$$\begin{aligned} &\lim_{n\to\infty}\|x_{n+1} - x_n\| \\ &= \lim_{n\to\infty}\|W_n x_n - x_n\| = 0 \Longrightarrow \lim_{n\to\infty}\|T_i x_n - x_n\| = 0 \quad \forall i = 1, 2, \dots, n. \end{aligned}$$

(iv) *If $0 < \ell \le \alpha_{i,n} \le k < 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$ for some ℓ, k with $0 < \ell < k < 1$ and $\lim_{n\to\infty}\gamma_{1,n} = 1$, $\lim_{n\to\infty}\varepsilon_{1,n} = 0$, then $\bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{n=1}^{\infty} F(W_n)$.*

Proof. (i) Let $n \in \mathbb{N}$, $x \in C$ and $p \in \bigcap_{i=1}^{\infty} F(T_i)$. We have for each $i = 1, 2, \ldots, n$ that

$$
\begin{aligned}
&\|W_n x - p\| = \|U_{1,n} x - p\| \\
&= \|\alpha_{1,n} T_1^n U_{2,n} x + (1 - \alpha_{1,n}) x - p\| \\
&\le \alpha_{1,n} \|T_1^n U_{2,n} x - p\| + (1 - \alpha_{1,n}) \|x - p\| \\
&\le \alpha_{1,n} k_{1,n} \|U_{2,n} x - p\| + \alpha_{1,n} c_{1,n} + (1 - \alpha_{1,n}) \|x - p\| \\
&\le \alpha_{1,n} k_{1,n} \|\alpha_{2,n} T_2^n U_{3,n} x + (1 - \alpha_{2,n}) x - p\| + \alpha_{1,n} c_{1,n} \\
&\quad + (1 - \alpha_{1,n}) \|x - p\| \\
&\le \alpha_{1,n} k_{1,n} \alpha_{2,n} \|T_2^n U_{3,n} x - p\| + \alpha_{1,n} k_{1,n} (1 - \alpha_{2,n}) \|x - p\| \\
&\quad + \alpha_{1,n} c_{1,n} + (1 - \alpha_{1,n}) \|x - p\| \\
&\le \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \|U_{3,n} x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} (1 - \alpha_{2,n}) + (1 - \alpha_{1,n})\big) \|x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} c_{2,n} + \alpha_{1,n} c_{1,n}\big) \\
&\ \vdots \\
&\le \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} k_{n-1,n} \|U_{n,n} x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} k_{n-2,n} (1 - \alpha_{n-1,n}) \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-3,n} k_{n-3,n} (1 - \alpha_{n-2,n}) \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} (1 - \alpha_{2,n}) + (1 - \alpha_{1,n})\big) \|x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} c_{n-1,n} \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} c_{n-2,n} \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} \alpha_{2,n} c_{2,n} + \alpha_{1,n} c_{1,n}\big) \\
&= \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} k_{n-1,n} \|\alpha_{n,n} T_n^n x + (1 - \alpha_{n,n}) x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} k_{n-2,n} (1 - \alpha_{n-1,n}) \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-3,n} k_{n-3,n} (1 - \alpha_{n-2,n}) \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} (1 - \alpha_{2,n}) + (1 - \alpha_{1,n})\big) \|x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} c_{n-1,n} \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} c_{n-2,n} \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} \alpha_{2,n} c_{2,n} + \alpha_{1,n} c_{1,n}\big) \\
&\le \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} k_{n-1,n} \alpha_{n,n} \|T_n^n x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} k_{n-1,n} (1 - \alpha_{n,n}) \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} k_{n-2,n} (1 - \alpha_{n-1,n}) \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} (1 - \alpha_{2,n}) + (1 - \alpha_{1,n})\big) \|x - p\| \\
&\quad + \big(\alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-1,n} c_{n-1,n} \\
&\quad + \alpha_{1,n} k_{1,n} \alpha_{2,n} k_{2,n} \cdots \alpha_{n-2,n} c_{n-2,n} \\
&\quad + \cdots + \alpha_{1,n} k_{1,n} \alpha_{2,n} c_{2,n} + \alpha_{1,n} c_{1,n}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \big(\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}k_{n,n}\\
&\quad+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}k_{n-1,n}(1-\alpha_{n,n})\\
&\quad+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-2,n}k_{n-2,n}(1-\alpha_{n-1,n})\\
&\quad+\cdots+\alpha_{1,n}k_{1,n}(1-\alpha_{2,n})+(1-\alpha_{1,n})\big)\|x-p\|\\
&\quad+\big(\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}c_{n,n}+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}c_{n-1,n}\\
&\quad+\cdots+\alpha_{1,n}k_{1,n}\alpha_{2,n}c_{2,n}+\alpha_{1,n}c_{1,n}\big)\\
&=\big(\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}(k_{n,n}-1)\\
&\quad+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}(k_{n-1,n}-1)\\
&\quad+\cdots+\alpha_{1,n}(k_{1,n}-1)+1\big)\|x-p\|\\
&\quad+\big(\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}c_{n,n}+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}c_{n-1,n}\\
&\quad+\cdots+\alpha_{1,n}k_{1,n}\alpha_{2,n}c_{2,n}+\alpha_{1,n}c_{1,n}\big).
\end{aligned}
$$

Put

$$
\begin{aligned}
\gamma_{1,n} &= \alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}(k_{n,n}-1)\\
&\quad+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}(k_{n-1,n}-1)\\
&\quad+\cdots+\alpha_{1,n}(k_{1,n}-1)+1
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\varepsilon_{1,n} &= \alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n,n}c_{n,n}\\
&\quad+\alpha_{1,n}k_{1,n}\alpha_{2,n}k_{2,n}\cdots\alpha_{n-1,n}c_{n-1,n}\\
&\quad+\cdots+\alpha_{1,n}k_{1,n}\alpha_{2,n}c_{2,n}+\alpha_{1,n}c_{1,n}.
\end{aligned}
\tag{3.2}
$$

Then $\gamma_{1,n}\geq 1$, $\varepsilon_{1,n}\geq 0$ and $\|W_nx-p\|\leq\gamma_{1,n}\|x-p\|+\varepsilon_{1,n}$.

(ii) Let $n\in\mathbb{N}$ and $x,y\in C$, we have

$$
\begin{aligned}
\|W_nx-W_ny\| &\leq \alpha_{1,n}\|T_1^nU_{2,n}x-T_1^nU_{2,n}y\|+(1-\alpha_{1,n})\|x-y\|\\
&\leq \alpha_{1,n}L_1\|U_{2,n}x-U_{2,n}y\|+(1-\alpha_{1,n})\|x-y\|\\
&\leq \alpha_{1,n}L_1\alpha_{2,n}\|T_2^nU_{3,n}x-T_2^nU_{3,n}y\|\\
&\quad+\big(\alpha_{1,n}L_1(1-\alpha_{2,n})+(1-\alpha_{1,n})\big)\|x-y\|\\
&\leq \alpha_{1,n}L_1\alpha_{2,n}L_2\|U_{3,n}x-U_{3,n}y\|\\
&\quad+\big(\alpha_{1,n}L_1(1-\alpha_{2,n})+(1-\alpha_{1,n})\big)\|x-y\|\\
&\ \ \vdots\\
&\leq \alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-1,n}L_{n-1}\|U_{n,n}x-U_{n,n}y\|\\
&\quad+\big(\alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-2,n}L_{n-2}(1-\alpha_{n-1,n})\\
&\quad+\alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-3,n}L_{n-3}(1-\alpha_{n-2,n})\\
&\quad+\cdots+\alpha_{1,n}L_1(1-\alpha_{2,n})+(1-\alpha_{1,n})\big)\|x-y\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_{1,n}L_1\cdots\alpha_{n-1,n}L_{n-1}\big(\alpha_{n,n}\|T_n^n x - T_n^n y\| + (1-\alpha_{n,n})\|x-y\|\big)\\
&\quad + \big(\alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-2,n}L_{n-2}(1-\alpha_{n-1,n})\\
&\quad + \alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-3,n}L_{n-3}(1-\alpha_{n-2,n})\\
&\quad + \cdots + \alpha_{1,n}L_1(1-\alpha_{2,n}) + (1-\alpha_{1,n})\big)\|x-y\|\\
&\leq \big(\alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n,n}L_n\\
&\quad + \alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-1,n}L_{n-1}(1-\alpha_{n,n})\\
&\quad + \alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-2,n}L_{n-2}(1-\alpha_{n-1,n})\\
&\quad + \alpha_{1,n}L_1\alpha_{2,n}L_2\cdots\alpha_{n-3,n}L_{n-3}(1-\alpha_{n-2,n})\\
&\quad + \cdots + \alpha_{1,n}L_1(1-\alpha_{2,n}) + (1-\alpha_{1,n})\big)\|x-y\|.\\
&\leq \big(L_1L_2\cdots L_n + L_1L_2\cdots L_{n-1}\\
&\quad + L_1L_2\cdots L_{n-2} + L_1L_2\cdots L_{n-3} + \cdots + L_1 + 1\big)\|x-y\|.
\end{aligned}$$

Put

$$\begin{aligned}
L^* \;=\;& L_1L_2\cdots L_n + L_1L_2\cdots L_{n-1} + L_1L_2\cdots L_{n-2} + L_1L_2\cdots L_{n-3}\\
&+\; \cdots + L_1 + 1.
\end{aligned}$$

It is easy to see that $L^* > 0$. Hence (ii) is proved.

(iii) Let $\{x_n\} \subset C$ be a bounded sequence and $\lim_{n\to\infty}\|W_nx_n - x_n\| = 0$. First, as $n \to \infty$, we observe

$$\|x_n - T_1^nU_{2,n}x_n\| = \frac{1}{\alpha_{1,n}}\|W_nx_n - x_n\| \leq \frac{1}{\ell}\|W_nx_n - x_n\| \to 0.$$

Let $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Put $M = \sup_{n\in N}\{\|x_n - T_1^nU_{2,n}x_n\| + 2\|T_1^nU_{2,n}x_n - p\|\}$ and $r = \sup_{n\in\mathbb{N}}\{\|T_2^nU_{3,n}x_n - p\|\} + \sup_{n\in\mathbb{N}}\{\|x_n - p\|\}$. By Lemma 2.7, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\|x_n - p\|^2 &\leq (\|x_n - T_1^nU_{2,n}x_n\| + \|T_1^nU_{2,n}x_n - p\|)^2\\
&= \|x_n - T_1^nU_{2,n}x_n\|(\|x_n - T_1^nU_{2,n}x_n\| + 2\|T_1^nU_{2,n}x_n - p\|)\\
&\quad + \|T_1^nU_{2,n}x_n - p\|^2\\
&\leq M\|x_n - T_1^nU_{2,n}x_n\| + \|T_1^nU_{2,n}x_n - p\|^2\\
&\leq M\|x_n - T_1^nU_{2,n}x_n\| + k_{1,n}^2\|U_{2,n}x_n - p\|^2\\
&\quad + 2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2\\
&\leq M\|x_n - T_1^nU_{2,n}x_n\| + k_{1,n}^2\|\alpha_{2,n}T_2^nU_{3,n}x_n + (1-\alpha_{2,n})x_n - p\|^2\\
&\quad + 2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2
\end{aligned}$$

$$\begin{aligned}
&\le M\|x_n - T_1^n U_{2,n}x_n\| + k_{1,n}^2\alpha_{2,n}\|T_2^n U_{3,n}x_n - p\|^2 \\
&\quad + k_{1,n}^2(1-\alpha_{2,n})\|x_n - p\|^2 + 2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2 \\
&\quad - k_{1,n}^2\alpha_{2,n}(1-\alpha_{2,n})g(\|T_2^n U_{3,n}x_n - x_n\|) \\
&\le M\|x_n - T_1^n U_{2,n}x_n\| + k_{1,n}^2\alpha_{2,n}k_{2,n}^2\|U_{3,n}x_n - p\|^2 \\
&\quad + 2k_{1,n}^2\alpha_{2,n}k_{2,n}c_{2,n}\|U_{3,n}x_n - p\| + k_{1,n}^2\alpha_{2,n}c_{2,n}^2 \\
&\quad + k_{1,n}^2(1-\alpha_{2,n})\|x_n - p\|^2 + 2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2 \\
&\quad - k_{1,n}^2\alpha_{2,n}(1-\alpha_{2,n})g(\|T_2^n U_{3,n}x_n - x_n\|) \\
&\le M\|x_n - T_1^n U_{2,n}x_n\| + k_{1,n}^2\alpha_{2,n}k_{2,n}^2\gamma_{3,n}^2\|x_n - p\|^2 \\
&\quad + 2k_{1,n}^2\alpha_{2,n}k_{2,n}^2\gamma_{3,n}\varepsilon_{3,n}\|x_n - p\| + k_{1,n}^2\alpha_{2,n}k_{2,n}^2\varepsilon_{3,n}^2 \\
&\quad + 2k_{1,n}^2\alpha_{2,n}k_{2,n}c_{2,n}\|U_{3,n}x_n - p\| + k_{1,n}^2\alpha_{2,n}c_{2,n}^2 \\
&\quad + k_{1,n}^2(1-\alpha_{2,n})\|x_n - p\|^2 + 2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2 \\
&\quad - k_{1,n}^2\alpha_{2,n}(1-\alpha_{2,n})g(\|T_2^n U_{3,n}x_n - x_n\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
& k_{1,n}^2\alpha_{2,n}(1-\alpha_{2,n})g(\|T_2^n U_{3,n}x_n - x_n\|) \\
\le\ & M\|x_n - T_1^n U_{2,n}x_n\| \\
& +2k_{1,n}^2\alpha_{2,n}k_{2,n}^2\gamma_{3,n}\varepsilon_{3,n}\|x_n - p\| + k_{1,n}^2\alpha_{2,n}k_{2,n}^2\varepsilon_{3,n}^2 \\
& +2k_{1,n}^2\alpha_{2,n}k_{2,n}c_{2,n}\|U_{3,n}x_n - p\| + k_{1,n}^2\alpha_{2,n}c_{2,n}^2 \\
& +2k_{1,n}c_{1,n}\|U_{2,n}x_n - p\| + c_{1,n}^2 \\
& +\big(k_{1,n}^2\alpha_{2,n}k_{2,n}^2\gamma_{3,n}^2 + k_{1,n}^2(1-\alpha_{2,n}) - 1\big)\|x_n - p\|^2.
\end{aligned} \tag{3.3}$$

Since $\lim_{n\to\infty}\|x_n - T_1^n U_{2,n}x_n\| = 0$, $\lim_{n\to\infty}\gamma_{3,n} = 1$, $\lim_{n\to\infty}\varepsilon_{3,n} = 0$ and $0 < \ell \le \alpha_{2,n} \le k < 1$, we obtain that $\lim_{n\to\infty} g(\|T_2^n U_{3,n}x_n - x_n\|) = 0$. By the property of g, it follows from (3.3) that $\lim_{n\to\infty}\|T_2^n U_{3,n}x_n - x_n\| = 0$. Similarly, we can show that

$$\lim_{n\to\infty}\|T_i^n U_{i+1,n}x_n - x_n\| = 0 \tag{3.4}$$

for all $i \in \mathbb{N}$. This implies for each $i = 1, 2, \ldots, n$ that

$$\begin{aligned}
\|U_{i,n}x_n - x_n\| &= \|\alpha_{i,n}T_i^n U_{i+1,n}x_n + (1-\alpha_{i,n})x_n - x_n\| \\
&= \alpha_{i,n}\|T_i^n U_{i+1,n}x_n - x_n\| \to 0
\end{aligned} \tag{3.5}$$

as $n \to \infty$. We observe that

$$\begin{aligned}
\|x_n - T_i^n x_n\| &\le \|x_n - T_i^n U_{i+1,n}x_n\| + \|T_i^n U_{i+1,n}x_n - T_i^n x_n\| \\
&\le \|x_n - T_i^n U_{i+1,n}x_n\| + L_i\|U_{i+1,n}x_n - x_n\|
\end{aligned} \tag{3.6}$$

for all $i = 1, 2, \ldots, n$. From (3.4), (3.5) and (3.6), we have

$$\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0.$$

Lemma 2.8 yields that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$.

(iv) It is easy to see that $\bigcap_{i=1}^{\infty} F(T_i) \subset \bigcap_{n=1}^{\infty} F(W_n)$. Let $z \in \bigcap_{n=1}^{\infty} F(W_n)$. Then $\|W_n z - z\| = 0$ for all $n \in \mathbb{N}$. By (iii), we have $z = T_i z$ for all $i = 1, 2, \ldots, n$. So we have $\bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{n=1}^{\infty} F(W_n)$. □

4. Strong convergence theorems

In this section, we prove strong convergence theorems for finding a common element of the solutions set of an equilibrium problem and the fixed points set of a countable family of generalized asymptotically quasi-nonexpansive and uniformly Lipschitzian mappings.

Theorem 4.1. *Let C be a closed and convex subset of a real Hilbert space H, let $F : C \times C \to R$ be a bifunction satisfying* (A1) − (A4) *and let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly L_i-Lipschitzian and generalized quasi-nonexpansive mappings of C into itself. Assume that $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \neq \emptyset$ and $\{\beta_n\} \subset [0, 1)$ such that $\limsup_{n\to\infty} \beta_n < 1$. Let W_n be the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_{n,n}, \alpha_{n-1,n}, \ldots, \alpha_{1,n}$ with $0 < \ell \leq \alpha_{i,n} \leq k < 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. Let $\{\gamma_{1,n}\} \subset [1, \infty)$, $\{\varepsilon_{1,n}\} \subset [0, \infty)$ be as in Lemma 3.1*(i) *with $\lim_{n\to\infty} \gamma_{1,n} = 1$ and $\lim_{n\to\infty} \varepsilon_{1,n} = 0$. For an initial point $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n) W_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \\ \qquad \leq \|x_n - z\|^2 - \beta_n(1 - \beta_n)\|W_n x_n - x_n\|^2 + (1 - \beta_n)\theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\theta_n = (\gamma_{1,n}^2 - 1)\nabla_n^2 + 2\gamma_{1,n}\varepsilon_{1,n}\nabla_n + \varepsilon_{1,n}^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in \Omega\} < \infty$. Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_0$.

Proof. We split the proof into six steps.

Step 1. Show that $\Omega \subset C_n$ for all $n \geq 1$.

By Lemma 2.9, we know that $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex. Moreover, by Lemma 2.4, C_n is also closed and convex for each $n \in \mathbb{N}$. $\Omega \subset C_1 = C$ is obviuos. Suppose that $\Omega \subset C_k$ for $k \geq 1$. For each $p \in \Omega$, by Lemma 3.1(i), we see that

$$\begin{aligned} \|y_k - p\|^2 &= \|\beta_k x_k + (1 - \beta_k) W_k x_k - p\|^2 \\ &= \beta_k\|x_k - p\|^2 + (1 - \beta_k)\|W_k x_k - p\|^2 - \beta_k(1 - \beta_k)\|W_k x_k - x_k\|^2 \\ &\leq \beta_k\|x_k - p\|^2 + (1 - \beta_k)\big(\gamma_{1,k}^2\|x_k - p\|^2 + 2\gamma_{1,k}\varepsilon_{1,k}\|x_k - p\| + \varepsilon_{1,k}^2\big) \\ &\quad - \beta_k(1 - \beta_k)\|W_k x_k - x_k\|^2 \\ &= \|x_k - p\|^2 - \beta_k(1 - \beta_k)\|W_k x_k - x_k\|^2 \\ &\quad + (1 - \beta_k)\big((\gamma_{1,k}^2 - 1)\|x_k - p\|^2 + 2\gamma_{1,k}\varepsilon_{1,k}\|x_k - p\| + \varepsilon_{1,k}^2\big) \\ &\leq \|x_k - p\|^2 - \beta_k(1 - \beta_k)\|W_k x_k - x_k\|^2 + (1 - \beta_k)\theta_k. \end{aligned}$$

So we have $p \in C_{k+1}$. By induction, we have $\Omega \subset C_n$ for all $n \in \mathbb{N}$.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Since Ω is a nonempty, closed and convex subset of C, there exists a unique element $z_0 = P_\Omega x_0 \in \Omega \subset C_n$. Since $x_n = P_{C_n} x_0$,

$$\|x_n - x_0\| \le \|z_0 - x_0\|. \tag{4.1}$$

Hence $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{u_n\}$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$,

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|. \tag{4.2}$$

From (4.1) and (4.2), we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.

By the construction of the set C_n, we know that $x_m = P_{C_m} x_0 \in C_m \subset C_n$ for $m > n$. By Lemma 2.2, it follows that

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \tag{4.3}$$

as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in C. So $x_n \to q \in C$ as $n \to \infty$.

Step 4. Show that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

From (4.3), we get

$$\|x_{n+1} - x_n\| \to 0$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$,

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &\le \|x_n - x_{n+1}\|^2 - \beta_n(1-\beta_n)\|W_n x_n - x_n\|^2 + (1-\beta_n)\theta_n \\ &\le \|x_n - x_{n+1}\|^2 + (1-\beta_n)\theta_n \to 0 \end{aligned}$$

as $n \to \infty$, which implies

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \tag{4.4}$$

as $n \to \infty$. Since T_{r_n} is firmly nonexpansive, for $p \in \Omega$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\| \tag{4.5}$$

and

$$\begin{aligned} &\|u_n - p\|^2 = \|T_{r_n} x_n - T_{r_n} p\|^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\}, \end{aligned}$$

hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\le \beta_n\|u_n - p\|^2 + (1-\beta_n)\|W_n u_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\gamma_{1,n}^2\|u_n - p\|^2 + 2\gamma_{1,n}\varepsilon_{1,n}\|x_n - p\| + \varepsilon_{1,n}^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\gamma_{1,n}^2\big(\|x_n - p\|^2 - \|x_n - u_n\|^2\big) \\
&\quad + \ 2\gamma_{1,n}\varepsilon_{1,n}\|x_n - p\| + \varepsilon_{1,n}^2 \\
&\le (\gamma_{1,n}^2 - 1)(1-\beta_n)\|x_n - p\|^2 + \|x_n - p\|^2 - (1-\beta_n)\gamma_{1,n}^2\|x_n - u_n\|^2 \\
&\quad + \ 2\gamma_{1,n}\varepsilon_{1,n}\|x_n - p\| + \varepsilon_{1,n}^2
\end{aligned}$$

which yields that

$$\begin{aligned}
&(1-\beta_n)\gamma_{1,n}^2\|x_n - u_n\|^2 \\
&\le (\gamma_{1,n}^2 - 1)(1-\beta_n)\|x_n - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad + \ 2\gamma_{1,n}\varepsilon_{1,n}\|x_n - p\| + \varepsilon_{1,n}^2.
\end{aligned}$$

Hence, from (4.4), we also have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.7}$$

Thus $u_n \to q$ as $n \to \infty$. From (4.4), (4.7) and $\limsup_{n\to\infty} \beta_n < 1$, we have

$$\|W_n u_n - u_n\| = \frac{1}{1-\beta_n}\|y_n - u_n\| \to 0 \tag{4.8}$$

as $n \to \infty$. By Lemma 3.1(iii), we conclude that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 5. Show that $q \in EP(F)$.

From (4.7) and $\liminf_{n\to\infty} r_n > 0$, we get

$$\lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{4.9}$$

From $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From the monotonicity of F, we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad \forall y \in C,$$

hence

$$\langle y - u_n, \frac{u_n - x_n}{r_n}\rangle \ge F(y, u_n), \quad \forall y \in C.$$

From (4.9) and (A4), we have

$$0 \ge F(y, q), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Thus $y_t \in C$ and $F(y_t, q) \leq 0$. So we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)f(y_t, q) \leq tF(y_t, y).$$

Dividing by t, we obtain

$$F(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \downarrow 0$ and from (A3), we get

$$F(q, y) \geq 0, \quad \forall y \in C.$$

Therefore $q \in EP(F)$ and hence $q \in \Omega$.

Step 6. Show that $q = P_\Omega x_0$.

Since $x_n = P_{C_n} x_0$ and $\Omega \subset C_n$, we obtain

$$\big\langle x_0 - x_n, x_n - p \big\rangle \geq 0 \quad \forall p \in \Omega. \tag{4.10}$$

By taking the limit in (4.10), we obtain

$$\big\langle x_0 - q, q - p \big\rangle \geq 0 \quad \forall p \in \Omega.$$

This shows that $q = P_\Omega x_0 = z_0$.

From Step 1 to Step 6, we conclude that $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_0$. This completes the proof. □

Corollary 4.2. *Let C be a closed and convex subset of a real Hilbert space H, let $F : C \times C \to R$ be a bifunction satisfying* (A1) – (A4) *and let T be a uniformly L-Lipschitzian and generalized quasi-nonexpansive mapping of C into itself with $k_n \geq 1$ and $c_n \geq 0$ for all $n \in \mathbb{N}$ such that* $\lim_{n\to\infty} k_n = 1$ *and* $\lim_{n\to\infty} c_n = 0$. *Assume that $\Omega := F(T) \cap EP(F) \neq \emptyset$ and $\{\alpha_n\} \subset [0, 1)$ such that* $\limsup_{n\to\infty} \alpha_n < 1$. *For an initial point $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\big\langle y - u_n, u_n - x_n \big\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n u_n + (1-\alpha_n) T^n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \\ \qquad\qquad \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T^n x_n - x_n\|^2 + (1-\alpha_n)\theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\theta_n = (k_n^2 - 1)\nabla_n^2 + 2k_n c_n \nabla_n + c_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : z \in \Omega\} < \infty$. Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_0$.

Proof. Putting, in Theorem 4.1, $\beta_n = 0$, $\alpha_{i,n} = \alpha_n$ for all $n \in \mathbb{N}$, $T_1 = T$ and $T_i = I$ for $i = 2, 3, \ldots, n$, where I is the identity mapping, we obtain the desired result. □

Corollary 4.3. *Let C be a closed and convex subset of a real Hilbert space H and let T be a uniformly L-Lipschitzian and generalized quasi-nonexpansive mapping of C into itself with $k_n \geq 1$ and $c_n \geq 0$ such that* $\lim_{n\to\infty} k_n = 1$ *and* $\lim_{n\to\infty} c_n = 0$. *Assume that $F(T) \neq \emptyset$ and $\{\alpha_n\} \subset [0, 1)$ with* $\limsup_{n\to\infty} \alpha_n$

< 1. *For an initial point* $x_0 \in H$ *with* $C_1 = C$ *and* $x_1 = P_{C_1}x_0$, *let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \\ \qquad\qquad \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T^n x_n - x_n\|^2 + (1-\alpha_n)\theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\theta_n = (k_n^2 - 1)\nabla_n^2 + 2k_n c_n \nabla_n + c_n^2$, $\nabla_n = \sup_{n\in\mathbb{N}}\{\|x_n - z\| : \ z \in F(T)\} < \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(T)}x_0$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ in Corollary 4.2. □

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