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APPROXIMATE EULER-LAGRANGE-JENSEN TYPE ADDITIVE MAPPING IN MULTI-BANACH SPACES: A FIXED POINT APPROACH

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Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam-Rassias stability of the following functional equation in multi-Banach spaces:

$$\sum_{1\le i<j\le n} f\Big(\frac{r_i x_i + r_j x_j}{k}\Big) = \frac{n-1}{k}\sum_{i=1}^{n} r_i f(x_i).$$

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [43] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [19]. It gave rise the stability theory for functional equations. The result of Hyers was extended by Aoki [1] in 1950, by considering the unbounded Cauchy differences. In 1978, Th. M. Rassias [39] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$ the mapping $f(tx)$ is continous in $t \in \mathbb{R}$. Găvruta [18] generalized the Rassias' result. Following the techniques of the proof of the corollary of Hyers [19] we observed that Hyers introduced (in 1941) the following Hyers continuity condition: about the continuity of the mapping for each fixed, and then he proved homogenouity of degree one and therefore the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers–Ulam stability problem forms (see [23]). Beginning around the year 1980 the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many

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interesting results concerning this problem (see [3], [6], [12]–[14], [21]–[34], [38], [40], [41]).

J. M. Rassias [37] following the spirit of the innovative approach of Hyers [19], Aoki [1] and Th. M. Rassias [39] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [36] for a number of other new results).

In 2003 Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [4] (see also [5], [6], [20], [35]). They could present a short and a simple proof (different of the "*direct method*", initiated by Hyers in 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [4], for Cauchy functional equation [6] and for quadratic functional equation [5].

The following functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \tag{1.1}$$

is called a *quadratic functional equation,* and every solution of equation (1.1) is said to be a *quadratic mapping.* F. Skof [42] proved the Hyers–Ulam stability of the quadratic functional equation (1.1) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [7], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation (1.1). C. Borelli and G. L. Forti [2] generalized the stability result of the quadratic functional equation (1.1).

Recently, Dales and Polyakov [9] introduced the notion of multi-normed spaces. This concept is somewhat similar to operator sequence spaces and has some connections with operator spaces and Banach lattices. Dales and Moslehian [8] investigated stability of Cauchy functional equation in multi-Banach spaces (see also [28], [44]).

In this paper, for a fixed positive integer $n \geq 2$, we introduce the following additive functional equation of Euler–Lagrange–Jensen type:

$$\sum_{1 \leq i < j \leq n} f\Big(\frac{r_i x_i + r_j x_j}{k}\Big) = \frac{n-1}{k} \sum_{i=1}^{n} r_i f(x_i), \tag{1.2}$$

where $r_1, \dots, r_n \in \mathbb{R}$ and k is a fixed non-zero integer. Every solution of the functional equation (1.2) is said to be a *generalized additive mapping of Euler–Lagrange–Jensen type.*

We will adopt the idea of Cădariu and Radu [4], [6], [35], to prove the generalized Hyers–Ulam–Rassias stability of additive functional equation of Euler–Lagrange–Jensen type on multi-Banach spaces.

2. Stability of Euler-Lagrange-Jensen type functional equation: fixed point method

Assume that $(E, \|\cdot\|)$ is a complex linear space, and let $m \in \mathbb{N}$. We denote by E^m the linear space $E \oplus E \oplus \cdots \oplus E$ consisting of m-tuples $(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m \in E$. The linear operations on E^m are defined coordinatewise. When we write $(0, \ldots, 0, x_i, 0, \ldots, 0)$ for an element in E^m, we understand that x_i appears in the i^{th} coordinate. The zero element of either E or E^m is denoted by 0. We denote by $\mathbb{N}_m$ the set $\{1, 2, \ldots, m\}$ and by σ_m the group of permutations on m symbols.

In this section, we recall the notion of a multi-normed space and some preliminaries concerning multi-normed spaces from [9].

Definition 2.1 ([9])**.** Let $(E, \|\cdot\|)$ be a complex normed space, and let $m \in \mathbb{N}$. A multi-norm of level m on $\{E^s \ : \ s \in \mathbb{N}_m\}$ is a sequence

$$(\|\cdot\|_s) = (\|\cdot\|_s : s \in \mathbb{N}_m)$$

such that $\|\cdot\|_s$ is a norm on E^s for each $s \in \mathbb{N}_m$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following Axioms $(A1)$-$(A4)$ are satisfied for each $s \in \mathbb{N}_m$ with $s \geq 2$:

(A1) for each $\sigma \in \sigma_s$ and $x_1, \ldots, x_s \in E$, we have

$$\|(x_{\sigma(1)}, \ldots, x_{\sigma(s)})\|_s = \|(x_1, \ldots, x_s)\|_s;$$

(A2) for each $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ and $x_1, \ldots, x_s \in E^s$, we have

$$\|\alpha_1 x_1, \ldots, \alpha_s x_s\|_s \leq \big(\max_{i \in \mathbb{N}_s} |\alpha_i|\big) \|x_1, \ldots, x_s)\|_s;$$

(A3) for each $x_1, \ldots, x_s \in E$, we have

$$\|x_1, \ldots, x_{s-1}, 0\|_s = \|x_1, \ldots, x_{s-1}\|_{s-1};$$

(A4) for each $x_1, \ldots, x_s \in E$, we have

$$\|x_1, \ldots, x_{s-1}, x_{s-1}\|_s = \|x_1, \ldots, x_{s-1}\|_{s-1};$$

In this case, we say that $((E^s, \|\cdot\|_s) : s \in \mathbb{N}_m)$ is a *multi-normed space of level m*.

Definition 2.2 ([9])**.** A multi-norm on $\{E^s : s \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_s) = (\|\cdot\|_s : s \in \mathbb{N})$$

such that $(\|\cdot\|_s : s \in \mathbb{N}_m)$ is a multi-norm of level m for each $m \in \mathbb{N}$. In this case, we say that $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is a multi-normed space.

Lemma 2.3 ([9])**.** *Let $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ be a multi-normed space. The following properties are immediate consequences of the axioms for multi-normed spaces.*

(i) *for all $x \in E$ and $s \in \mathbb{N}$, we have*

$$\|(x, \ldots, x)\|_s = \|x\|.$$

(ii) *for all $s \in \mathbb{N}$ and all $x_1, \ldots, x_s \in E$, we have*

$$\max_{i\in\mathbb{N}_s} \|x_i\| \le \|(x_1, \ldots, x_s)\|_s \le \sum_{i=1}^{s} \|x_i\| \le s \max_{i\in\mathbb{N}_s} \|x_i\|.$$

The following lemma is a consequence of (ii).

Lemma 2.4 ([9]). *Suppose that $(E, \|\cdot\|)$ is a Banach space. Then $(E^s, \|\cdot\|_s)$ is a Banach space for each $s \in \mathbb{N}$.*

Definition 2.5 ([9]). Let $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ be a multi-normed space for which $(E, \|\cdot\|)$ is a Banach space. Then $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is called a multi-Banach space.

Now, we recall two important examples of multi-norms for arbitrary space $(E, \|\cdot\|)$. For other examples we refer to readers to [9].

Example 2.6. Let $(E, \|\cdot\|)$ be a normed space. For $m \in \mathbb{N}$, define $\|\cdot\|_m$ on E^m by

$$\|(x_1, \ldots, x_m)\|_m = \max_{i\in\mathbb{N}_m} \|x_i\| \quad (x_1, \ldots, x_m \in E).$$

It is immediate that $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is a multi-normed space. The sequence $(\|\cdot\|_m : m \in \mathbb{N})$ is called *minimum multi-norm*. The terminology 'minimum' is justified by Lemma 2.3.

Example 2.7. Let $(E, \|\cdot\|)$ be a normed space and let $\{(\|\cdot\|_m^\alpha : m \in \mathbb{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^s : s \in \mathbb{N}\}$. For $s \in \mathbb{N}$, define

$$|||(x_1, \ldots, x_s)|||_s = \sup_{\alpha\in A} \|(x_1, \ldots, x_s)\|_s^\alpha \quad (x_1, \ldots, x_s \in E).$$

Then $(|||\cdot|||_m : m \in \mathbb{N})$ is a multi-norm on $\{E^s : s \in \mathbb{N}\}$, which is called *maximum multi-norm*.

We recall two fundamental results in fixed point theory.

Theorem 2.8 ([4]). *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \le Lf(x, y), \qquad \forall x, y \in X$$

for some Lipschitz constant $L < 1$. Then

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, i.e.,*

$$\lim_{n\to\infty} J^n x = x^*$$

for any starting point $x \in X$;

(3) *one has the following estimation inequalities*:

$$d(J^n x, x^*) \quad \le \quad L^n d(x, x^*),$$

$$d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1} x),$$
$$d(x, x^*) \leq \frac{1}{1-L} d(x, Jx)$$

for all nonnegative integers n *and all* $x \in X$.

Definition 2.9. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.10 ([10])**.** *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper, n will be a positive integer such that $n \geq 2$, k will be a fixed non-zero integer and $r_1, \ldots, r_n$ will be real numbers such that $r_i, r_j \neq 0$ for fixed $1 \leq i < j \leq n$.

Lemma 2.11. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be linear spaces and suppose that* $r_1, \ldots, r_n$ *are real numbers with* $\sum_{i=1}^n r_i \neq \frac{nk}{2}$. *Assume that a mapping* $L : \mathcal{X} \to \mathcal{Y}$ *satisfies the functional equation* (1.2) *for all* $x_1, \ldots, x_n \in \mathcal{X}$. *Then the mapping* L *is Cauchy additive. Moreover,* $L(\frac{r_j x}{k}) = \frac{r_j}{k} L(x)$ *for all* $x \in \mathcal{X}$ *and all* $1 \leq j \leq n$.

Proof. Since $\sum_{i=1}^n r_i \neq \frac{nk}{2}$, putting $x_1 = \cdots = x_n = 0$ in (1.2), we get $L(0) = 0$. Letting $x_m = 0$ in (1.2) for all $1 \leq m \leq n$ with $m \neq i, j$, we get

$$L(\frac{r_i x_i + r_j x_j}{k}) + (n-2) L(\frac{r_i x_i}{k}) + (n-2) L(\frac{r_j x_j}{k}) = \frac{n-1}{k}\big(r_i L(x_i) + r_j L(x_j)\big) \tag{2.1}$$

for all $x_i, x_j \in \mathcal{X}$. Letting $x_j = 0$ in (2.1), we have

$$L(\frac{r_i x_i}{k}) = \frac{r_i}{k} L(x_i) \tag{2.2}$$

for all $x_i \in \mathcal{X}$. Similarly, by putting $x_i = 0$ in (2.1), we get

$$L(\frac{r_j x_j}{k}) = \frac{r_j}{k} L(x_j) \tag{2.3}$$

for all $x_j \in \mathcal{X}$. It follows from (2.1), (2.2) and (2.3) that

$$L(\frac{r_ix_i + r_jx_j}{k}) + (n-2)L(\frac{r_ix_i}{k}) + (n-2)L(\frac{r_jx_j}{k}) = (n-1)\big(L(\frac{r_ix_i}{k}) + L(\frac{r_jx_j}{k})\big) \tag{2.4}$$

for all $x_i, x_j \in \mathcal{X}$. Replacing x_i and x_j by kx/r_i and ky/r_j in (2.4), respectively, we get

$$L(x+y) + (n-2)L(x) + (n-2)L(y) = (n-1)\big(L(x) + L(y)\big)$$

for all $x, y \in \mathcal{X}$. Therefore, L is additive. Moreover, let $x \in \mathcal{X}$ and $1 \le j \le n$. Setting $x_j = x$ and $x_i = 0$ for all $1 \le i \le n$ with $i \ne j$ in (1.2), we get that $L(\frac{r_jx}{k}) = \frac{r_j}{k}L(x)$. □

Using the same proof as Lemma 2.11, we have an alternative result of Lemma 2.11 when $\sum_{i=1}^n r_i = \frac{nk}{2}$.

Lemma 2.12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces. Assume that a mapping $L : \mathcal{X} \to \mathcal{Y}$ with $L(0) = 0$ satisfies the functional equation (1.2) for all $x_1, \dots, x_n \in \mathcal{X}$. Then the mapping L is Cauchy additive. Moreover, $L(\frac{r_jx}{k}) = \frac{r_j}{k}L(x)$ for all $x \in \mathcal{X}$ and all $1 \le j \le n$.*

We will use the following lemma in the proof of the next theorems.

Lemma 2.13 ([27])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1 := \{\,\lambda \in \mathbb{C} : |\lambda| = 1\,\}$. Then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is $\mathbb{C}$-linear.*

Let X and Y be vector spaces. For a given mapping $f : X \to Y$, we define

$$D_{\mu,r_1,\dots,r_n}f(x_1,\dots,x_n) = \sum_{1\le i<j\le n} f\left(\frac{\mu r_ix_i + \mu r_jx_j}{k}\right) - \left(\frac{n-1}{k}\right)\sum_{i=1}^n \mu r_i f(x_i)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \dots, x_n \in X$.

Now, we prove the generalized Hyers–Ulam–Rassias stability of Euler-Lagrange-Jensen type additive mapping on multi-Banach spaces for the functional equation $D_{\mu,r_1,\dots,r_n}f(x_1,\dots,x_n) = 0$.

Theorem 2.14. *Let E be a linear space and let $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $s \in \mathbb{N}$ and $f : E \to F$ is a mapping satisfying $f(0) = 0$ for which there exists a control function $\varphi : E^{ns} \to [0,\infty)$ such that*

$$\|D_{\mu,r_1,\dots,r_n}f(X^{(1)}), \dots, D_{\mu,r_1,\dots,r_n}f(X^{(s)})\|_s \le \varphi(X^{(1)},\dots,X^{(s)}) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $X^{(1)} = (x_1^{(1)},\dots,x_n^{(1)}), \dots, X^{(s)} = (x_1^{(s)},\dots,x_n^{(s)}) \in E^n$. If there exists a Lipschitz constant $L < 1$ such that

$$\varphi\Big(X^{(1)},\dots,X^{(s)}\Big) \le 2L\varphi\Big(\frac{X^{(1)}}{2},\dots,\frac{X^{(s)}}{2}\Big)$$

for all $X^{(1)}, \dots, X^{(s)} \in E^n$*, then there exists a unique linear mapping* $T : E \to F$ *such that*

$$\begin{aligned}
&\|f(x_1) - T(x_1), \dots, f(x_s) - T(x_s)\|_s \\
&\leq \frac{1}{2-2L}\Bigg[\varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, \frac{kx_s}{r_j}\Big)\right) \\
&\quad + \varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, 0\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, 0\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, 0\Big)\right) \\
&\quad + \varphi\left(X_{i,j}\Big(0, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(0, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(0, \frac{kx_s}{r_j}\Big)\right)\Bigg]
\end{aligned} \tag{2.6}$$

for all $x_1, \dots, x_s \in E$*, where*

$$X_{i,j}(x, y) = (0, \dots, 0, \underbrace{x}_{i\,th}, 0, \dots, 0, \underbrace{y}_{j\,th}, 0, \dots, 0)$$

for all $x, y \in E$*.*

Proof. For convenience, set

$$\begin{aligned}
&\varphi_{i,j}(x_1, x_2, \dots, x_s) \\
&= \varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, \frac{kx_s}{r_j}\Big)\right) \\
&\quad + \varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, 0\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, 0\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, 0\Big)\right) \\
&\quad + \varphi\left(X_{i,j}\Big(0, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(0, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(0, \frac{kx_s}{r_j}\Big)\right),
\end{aligned}$$

where $x_1, \dots, x_s \in E$ and $1 \leq i < j \leq n$. Consider the set $\mathcal{X} := \{g : E \to F,\ g(0) = 0\}$ and introduce the *generalized metric* on $\mathcal{X}$:

$$\begin{aligned}
d(g, h) = \inf\{C \in \mathbb{R}_+ : \quad &\|g(x_1) - h(x_1), \dots, g(x_s) - h(x_s)\|_s \\
&\leq C\varphi_{i,j}(x_1, x_2, \dots, x_s), \quad \forall\, x_1, x_2, \dots, x_s \in E\}.
\end{aligned}$$

It is easy to show that $(\mathcal{X}, d)$ is complete. Now we consider the linear mapping $J : \mathcal{X} \to \mathcal{X}$ such that $Jg(x) := \frac{1}{2}g(2x)$ for all $x \in E$. For any $g, h \in \mathcal{X}$, we have

$$\begin{aligned}
&d(g, h) < C \\
\Rightarrow\quad &\|g(x_1) - h(x_1), \dots, g(x_s) - h(x_s)\|_s \leq C\varphi_{i,j}(x_1, x_2, \dots, x_s) \\
&(x_1, x_2, \dots, x_s \in E) \\
\Rightarrow\quad &\Big\|\frac{1}{2}g(2x_1) - \frac{1}{2}h(2x_1), \dots, \frac{1}{2}g(2x_s) - \frac{1}{2}h(2x_s)\Big\|_s \leq \frac{1}{2}C\varphi_{i,j}(2x_1, 2x_2, \dots, 2x_s) \\
\Rightarrow\quad &\Big\|\frac{1}{2}g(2x_1) - \frac{1}{2}h(2x_1), \dots, \frac{1}{2}g(2x_s) - \frac{1}{2}h(2x_s)\Big\|_s \leq LC\varphi_{i,j}(x_1, x_2, \dots, x_s) \\
\Rightarrow\quad &d(Jg, Jh) \leq LC.
\end{aligned}$$

Therefore, we see that

$$d(Jg, Jh) \leq L\, d(g, h), \qquad \forall g, h \in \mathcal{X}.$$

This means J is a strictly contractive self-mapping of $\mathcal{X}$, with the Lipschitz constant L.

Letting $\mu = 1$ and for each $1 \leq r \leq n$ with $r \neq i, j$ and each $1 \leq m \leq s$, let $x_i^{(m)} = x_m$, $x_j^{(m)} = y_m$ and $x_r^{(m)} = 0$ in (2.5), we get

$$\begin{aligned}
&\Big\| f(\frac{r_i x_1 + r_j y_1}{k}) + (n-2) f(\frac{r_i x_1}{k}) + (n-2) f(\frac{r_j y_1}{k}) \\
&\quad - \frac{n-1}{k}\big(r_i f(x_1) + r_j f(y_1)\big), f(\frac{r_i x_2 + r_j y_2}{k}) + (n-2) f(\frac{r_i x_2}{k}) \\
&\quad + (n-2) f(\frac{r_j y_2}{k}) - \frac{n-1}{k}\big(r_i f(x_2) + r_j f(y_2)\big), \dots, f(\frac{r_i x_s + r_j y_s}{k}) \\
&\quad + (n-2) f(\frac{r_i x_s}{k}) + (n-2) f(\frac{r_j y_s}{k}) - \frac{n-1}{k}\big(r_i f(x_s) + r_j f(y_s)\big) \Big\|_s \\
&\leq \varphi\Big(X_{i,j}(x_1, y_1), X_{i,j}(x_2, y_2), \dots, X_{i,j}(x_s, y_s)\Big)
\end{aligned} \tag{2.7}$$

for all $x_1, y_1, x_2, y_2, \dots, x_s, y_s \in E$. Letting $y_1 = y_2 = \cdots = y_s = 0$ in (2.7), we have

$$\begin{aligned}
&\Big\| (n-1)\big(f(\frac{r_i x_1}{k}) - \frac{r_i}{k} f(x_1)\big), (n-1)\big(f(\frac{r_i x_2}{k}) - \frac{r_i}{k} f(x_2)\big), \dots, \\
&\quad (n-1) f(\frac{r_i x_s}{k}) - \frac{r_i}{k} f(x_s)\big) \Big\|_s \\
&\leq \varphi\Big(X_{i,j}(x_1, 0), X_{i,j}(x_2, 0), \dots, X_{i,j}(x_s, 0)\Big)
\end{aligned} \tag{2.8}$$

for all $x_1, x_2, \dots, x_s \in E$. Similarly, letting $x_1 = x_2 = \cdots = x_s = 0$ in (2.7), we get

$$\begin{aligned}
&\Big\| (n-1)\big(f(\frac{r_i y_1}{k}) - \frac{r_i}{k} f(y_1)\big), (n-1)\big(f(\frac{r_i y_2}{k}) - \frac{r_i}{k} f(y_2)\big), \dots, \\
&\quad (n-1) f(\frac{r_i x_s}{k}) - \frac{r_i}{k} f(y_s)\big) \Big\|_s \\
&\leq \varphi\Big(X_{i,j}(0, y_1), X_{i,j}(0, y_2), \dots, X_{i,j}(0, y_s)\Big)
\end{aligned} \tag{2.9}$$

for all $y_1, y_2, \dots, y_s \in E$. It follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned}
&\Big\| f(\frac{r_i x_1 + r_j y_1}{k}) - f(\frac{r_i x_1}{k}) - f(\frac{r_j y_1}{k}), \\
&\quad f(\frac{r_i x_2 + r_j y_2}{k}) - f(\frac{r_i x_2}{k}) - f(\frac{r_j y_2}{k}), \dots, \\
&\quad f(\frac{r_i x_s + r_j y_s}{k}) - f(\frac{r_i x_s}{k}) - f(\frac{r_j y_s}{k}) \Big\|_s
\end{aligned} \tag{2.10}$$

$$\leq \varphi\Big(X_{i,j}(x_1, y_1), X_{i,j}(x_2, y_2), \ldots, X_{i,j}(x_s, y_s)\Big) + \varphi\Big(X_{i,j}(x_1, 0), X_{i,j}(x_2, 0), \ldots, X_{i,j}(x_s, 0)\Big) + \varphi\Big(X_{i,j}(0, y_1), X_{i,j}(0, y_2), \ldots, X_{i,j}(0, y_s)\Big)$$

for all $x_1, y_1, x_2, y_2, \ldots, x_s, y_s \in E$. Replacing x_m and y_m by kx_m/r_i and kx_m/r_j, for all $1 \leq m \leq s$ in (2.10), respectively, we get

$$\Big\|f(2x_1) - 2f(x_1), f(2x_2) - 2f(x_2), \ldots, f(2x_s) - 2f(x_s)\Big\|_s \leq \varphi_{i,j}(x_1, x_2, \ldots, x_s) \tag{2.11}$$

for all $x_1, x_2, \ldots, x_s \in E$. So

$$\Big\|\frac{1}{2}f(2x_1) - f(x_1), \frac{1}{2}f(2x_2) - f(x_2), \ldots, \frac{1}{2}f(2x_s) - f(x_s)\Big\|_s \leq \frac{1}{2}\varphi_{i,j}(x_1, x_2, \ldots, x_s)$$

for all $x_1, x_2, \ldots, x_s \in E$. Hence $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 2.10, there exists a mapping $T : E \to F$ such that

(1) T is a fixed point of J, i.e.,

$$T(x) = \frac{1}{2}T(2x) \tag{2.12}$$

for all $x \in E$. The mapping T is a unique fixed point of J in the set

$$Y = \{g \in \mathcal{X} : d(f, g) < \infty\}.$$

This implies that T is a unique mapping satisfying (2.12) such that there exists $C \in (0, \infty)$ satisfying

$$\|T(x_1) - f(x_1), \ldots, T(x_s) - f(x_s)\|_s \leq C\varphi_{i,j}(x_1, x_2, \ldots, x_s)$$

for all $x_1, x_2, \ldots, x_s \in E$.

(2) $d(J^m f, T) \to 0$ as $m \to \infty$. This implies the equality

$$\lim_{m\to\infty} \frac{1}{2^m} f\left(2^m x\right) = T(x) \tag{2.13}$$

for all $x \in E$.

(3) $d(f, T) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, T) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (2.6) holds.

Letting $\mu = 1$ and replacing $X^{(1)} = \cdots = X^{(s)} = (x_1, x_2, \ldots, x_n) := X$ in (2.5), by using the properties of norm in multi-normed spaces and (2.13), we have

$$\|D_{1,r_1,\ldots,r_n} T(x_1, \ldots, x_n), \ldots, D_{1,r_1,\ldots,r_n} T(x_1, \ldots, x_n)\|_s$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \frac{1}{2^m}\|D_{1,r_1,\ldots,r_n} f(2^m x_1,\ldots,2^m x_n),\ldots,D_{1,r_1,\ldots,r_n} f(2^m x_1,\ldots,2^m x_n)\|_s \\
&= \lim_{m\to\infty} \frac{1}{2^m}\|D_{1,r_1,\ldots,r_n} f(2^m x_1,\ldots,2^m x_n)\|_s \\
&\leq \lim_{m\to\infty} \frac{1}{2^m}\varphi(2^m X,\ldots,2^m X) \\
&\leq \lim_{m\to\infty} L^m \varphi(X,\ldots,X) = 0
\end{aligned}
$$

for all $x_1, x_2, \ldots, x_n \in E$. So

$$
\sum_{1\leq i<j\leq n} T\left(\frac{r_i x_i + r_j x_j}{k}\right) = \left(\frac{n-1}{k}\right)\sum_{i=1}^{n} r_i T(x_i).
$$

By Lemma 2.11, the mapping $T : E \to F$ is Cauchy additive, i.e., $T(x+y) = T(x) + T(y)$ for all $x, y \in E$.

Now, we show that the mapping $T : E \to F$ is $\mathbb{C}$-linear. Letting $X^{(1)} = \cdots = X^{(s)} = X = (0,\ldots,0,\underbrace{x}_{i\,th},0,\ldots,0)$ in (2.5) and using the properties of norm in multi-normed spaces, we have

$$
\begin{aligned}
&\|D_{\mu,r_1,\ldots,r_n} T(0,\ldots,0,\underbrace{x}_{i\,th},0,\ldots,0)\|_s \\
&= \|D_{\mu,r_1,\ldots,r_n} T(X^{(1)}),\ldots,D_{\mu,r_1,\ldots,r_n} T(X^{(s)})\|_s \\
&= \|(n-1)T\left(\frac{\mu r_i x}{k}\right) - \frac{(n-1)\mu r_i}{k}T(x),\ldots, \\
&\qquad (n-1)T\left(\frac{\mu r_i x}{k}\right) - \frac{(n-1)\mu r_i}{k}T(x)\|_s \\
&= \lim_{m\to\infty} \frac{1}{2^m}\|(n-1)f\left(\frac{\mu r_i 2^m x}{k}\right) - \frac{(n-1)\mu r_i}{k}f(2^m x),\ldots, \\
&\qquad\qquad (n-1)f\left(\frac{\mu r_i 2^m x}{k}\right) - \frac{(n-1)\mu r_i}{k}f(2^m x)\|_s \\
&= \lim_{m\to\infty} \frac{1}{2^m}\|(n-1)f\left(\frac{\mu r_i 2^m x}{k}\right) - \frac{(n-1)\mu r_i}{k}T(2^m x)\| \\
&\leq \lim_{m\to\infty} \frac{1}{2^m}\varphi(2^m X,\ldots,2^m X) \\
&\leq \lim_{m\to\infty} L^m \varphi(X,\ldots,X) = 0
\end{aligned}
$$

for all $x \in E$ and all $\mu \in \mathbb{T}^1$. So

$$
T\left(\frac{\mu r_i x}{k}\right) = \frac{\mu r_i}{k}T(x)
$$

for all $x \in E$ and all $\mu \in \mathbb{T}^1$. Since

$$
T(\frac{r_i x}{k}) = \frac{r_i}{k}T(x)
$$

for all $x \in E$ and all $r_i \neq 0$. Therefore by Lemma 2.13, the mapping T is $\mathbb{C}$-linear. □

Corollary 2.15. *Let E be a linear space and let $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $s \in \mathbb{N}$ and $0 < p < 1$ and $f : E \to F$ is a mapping with $f(0) = 0$ satisfying*

$$\|D_{\mu,r_1,\dots,r_n} f(X^{(1)}), \dots, D_{\mu,r_1,\dots,r_n} f(X^{(s)})\|_s \leq \epsilon \sum_{m=1}^{s} \sum_{t=1}^{n} \|x_t^{(m)}\|^p$$

for all $\mu \in \mathbb{T}^1$ and all $X^{(1)}, \dots, X^{(s)} \in E^n$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x_1) - T(x_1), \dots, f(x_s) - T(x_s)\|_s \leq \frac{\epsilon k^p}{1 - 2^{p-1}} \left(\frac{1}{r_i^p} + \frac{1}{r_j^p} \right) \sum_{m=1}^{s} \|x_m\|^p$$

for all $x_1, \dots, x_s \in E$.

Proof. The proof follows from Theorem 2.14 by taking

$$\varphi(X^{(1)}, \dots, X^{(s)}) = \epsilon \sum_{m=1}^{s} \sum_{t=1}^{n} \|x_t^{(m)}\|^p$$

for all $X^{(1)}, \dots, X^{(s)} \in E^n$. We can choose $L = \frac{1}{2^{1-p}}$ to get the desired result. □

Theorem 2.16. *Let E be a linear space and let $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $s \in \mathbb{N}$ and $f : E \to F$ is a mapping satisfying $f(0) = 0$ for which there exists a control function $\varphi : E^{ns} \to [0, \infty)$ satisfying (2.5) for all $\mu \in \mathbb{T}^1$ and all $X^{(1)}, \dots, X^{(s)} \in E^n$. If there exists a Lipschitz constant $L < 1$ such that*

$$\varphi\Big(X^{(1)}, \dots, X^{(s)}\Big) \leq \frac{1}{2} L \varphi\Big(2X^{(1)}, \dots, 2X^{(s)}\Big)$$

for all $X^{(1)}, \dots, X^{(s)} \in E^n$, then there exists a unique linear mapping $T : E \to F$ such that

(2.14)

$$\begin{aligned}
&\|f(x_1) - T(x_1), \dots, f(x_s) - T(x_s)\|_s \\
&\leq \frac{L}{2 - 2L} \Bigg[\varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, \frac{kx_s}{r_j}\Big) \right) \\
&\quad + \varphi\left(X_{i,j}\Big(\frac{kx_1}{r_i}, 0\Big), X_{i,j}\Big(\frac{kx_2}{r_i}, 0\Big), \dots, X_{i,j}\Big(\frac{kx_s}{r_i}, 0\Big) \right) \\
&\quad + \varphi\left(X_{i,j}\Big(0, \frac{kx_1}{r_j}\Big), X_{i,j}\Big(0, \frac{kx_2}{r_j}\Big), \dots, X_{i,j}\Big(0, \frac{kx_s}{r_j}\Big) \right) \Bigg]
\end{aligned}$$

for all $x_1, \ldots, x_s \in E$, *where*

$$X_{i,j}(x,y) = (0, \ldots, 0, \underbrace{x}_{i\,th}, 0, \ldots, 0, \underbrace{y}_{j\,th}, 0, \ldots, 0)$$

for all $x, y \in E$.

Proof. Similar to the proof of Theorem 2.14, we consider the linear mapping $J : \mathcal{X} \to \mathcal{X}$ such that $Jg(x) := 2g\big(\frac{1}{2}x\big)$ for all $x \in E$. We can conclude that J is a strictly contractive self-mapping of $\mathcal{X}$, with the Lipschitz constant L.

It follows from (2.11) that

$$\Big\|f(x_1) - 2f(\frac{1}{2}x_1), f(x_2) - 2f(\frac{1}{2}x_2), \ldots, f(x_s) - 2f(\frac{1}{2}x_s)\Big\|_s \\ \leq \frac{L}{2}\varphi_{i,j}(x_1, x_2, \ldots, x_s)$$

for all $x_1, x_2, \ldots, x_s \in E$. Hence $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 2.10, there exists a mapping $T : E \to F$ such that

(1) T is a fixed point of J, i.e.,

$$T(x) = 2T(\frac{x}{2}) \tag{2.15}$$

for all $x \in E$. The mapping T is a unique fixed point of J in the set

$$Y = \{g \in \mathcal{X} : d(f,g) < \infty\}.$$

This implies that T is a unique mapping satisfying (2.15) such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x_1) - T(x_1), \ldots, f(x_s) - T(x_s)\|_s \leq C\varphi_{i,j}(x_1, x_2, \ldots, x_s)$$

for all $x_1, x_2, \ldots, x_s \in E$.

(2) $d(J^m f, T) \to 0$ as $m \to \infty$. This implies the equality

$$\lim_{m\to\infty} 2^m f\left(\frac{x}{2^m}\right) = T(x)$$

for all $x \in E$.

(3) $d(f,T) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f,T) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.14) holds.

The rest of the proof is similar to the proof of Theorem 2.14. □

Corollary 2.17. *Let* E *be a linear space and let* $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ *be a multi-Banach space. Suppose that* $s \in \mathbb{N}$ *and* $p > 1$ *and* $f : E \to F$ *is a mapping with* $f(0) = 0$ *satisfying*

$$\|D_{\mu,r_1,\ldots,r_n} f(X^{(1)}), \ldots, D_{\mu,r_1,\ldots,r_n} f(X^{(s)})\|_s \leq \epsilon \sum_{m=1}^{s}\sum_{t=1}^{n} \|x_t^{(m)}\|^p$$

for all $\mu \in \mathbb{T}^1$ *and all* $X^{(1)}, \ldots, X^{(s)} \in E^n$. *Then there exists a unique linear mapping* $T : E \to F$ *such that*

$$\|f(x_1) - T(x_1), \ldots, f(x_s) - T(x_s)\|_s \leq \frac{\epsilon k^p}{2^{p-1} - 1} \left(\frac{1}{r_i^p} + \frac{1}{r_j^p} \right) \sum_{m=1}^{s} \|x_m\|^p$$

for all $x_1, \ldots, x_s \in E$.

Proof. The proof follows from Theorem 2.16 by taking

$$\varphi(X^{(1)}, \ldots, X^{(s)}) = \epsilon \sum_{m=1}^{s} \sum_{t=1}^{n} \|x_t^{(m)}\|^p$$

for all $X^{(1)}, \ldots, X^{(s)} \in E^n$. We can choose $L = \frac{1}{2^{p-1}}$ to get the desired result. □

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