Commun. Korean Math. Soc. **28** (2013), No. 2, pp. 297–301
http://dx.doi.org/10.4134/CKMS.2013.28.2.297

A REDUCIBILITY OF SRIVASTAVA'S TRIPLE HYPERGEOMETRIC SERIES $F^{(3)}[x, y, z]$

Junesang Choi, Xiaoxia Wang, and Arjun K. Rathie

Abstract. When certain general single or multiple hypergeometric functions were introduced, their reduction formulas have naturally been investigated. Here, in this paper, we aim at presenting a very interesting reduction formula for the Srivastava's triple hypergeometric function $F^{(3)}[x, y, z]$ by applying the so-called Beta integral method to the Henrici's triple product formula for hypergeometric series.

1. Introduction

In the usual notation, let $\mathbb{C}$ denote the set of *complex* numbers. For

$$\alpha_j \in \mathbb{C}\ (j = 1, \ldots, p) \text{ and } \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \left(\mathbb{Z}_0^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, \ldots\}\right),$$

the *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters α_1, …, α_p and q denominator parameters $\beta_1, \ldots, \beta_q$ is defined by (see, for example, [6, Chapter 4]; see also [9, pp. 71–72]):

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$$

$$\Big(p,\, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\};\ p \leqq q + 1;\ p \leqq q \text{ and } |z| < \infty;$$
$$p = q + 1 \text{ and } |z| < 1;\ p = q + 1,\ |z| = 1 \text{ and } \Re(\omega) > 0\Big),$$

Received April 20, 2012; Revised December 28, 2012.
2010 *Mathematics Subject Classification.* Primary 33C20, 33B20; Secondary 33C90, 33C05.
Key words and phrases. generalized hypergeometric function ${}_pF_q$, Gamma function, Pochhammer symbol, Beta integral, Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$, Henrici's formula.

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \ \big(\alpha_j \in \mathbb{C}\,(j=1,\ldots,p);\ \beta_j \in \mathbb{C}\setminus \mathbb{Z}_0^-\ (j=1,\ldots,q)\big) \tag{1.2}$$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}). \end{cases} \tag{1.3}$$

A unification of Lauricella's 14 triple hypergeometric series $F_1, \ldots, F_{14}$ (*cf.* [10, pp. 41–43]) and the additional Srivastava's triple hypergeometric series H_A, H_B, H_C was introduced by Srivastava [7] (*cf.* [10, p. 43]) who defined a general triple hypergeometric series $F^{(3)}[x,y,z]$ (*cf.* [7, p. 428]; see also [10, pp. 44–45]):

$$\begin{aligned} F^{(3)}[x,y,z] &\equiv F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') : (c); (c'); (c''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} x, y, z\right] \\ &= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p)\, \frac{x^m}{m!}\, \frac{y^n}{n!}\, \frac{z^p}{p!}, \end{aligned} \tag{1.4}$$

where, for convenience,

$$\begin{aligned} \Lambda(m,n,p) &= \frac{\prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{n+p} \prod_{j=1}^{B''} (b''_j)_{p+m}}{\prod_{j=1}^{E} (e_j)_{m+n+p} \prod_{j=1}^{G} (g_j)_{m+n} \prod_{j=1}^{G'} (g'_j)_{n+p} \prod_{j=1}^{G''} (g''_j)_{p+m}} \\ &\quad \cdot \frac{\prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c'_j)_n \prod_{j=1}^{C''} (c''_j)_p}{\prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h'_j)_n \prod_{j=1}^{H''} (h''_j)_p}, \end{aligned} \tag{1.5}$$

and, (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b''), and so on.

When general hypergeometric functions in one, two and more variables were given, their reduction formulas have naturally been investigated. For various interesting reduction formulas, see, for example, [10, pp. 32, 34, 38, and so on], and see also [1], [2] and [8]. Here, we would like to mention an interesting reduction formula for the Srivastava's triple hypergeometric series $F^{(3)}[-x/(1-z)(1-x), -y/(1-x)(1-y), -z/(1-y)(1-z)]$ (see [10, p. 272, Equation (17);

p. 272, Equation (121)]):

(1.6)

$$F^{(3)}\left[\begin{matrix} - :: \beta; \beta'; \alpha : -; -; -; \\ - :: -; -; - : \alpha; \beta; \beta'; \end{matrix} \; -\frac{x}{(1-z)(1-x)}, -\frac{y}{(1-x)(1-y)}, -\frac{z}{(1-y)(1-z)}\right]$$
$$= (1+xyz)^{-1}\,(1-x)^{\beta}\,(1-y)^{\beta'}\,(1-z)^{\alpha}.$$

Here, in this paper, we aim at presenting a very interesting reduction formula for the Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ by applying the so-called Beta integral method (see, for example, [5]) to the Henrici's triple product formula for hypergeometric series [3]:

$$(1.7)\qquad \begin{aligned} &{}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; x\right] {}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; \omega x\right] {}_0F_1\left[\begin{matrix} -; \\ 6c; \end{matrix}\; \omega^2 x\right] \\ &= {}_2F_7\left[\begin{matrix} 3c-\frac{1}{4},\; 3c+\frac{1}{4}; \\ 6c,\, 2c,\, 2c+\frac{1}{3},\, 2c+\frac{2}{3},\, 4c-\frac{1}{3},\, 4c,\, 4c+\frac{1}{3}; \end{matrix}\; \left(\frac{4x}{9}\right)^3\right], \end{aligned}$$

where $\omega = \exp(\frac{2\pi i}{3})$.

2. Main theorem

We begin by stating our main theorem.

Theorem. *The following interesting reduction formula for the Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ holds true.*

(2.1)

$$F^{(3)}\left[\begin{matrix} e :: & -; & -; & - : & -; & -; & -; \\ d :: & -; & -; & - : & 6c; & 6c; & 6c; \end{matrix}\; 1,\, \omega,\, \omega^2\right]$$
$$= {}_5F_{10}\left[\begin{matrix} 3c-\frac{1}{4},\; 3c+\frac{1}{4},\; \frac{e}{3},\; \frac{e}{3}+\frac{1}{3},\; \frac{e}{3}+\frac{2}{3}; \\ 6c,\, 2c,\, 2c+\frac{1}{3},\, 2c+\frac{2}{3},\, 4c-\frac{1}{3},\, 4c,\, 4c+\frac{1}{3},\, \frac{d}{3},\, \frac{d}{3}+\frac{1}{3},\, \frac{d}{3}+\frac{2}{3}; \end{matrix}\; \left(\frac{4}{9}\right)^3\right],$$

where $\omega = \exp(\frac{2\pi i}{3})$.

Proof. Multiply the left-hand side of (1.7) by $x^{e-1}(1-x)^{d-e-1}$, integrate the resulting equation with respect to x between 0 to 1, express the involved three ${}_0F_1$ as series, change the order of summation and integration which is easily seen to be justified due to the uniform convergence of the involved series, finally use the following well-known relationship between the Beta function $B(\alpha, \beta)$ and the Gamma function Γ (see, for example, [9, Section 1.1]):

$$(2.2)\qquad B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-), \end{cases}$$

and, after a little simplification, we get

$$\frac{\Gamma(e)\Gamma(d-e)}{\Gamma(d)} \sum_{m,n,p=0}^{\infty} \frac{(e)_{m+n+p}}{(d)_{m+n+p}} \frac{\omega^n\,\omega^{2p}}{(6c)_m\,(6c)_n\,(6c)_p\,m!\,n!\,p!}.$$

In view of the Srivastava's function (1.4), we see that the last triple series is interpreted as follows:

$$\frac{\Gamma(e)\Gamma(d-e)}{\Gamma(d)} F^{(3)}\left[\begin{matrix} e :: & -; & -; & - : & -; & -; & -; \\ d :: & -; & -; & - : & 6c; & 6c; & 6c; \end{matrix}\; 1,\,\omega,\,\omega^2\right]. \tag{2.3}$$

On the other hand, multiply the right-hand side of (1.7) by $x^{e-1}(1-x)^{d-e-1}$ and proceed as above. Then, applying the multiplication formula for the Gamma function (in case of $m = 3$):

$$\begin{gathered}\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)}\, m^{mz-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left(z + \frac{j-1}{m}\right) \\ \left(z \neq 0,\, -\frac{1}{m},\, -\frac{2}{m},\, \cdots;\; m \in \mathbb{N}\right)\end{gathered} \tag{2.4}$$

to the last resulting series, we have

$$\begin{aligned}&\frac{\Gamma(e)\Gamma(d-e)}{\Gamma(d)} \sum_{n=0}^{\infty} \frac{\left(3c-\frac{1}{4}\right)_n \left(3c+\frac{1}{4}\right)_n \left(\frac{e}{3}\right)_n}{n!\,(6c)_n\,(2c)_n\,\left(2c+\frac{1}{3}\right)_n \left(2c+\frac{2}{3}\right)_n \left(4c-\frac{1}{3}\right)_n (4c)_n} \\ &\qquad \cdot \frac{\left(\frac{e}{3}+\frac{1}{3}\right)_n \left(\frac{e}{3}+\frac{2}{3}\right)_n \left(\frac{4}{9}\right)^{3n}}{\left(4c+\frac{1}{3}\right)_n \left(\frac{d}{3}\right)_n \left(\frac{d}{3}+\frac{1}{3}\right)_n \left(\frac{d}{3}+\frac{2}{3}\right)_n}.\end{aligned}$$

If we express this last series in terms of the generalized hypergeometric series (1.1), we have

$$\frac{\Gamma(e)\Gamma(d-e)}{\Gamma(d)}\, {}_5F_{10}\left[\begin{matrix} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \end{matrix}\ \left(\frac{4}{9}\right)^3\right]. \tag{2.5}$$

Finally, equating the two expressions (2.3) and (2.5) proves (2.1). □

Remark. Karlsson and Srivastava [4] generalized (1.7) by utilizing certain known transformations of hypergeometric functions and deduced (1.7) as a special case. Krattenthaler and Rao [5] made a systematic use of the so-called Beta integral method, a method of deriving new hypergeometric identities from old ones by mainly using the Beta integral in (2.2) based on the Mathematica Package HYP, to illustrate several interesting identities for the hypergeometric series and Kampé de Fériet series of unit arguments.

Acknowledgements. The present investigation was supported, in part, by Shanghai Leading Academic Discipline Project S30104.

References

[1] R. G. Buschman and H. M. Srivastava, *Series identities and reducibility of Kampé de Fériet functions*, Math. Proc. Cambridge Philos. Soc. **91** (1982), no. 3, 435–440.

[2] C. C. Grosjean and H. M. Srivastava, *Some transformation and reduction formulas for hypergeometric series in several variables*, J. Comput. Appl. Math. **37** (1991), no. 1-3, 287–299.

[3] P. Henrici, *A triple product theorem for hypergeometric series*, SIAM J. Math. Anal. **18** (1987), no. 6, 1513–1518.

[4] P. W. Karlsson and H. M. Srivastava, *A note on Henrici's triple product theorem*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 85–88.

[5] C. Krattenthaler and K. S. Rao, *Automatic generation of hypergeometric identities by the beta integral method*, J. Comput. Appl. Math. **160** (2003), no. 1-2, 159–173.

[6] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[7] H. M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc. **63** (1967), 425–429.

[8] ———, *On the reducibility of Appell's function* F_4, Canad. Math. Bull. **16** (1973), 295–298.

[9] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[10] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 780-714, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Xiaoxia Wang
Department of Mathematics
Shanghai University
Shanghai 200444, P. R. China
E-mail address: xiaoxiawang@shu.edu.cn

Arjun K. Rathie
Department of Mathematics
School of Mathematical & Physical Sciences
Central University of Kerala
Riverside Transit Campus
Padennakad P.O. Nileshwar, Kasaragod-671 328, India
E-mail address: akrathie@gmail.com