Commun. Korean Math. Soc. **28** (2013), No. 2, pp. 269–283
http://dx.doi.org/10.4134/CKMS.2013.28.2.269

A FIXED POINT APPROACH TO THE STABILITY OF THE GENERALIZED POLYNOMIAL FUNCTIONAL EQUATION OF DEGREE 2

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Abstract. In this paper, we investigate a stability of the functional equation

$$\sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix+y) = 0$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

In 1940, S. M. Ulam [24] raised a question concerning the stability of homomorphisms: Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$, and a positive number ε, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $F : G_1 \to G_2$ with

$$d(f(x), F(x)) < \varepsilon$$

for all $x \in G_1$? When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are *stable*. In the next year, D. H. Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [22] for linear mappings, by considering the stability problem with unbounded Cauchy differences. The paper of Th. M. Rassias had much influence in the development of stability problems. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [5], [10]-[19]).

Received April 14, 2011; Revised June 1, 2012.
2010 *Mathematics Subject Classification.* Primary 39B52.
Key words and phrases. generalized polynomial functional equation of degree 2, fixed point method, Hyers-Ulam stability.

Almost all subsequent proofs, in this very active area, have used Hyers' method. Namely, the solution F of a functional equation, starting from the given mapping f, is explicitly constructed by the formulae

$$F(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \text{ or } F(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}).$$

We call it a *direct method*. In 2003, L. Cădariu and V. Radu [2] observed that the existence of the solution F of a functional equation and the estimation of the difference with the given mapping f can be obtained from the fixed point theory alternative. They applied this method to prove stability theorems of *Jensen's functional equation*:

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = 0. \tag{1.1}$$

This method is called a *fixed point method*. In 2005, L. Cădariu [3] obtained a stability of the *quadratic functional equation*:

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{1.2}$$

by using the fixed point method. If we consider the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = ax+b$ and $f_2(x) = ax^2$, where a and b are real constants, then f_1 satisfies the equation (1.1) and f_2 satisfies the equation (1.2), respectively. Now we consider the functional equation

$$\sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix+y) = 0 \tag{1.3}$$

which is called the *generalized polynomial functional equation of degree 2*. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^2 + bx + c$ satisfies this functional equation. We call a solution of (1.3) a *general quadratic mapping*. On the other hand, a solution of (1.1) with the condition $f(0) = 0$ is called an *additive mapping* and a solution of (1.2) a *quadratic mapping*, respectively. In [14], Lee obtained a stability of the functional equation (1.3) by handling the odd part and the even part of the given mapping f, respectively. In the processing, Lee needed to take an additive mapping A which is close to the odd part $\frac{f(x)-f(-x)}{2}$ of f and a quadratic mapping Q which is approximate to the even part $\frac{f(x)+f(-x)}{2} - f(0)$ of it, and then combining A and Q to prove the existence of a general quadratic mapping F which is close to the given mapping f.

In this paper, we will prove the stability of a generalized polynomial functional equation of degree 2 (1.3) by using the fixed point theory. In the previous results of stability problems of (1.3), as we mentioned above, we get a solution by using the direct method to the odd part and even part, respectively. Instead of splitting the given mapping $f : X \to Y$ into two parts, in this paper, we can take the desired solution F at once. Precisely, we introduce a strictly contractive mapping with Liptshitz constant $0 < L < 1$. Using the fixed point theory in the sense of L. Cădariu and V. Radu, together with suitable conditions, we can show that the contractive mapping has a fixed point. Actually the fixed point F becomes the precise solution of the functional equation (1.3). Recently, using such an idea, the authors and S.-M. Jung have proved several kinds of stability results of functional equations (see [7]-[9], [20], [21]). In Section 2, we consider the fundamental result in the fixed point theory and construct some strictly contractive self-mappings. In Section 3, we prove several stability results of the functional equation (1.3) using the fixed point theory (see Theorem 3.2, Theorem 3.3, Theorem 3.4, and Theorem 3.5). In Section 4, we use the results in the previous sections to get a stability of Jensen's functional equation (1.1) and that of the quadratic functional equation (1.2), respectively.

2. Preliminaries

We recall the fundamental result in the fixed point theory.

Theorem 2.1 ([4] or [23]). *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $A : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(A^n x, A^{n+1}x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\}$$

or there exists a nonnegative integer k such that:
(1) $d(A^n x, A^{n+1}x) < +\infty$ *for all* $n \geq k$;
(2) *the sequence* $\{A^n x\}$ *is convergent to a fixed point* y^* *of* A;
(3) y^* *is the unique fixed point of* A *in* $Y := \{y \in X, d(A^k x, y) < +\infty\}$;
(4) $d(y, y^*) \leq (1/(1-L))d(y, Ay)$ *for all* $y \in Y$.

Throughout this paper, let V be a (real or complex) linear space and Y a Banach space. For a mapping $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$, we will introduce a generalized metric d_φ on the set $S := \{g : V \to Y \mid g(0) = 0\}$ by following

$$d_\varphi(g, h) := \inf \big\{K \in \mathbb{R}^+ \bigm| \|g(x) - h(x)\| \leq K\big(\varphi(x, -x) + \varphi(-x, x)\big) \text{ for all } x \in V\backslash\{0\}\big\}$$

for $g, h \in S$. It is easy to see that (S, d_φ) is complete.

Lemma 2.2. *Let* $0 < L < 1$ *and let* $\varphi, \tilde{\varphi} : (V\backslash\{0\})^2 \to [0, \infty)$ *satisfy*

$$\varphi(2x, 2y) \leq 2L\varphi(x, y), \tag{2.1}$$

$$L\tilde{\varphi}(2x, 2y) \geq 4\tilde{\varphi}(x, y) \tag{2.2}$$

for all $x, y \in V\backslash\{0\}$*, respectively. Consider the mappings* $A, \tilde{A} : S \to S$ *defined by*

$$Ag(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8},$$
$$\tilde{A}g(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and $x \in V$. Then A and $\tilde{A}$ are strictly contractive self mappings of S with the Lipschitz constant L with respect to the generalized metric d_φ and $d_{\tilde{\varphi}}$, respectively.

Proof. By the induction on $n \in \mathbb{N}$, we get

$$\begin{aligned} A^n g(x) &= \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}, \\ \tilde{A}^n g(x) &= 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right) \end{aligned}$$

for all $x \in V$. For any $g, h \in S$, let $d_\varphi(g,h) < K$. Then

$$\begin{aligned} \|Ag(x) - Ah(x)\| &\le \frac{3}{8}\left\|g(2x) - h(2x)\right\| + \frac{1}{8}\left\|g(-2x) - h(-2x)\right\| \\ &\le \frac{1}{2}K\left(\varphi(2x,-2x) + \varphi(-2x,2x)\right) \\ &\le LK\left(\varphi(x,-x) + \varphi(-x,x)\right) \end{aligned}$$

for all $x \in V \backslash \{0\}$. This implies that

$$d_\varphi(Ag, Ah) \le LK$$

as well as

$$d_\varphi(Ag, Ah) \le L d_\varphi(g,h)$$

for all $g, h \in S$. On the other hand, if $d_{\tilde{\varphi}}(g,h) < K$, then

$$\begin{aligned} \|\tilde{A}g(x) - \tilde{A}h(x)\| &\le 3\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\le 4K\left(\tilde{\varphi}\left(\frac{x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(-\frac{x}{2}, \frac{x}{2}\right)\right) \\ &\le LK\left(\tilde{\varphi}(x,-x) + \tilde{\varphi}(-x,x)\right) \end{aligned}$$

for all $x \in V \backslash \{0\}$. So we also have

$$d_{\tilde{\varphi}}(\tilde{A}g, \tilde{A}h) \le L d_{\tilde{\varphi}}(g,h)$$

for all $g, h \in S$. □

3. The stability of Eq.(1.3)

In this section, we consider stability results of the functional equation (1.3). For a given mapping $f : V \to Y$, we use the following abbreviation

$$Df(x,y) := \sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix+y)$$

for all $x, y \in V$.

Lemma 3.1. *If $f : V \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in V \backslash \{0\}$, then f is a general quadratic mapping.*

Proof. Notice that we get

$$\begin{aligned} 3Df(x,0) &= -Df(x,x) + Df(2x,-2x) - 3Df(x,-x) - Df(x,-2x) \\ &= 0 \end{aligned}$$

for all $x \in V \setminus \{0\}$. Moreover, we have

$$Df(0,y) = \sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(y) = 0$$

for all $y \in V$. So we can say that $Df(x,y) = 0$ for all $x, y \in V$. □

Now we can prove the stability of the functional equation $Df \equiv 0$ using the fixed point theory.

Theorem 3.2. *Let* $\varphi : (V\backslash\{0\})^2 \to [0,\infty)$ *satisfy the condition* (2.1) *for given* $0 < L < 1$ *and let* $f : V \to Y$. *If*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.1}$$

for all $x, y \in V\backslash\{0\}$, *then there exists a unique general quadratic mapping* $F : V \to Y$ *such that*

$$\|f(x) - F(x)\| \le \frac{3}{8(1-L)} \left(\varphi(x,-x) + \varphi(-x,x)\right) \tag{3.2}$$

for all $x \in V\backslash\{0\}$. *In particular, the mapping* F *is represented by*

$$F(x) = \lim_{n\to\infty} \left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right) + f(0) \tag{3.3}$$

for all $x \in V$. *Moreover, if* $0 < L < \frac{1}{2}$ *and* φ *is continuous, then* $f \equiv F$, *i.e.,* f *is itself a general quadratic mapping.*

Proof. Consider the mapping $\tilde{f} : V \to Y$ such that $\tilde{f}(x) = f(x) - f(0)$ for all $x \in V$. Then $\tilde{f}(0) = 0$ and

$$D\tilde{f}(x,y) = Df(x,y)$$

for all $x, y \in V\backslash\{0\}$. Notice that, if we consider the mapping A in Lemma 2.2, then we have

$$\begin{aligned} \|\tilde{f}(x) - A\tilde{f}(x)\| &= \frac{1}{8} \left\| -3D\tilde{f}(x,-x) - D\tilde{f}(-x,x) \right\| \\ &\le \frac{3}{8} \left(\varphi(x,-x) + \varphi(-x,x)\right) \end{aligned}$$

for all $x \in V\backslash\{0\}$, i.e., $d_\varphi\left(\tilde{f}, A\tilde{f}\right) \le \frac{3}{8} < \infty$. By Lemma 2.2, this implies that

$$d_\varphi\left(A^n\tilde{f}, A^{n+1}\tilde{f}\right) < \infty$$

for all $n \geq 0$. So we can apply (2) and (3) of Theorem 2.1 to get a unique fixed point $\tilde{F} : V \to Y$ of the strictly contractive mapping A, which is defined by

$$\tilde{F}(x) := \lim_{n\to\infty} A^n\tilde{f}(x) = \lim_{n\to\infty}\left(\frac{\tilde{f}(2^n x)+\tilde{f}(-2^n x)}{2\cdot 4^n}+\frac{\tilde{f}(2^n x)-\tilde{f}(-2^n x)}{2^{n+1}}\right) \tag{3.4}$$

for all $x \in V$. Since

$$d_\varphi(\tilde{f},\tilde{F}) \leq \frac{1}{1-L} d_\varphi(\tilde{f}, A\tilde{f}) \leq \frac{3}{8(1-L)}$$

we have

$$\|\tilde{F}(x) - \tilde{f}(x)\| \leq \frac{3}{8(1-L)}\left(\varphi(x,-x) + \varphi(-x,x)\right) \tag{3.5}$$

for all $x \in V\backslash\{0\}$. Replacing x by $2^n x$ and y by $2^n y$ in (3.1), we obtain

$$\begin{aligned}\|DA^n\tilde{f}(x,y)\| &\leq \frac{1}{2^{n+1}}\left(\left\|D\tilde{f}(2^n x, 2^n y)\right\| + \left\|D\tilde{f}(-2^n x, -, 2^n y)\right\|\right)\\ &\quad + \frac{1}{2\cdot 4^n}\left(\left\|D\tilde{f}(2^n x, 2^n y)\right\| + \left\|D\tilde{f}(-2^n x, -2^n y)\right\|\right)\\ &\leq \left(\frac{1}{2^{n+1}} + \frac{1}{2\cdot 4^n}\right)\left(\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)\right)\\ &\leq \left(\frac{1}{2^{n+1}} + \frac{1}{2\cdot 4^n}\right) 2^n L^n \left(\varphi(x,y) + \varphi(-x,-y)\right).\end{aligned}$$

The right hand side tends to 0 as $n \to \infty$, since $0 < L < 1$. This implies that $D\tilde{F}(x,y) = 0$ for all $x, y \in V\backslash\{0\}$. From Lemma 3.1, we also have

$$D\tilde{F}(x,y) = 0$$

for all $x, y \in V$. Put $F = \tilde{F} + f(0)$. Then (3.2) and (3.3) follow from (3.5) and (3.4), respectively. Now let $0 < L < \frac{1}{2}$ and φ be continuous. Then we get

$$\begin{aligned}\lim_{n\to\infty}\varphi((a\cdot 2^n + b)x, (c\cdot 2^n + d)y) &\leq \lim_{n\to\infty}\left((2L)^n\varphi\left(\frac{a\cdot 2^n + b}{2^n}x, \frac{c\cdot 2^n + d}{2^n}y\right)\right)\\ &= 0\cdot\varphi(ax, cy) = 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$ and for any fixed integers a, b, c, d with $a, c \neq 0$. Therefore, we obtain

$$\begin{aligned}3\|f(x) - F(x)\| \leq &\lim_{n\to\infty}\big(\|Df\left((2^n+1)x, -2^n x\right) - DF\left((2^n+1)\,x, -2^n x\right)\|\\ &+ \left\|(F-f)\left(\left(2^{n+1}+3\right)x\right)\right\| + 3\left\|(f-F)\left((2^n+2)\,x\right)\right\|\\ &+ \left\|(f-F)\left(-2^n x\right)\right\|\big)\\ \leq &\lim_{n\to\infty}\varphi\left((2^n+1)\,x, -2^n x\right)\\ &+ \frac{3}{8(1-L)}\lim_{n\to\infty}\big(\varphi\left(-2^n x, 2^n x\right) + \varphi\left(2^n x, -2^n x\right)\\ &+ 3\varphi\left((2^n+2)\,x, -(2^n+2)x\right) + 3\varphi\left(-\left(2^n+2\right)x, \left(2^n+2\right)x\right)\end{aligned}$$

$$
\begin{aligned}
&+\varphi\left(\left(2^{n+1}+3\right)x,-\left(2^{n+1}+3\right)x\right)\\
&+\varphi\left(-\left(2^{n+1}+3\right)x,\left(2^{n+1}+3\right)x\right)\big)\\
=\ &0
\end{aligned}
$$

for all $x \in V\backslash\{0\}$. Since $f(0)=F(0)$, we have shown that $f \equiv F$. □

Theorem 3.3. *Suppose that $f : V \to Y$ satisfies the inequality* (3.1) *for all $x, y \in V\backslash\{0\}$, where φ has the property* (2.2) *with $0 < L < 1$. Then there exists a unique general quadratic mapping $F : V \to Y$ such that*

$$
\|f(x)-F(x)\| \le \frac{L}{4(1-L)}\big(\varphi(x,-x)+\varphi(-x,x)\big) \tag{3.6}
$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$
\begin{aligned}
F(x)=\lim_{n\to\infty}\Bigg(&2^{n-1}\left(f\left(\frac{x}{2^n}\right)-f\left(\frac{-x}{2^n}\right)\right)\\
&+\frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{-x}{2^n}\right)-2f(0)\right)\Bigg)+f(0)
\end{aligned} \tag{3.7}
$$

for all $x \in V$.

Proof. Let $\tilde f = f - f(0)$. Then $\tilde f$ satisfies (3.1), $\tilde f(0)=0$, and $D\tilde f = Df$. If we consider the mapping $\tilde A$ as in Lemma 2.2, then we see that

$$
\begin{aligned}
\left\|\tilde f(x)-\tilde A\tilde f(x)\right\| &= \left\|D\tilde f\left(\frac{x}{2},-\frac{x}{2}\right)\right\|\\
&\le \varphi\left(\frac{x}{2},-\frac{x}{2}\right)+\varphi\left(-\frac{x}{2},\frac{x}{2}\right)\\
&\le \frac{L}{4}\big(\varphi(x,-x)+\varphi(-x,x)\big)
\end{aligned}
$$

for all $x \in V\backslash\{0\}$, which implies that $d_\varphi(\tilde f,\tilde A\tilde f) \le \frac{L}{4} < \infty$. By Lemma 2.2, we get

$$
d_\varphi\left(\tilde A^n\tilde f,\tilde A^{n+1}\tilde f\right)<\infty
$$

for all $n \ge 0$. We can apply (2) and (3) of Theorem 2.1 to get a unique fixed point $\tilde F : V \to Y$ of the strictly contractive mapping $\tilde A$, which is defined by

$$
\begin{aligned}
\tilde F(x) &:= \lim_{n\to\infty}\tilde A^n\tilde f(x)\\
&= \lim_{n\to\infty}\left(2^{n-1}\left(\tilde f\left(\frac{x}{2^n}\right)-\tilde f\left(-\frac{x}{2^n}\right)\right)+\frac{4^n}{2}\left(\tilde f\left(\frac{x}{2^n}\right)+\tilde f\left(-\frac{x}{2^n}\right)\right)\right)
\end{aligned} \tag{3.8}
$$

for all $x \in V$. Moreover, we can say that

$$
d_\varphi(\tilde f,\tilde F) \le \frac{1}{1-L}d_\varphi(\tilde f,A\tilde f) \le \frac{L}{4(1-L)}
$$

that is

$$
\|\tilde F(x)-\tilde f(x)\| \le \frac{L}{4(1-L)}\big(\varphi(x,-x)+\varphi(-x,x)\big) \tag{3.9}
$$

for all $x \in V\backslash\{0\}$. Replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (3.1), we obtain

$$\begin{aligned}\|D\tilde{A}^n\tilde{f}(x,y)\| &= \Big\|2^{n-1}\left(D\tilde{f}\left(\frac{x}{2^n},\frac{y}{2^n}\right) - D\tilde{f}\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right)\\ &\quad + \frac{4^n}{2}\left(D\tilde{f}\left(\frac{x}{2^n},\frac{y}{2^n}\right) + D\tilde{f}\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right)\Big\|\\ &\le \left(2^{n-1}+\frac{4^n}{2}\right)\left(\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)+\varphi\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right)\\ &\le \frac{L^n}{4^n}\left(2^{n-1}+\frac{4^n}{2}\right)(\varphi(x,y)+\varphi(-x,-y))\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. In a similar way of the proof of Theorem 3.2, this implies that

$$D\tilde{F}(x,y) = 0$$

for all $x, y \in V$. Put $F = \tilde{F} + f(0)$. Then (3.6) and (3.7) follow from (3.9) and (3.8), respectively, too. Since the uniqueness of F is clear in the fixed point theory, we have proved this theorem. □

Theorem 3.4. *Let $\varphi : (V\backslash\{0\})^2 \to [0,\infty)$ satisfy the condition* (2.1) *for given $0 < L < 1$ with $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V\backslash\{0\}$. If f satisfies the inequality* (3.1) *for all $x, y \in V\backslash\{0\}$, then there exists a unique general quadratic mapping $F : V \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{1}{2(1-L)}\varphi(x,-x)$$

for all $x \in V\backslash\{0\}$. Moreover, if $0 < L < \frac{1}{2}$ and φ is continuous, then $f \equiv F$, i.e., f is itself a general quadratic mapping.

Proof. If we consider the mappings A, $\tilde{f}$ and $\tilde{F}$ in Theorem 3.2, then we have

$$\begin{aligned}\|\tilde{f}(x) - A\tilde{f}(x)\| &= \left\|-\frac{3}{8}D\tilde{f}(x,-x) + \frac{1}{8}D\tilde{f}(-x,x)\right\|\\ &\le \frac{1}{4}(\varphi(x,-x)+\varphi(-x,x))\end{aligned}$$

and so

$$\|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\| = \frac{1}{1-L}\|\tilde{f}(x) - A\tilde{f}(x)\| \le \frac{\varphi(x,-x)}{2(1-L)}$$

for all $x \in V\backslash\{0\}$. □

Theorem 3.5. *Suppose that $f : V \to Y$ satisfies the inequality* (3.1) *for all $x, y \in V\backslash\{0\}$, where φ has the property* (2.2) *with $0 < L < 1$. If $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V\backslash\{0\}$, then there exists a unique general quadratic mapping $F : V \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{L}{4(1-L)}\varphi(x,-x)$$

for all $x \in V\backslash\{0\}$.

Proof. If we consider the mappings $\tilde{A}$, $\tilde{f}$, and $\tilde{F}$ in Theorem 3.3, then we have

$$\left\|\tilde{f}(x)-\tilde{A}\tilde{f}(x)\right\| = \left\|D\tilde{f}\left(\frac{x}{2},-\frac{x}{2}\right)\right\| \leq \frac{L}{8}\big(\varphi(x,-x)+\varphi(-x,x)\big)$$

and so

$$\|f(x)-F(x)\| = \|\tilde{f}(x)-\tilde{F}(x)\| = \frac{1}{1-L}\|\tilde{f}(x)-\tilde{A}\tilde{f}(x)\| \leq \frac{L\varphi(x,-x)}{4(1-L)}$$

for all $x \in V\backslash\{0\}$. □

Now we obtain the Hyers-Ulam stability results in the framework of normed spaces using Theorem 3.4 and Theorem 3.5.

Corollary 3.6. *Let X be a normed space and Y a Banach space. Suppose that, for $\theta \geq 0$ and $p \in \mathbb{R}\backslash[1,2]$, the mapping $f : X \to Y$ satisfies an inequality of the form*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general quadratic mapping $F : X \to Y$ such that

$$\|f(x)-F(x)\| \leq \begin{cases} \frac{2\theta}{2-2^p}\|x\|^p & \text{if } 0 \leq p < 1 \\ \frac{2\theta}{2^p-4}\|x\|^p & \text{if } p > 2 \end{cases}$$

for all $x \in X\backslash\{0\}$ and f is itself a general quadratic mapping if $p < 0$.

Proof. Let $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X\backslash\{0\}$. If $p < 1$, then φ satisfies (2.1) with $L = 2^{p-1} < 1$. In particular, if $p < 0$, then $0 < L < \frac{1}{2}$ and it is clear that φ is continuous on $(X\backslash\{0\})^2$. On the other hand if $p > 2$, then φ satisfies (2.2) with $L = 2^{2-p} < 1$. So we can prove this corollary by using Theorem 3.4 and Theorem 3.5, respectively. □

Corollary 3.7. *Let X be a normed space and Y a Banach space. Suppose that, for $\theta \geq 0$ and $p+q \in \mathbb{R}\backslash[1,2]$, the mapping $f : X \to Y$ satisfies an inequality of the form*

$$\|Df(x,y)\| \leq \theta\|x\|^p\|y\|^q$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general quadratic mapping $F : X \to Y$ such that

$$\|f(x)-F(x)\| \leq \begin{cases} \frac{\theta}{2-2^{p+q}}\|x\|^{p+q} & \text{if } 0 \leq p+q < 1 \\ \frac{\theta}{2^{p+q}-4}\|x\|^{p+q} & \text{if } p+q > 2 \end{cases}$$

for all $x \in X\backslash\{0\}$ and f is itself a general quadratic mapping if $p+q < 0$.

Proof. Let $\varphi(x,y) := \theta\|x\|^p\|y\|^q$ for all $x, y \in X\backslash\{0\}$. If $p+q < 1$, then φ satisfies (2.1) with $L = 2^{p+q-1} < 1$. In particular, if $p+q < 0$, then $0 < L < \frac{1}{2}$ and it is clear that φ is continuous on $(X\backslash\{0\})^2$. Therefore we can prove the corollary in this case by Theorem 3.4. On the other hand, if $p+q > 2$, then φ satisfies (2.2) with $L = 2^{2-p+q} < 1$. By Theorem 3.5, the proof of this corollary completes. □

4. Applications to Jensen's functional equation and the quadratic functional equation

For a given mapping $f : V \to Y$, we use the following abbreviations

$$Jf(x, y) := 2f\left(\frac{x+y}{2}\right) - f(x) - f(y),$$
$$Qf(x, y) := f(x+y) + f(x-y) - 2f(x) - 2f(y)$$

for all $x, y \in V$. Using the previous results, we can prove stability results about Jensen's functional equation $Jf \equiv 0$ and the quadratic functional equation $Qf \equiv 0$ by followings.

Corollary 4.1. *Let $\psi_i : V^2 \to [0, \infty)$, $i = 1, 2$, be given functions. Suppose that for each $i = 1, 2$, $f_i : V \to Y$ satisfies*

$$\|Jf_i(x, y)\| \le \psi_i(x, y) \tag{4.1}$$

for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that ψ_1 has the property (2.1) and ψ_2 satisfies (2.2) for all $x, y \in V$, then there exist unique Jensen mappings $F_i : V \to Y$, $i = 1, 2$, such that

$$\|f_1(x) - F_1(x)\| \le \frac{3\Psi_1(x)}{8(1-L)}, \tag{4.2}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L}{4(1-L)}\Psi_2(x) \tag{4.3}$$

for all $x \in V$, where $\Psi_i(x) := \psi_i(x, -x) + \psi_i(2x, 0) + \psi_i(-x, x) + \psi_i(-2x, 0)$. In particular, the desired mappings F_1, F_2 are represented by

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{2^n} + f_1(0), \tag{4.4}$$

$$F_2(x) = \lim_{n\to\infty} 2^n \left(f_2\left(\frac{x}{2^n}\right) - f_2(0)\right) + f_2(0) \tag{4.5}$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{2}$ and ψ_1 is continuous, then f_1 is itself a Jensen mapping.

Proof. Notice that for $f_i : V \to Y$, $i = 1, 2$, we have

$$\begin{aligned}\|Df_i(x, y)\| &= \|Jf_i(2x + y, y) - Jf_i(3x + y, x + y)\| \\ &\le \psi_i(2x + y, y) + \psi_i(3x + y, x + y)\end{aligned}$$

for all $x, y \in V$. Put $\varphi_i(x, y) := \psi_i(2x + y, y) + \psi_i(3x + y, x + y)$, $i = 1, 2$, for all $x, y \in V$, then φ_1 satisfies (2.1) and φ_2 satisfies (2.2). Observe that $\|Df_i(x, y)\| \le \varphi_i(x, y)$, $i = 1, 2$, for all $x, y \in V$, respectively. According to Theorem 3.2, we can take the unique general quadratic mapping F_1 by

$$F_1(x) := \lim_{n\to\infty} \left(\frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}}\right) + f_1(0) \tag{4.6}$$

which satisfies (4.2) clearly. Observe that

$$\begin{aligned}\left\|\frac{f_1(2^n x)+f_1(-2^n x)-2f_1(0)}{2^{n+1}}\right\| &= \frac{1}{2^{n+1}}\|Jf_1(2^n x,-2^n x)\| \\ &\le \frac{1}{2^{n+1}}\psi_1(2^n x,-2^n x) \\ &\le \frac{L^n}{2}\psi_1(x,-x)\end{aligned}$$

for all $x \in V$. Letting $n \to \infty$, we get

$$\lim_{n\to\infty}\frac{f_1(2^n x)+f_1(-2^n x)}{2^{n+1}}=0$$

for all $x, y \in V$. Together with (4.6), this implies (4.4). Notice that

$$\left\|\frac{Jf_1(2^n x,2^n y)}{2^n}\right\| \le \frac{\psi_1(2^n x,2^n y)}{2^n} \le L^n\psi_1(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$, we obtain

$$JF_1(x,y)=0$$

for all $x, y \in V$. In particular, consider the case $0 < L < \frac{1}{2}$ and ψ_1 is continuous. Then φ_1 is continuous on $(V\backslash\{0\})^2$ and we can say that $f_1 \equiv F_1$ by Theorem 3.2. On the other hand, according Theorem 3.3, we can get

$$\begin{aligned}F_2(x) := \lim_{n\to\infty}\Bigg(&2^{n-1}\left(f_2\left(\frac{x}{2^n}\right)-f_2\left(\frac{-x}{2^n}\right)\right) \\ &+\frac{4^n}{2}\left(f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)-2f_2(0)\right)\Bigg)+f_2(0)\end{aligned} \tag{4.7}$$

which is the unique general quadratic mapping satisfying (4.3). From (4.1) and (2.2), we have

$$\begin{aligned}\lim_{n\to\infty}2^{2n-1}\left\|f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)-2f_2(0)\right\| &= \lim_{n\to\infty}2^{2n-1}\left\|Jf_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty}2^{2n-1}\psi_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\psi_2(x,-x) \\ &= 0\end{aligned}$$

as well as

$$\lim_{n\to\infty}2^{n-1}\left(f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)-2f_2(0)\right)=0$$

for all $x \in V$. So we get (4.5) following (4.7). Observe that

$$\left\|2^n Jf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 2^n\psi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le \frac{L^n}{2^n}\psi_2(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$, then we get

$$JF_2(x,y)=0$$

for all $x, y \in V$. $\square$

Corollary 4.2. *Let $\psi_i : V^2 \to [0,\infty)$, $i = 1,2$, be given functions. Suppose that each $f_i : V \to Y$, $i = 1,2$, satisfies*

$$\|Qf_i(x,y)\| \le \psi_i(x,y) \tag{4.8}$$

for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that the mappings ψ_1 and ψ_2 have the property (2.1) and (2.2) for all $x, y \in V$, respectively, then there exist unique quadratic mappings $F_1, F_2 : V \to Y$ such that

$$\|f_1(x) - f_1(0) - F_1(x)\| \le \frac{3\Psi_1(x)}{8(1-L)}, \tag{4.9}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L}{4(1-L)}\Psi_2(x) \tag{4.10}$$

for all $x \in V \backslash \{0\}$, where

$$\begin{aligned}\Psi_i(x) := \psi_i(x,x) + \psi_i(x,0) + 2\psi_i(0,0) + \psi_i(0,-x)\\ + \psi_i(-x,-x) + \psi_i(-x,0) + \psi_i(0,x),\end{aligned}$$

respectively. In particular, the desired mappings F_1 and F_2 are represented by

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{4^n}, \tag{4.11}$$

$$F_2(x) = \lim_{n\to\infty} 4^n f_2\left(\frac{x}{2^n}\right) \tag{4.12}$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{2}$ and ψ_1 is continuous, then $f_1 - f_1(0)$ is itself a quadratic mapping.

Proof. Notice that

$$\begin{aligned}\|Df_i(x,y)\| &= \|Qf_i(x,2x+y) - Qf_i(x,x+y) + Qf_i(0,y) - Qf_i(0,x+y)\|\\ &\le \psi_i(x,2x+y) + \psi_i(x,x+y) + \psi_i(0,y) + \psi_i(0,x+y)\end{aligned}$$

for all $x, y \in V$, $i = 1,2$. Put

$$\varphi_i(x,y) := \psi_i(x,2x+y) + \psi_i(x,x+y) + \psi_i(0,y) + \psi_i(0,x+y)$$

for all $x, y \in V$, then $\|Df_i(x,y)\| \le \varphi_i(x,y)$ for all $x, y \in V$, respectively. Moreover, φ_1 satisfies (2.1) and φ_2 holds (2.2). Therefore, according to Theorem 3.2, there exists a unique mapping $F_1 : V \to Y$ satisfying (4.9), which is represented by

$$F_1(x) := \lim_{n\to\infty}\left(\frac{f_1(2^n x) + f_1(-2^n x)}{2\cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}}\right).$$

Observe that

$$\lim_{n\to\infty}\left\|\frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}}\right\| = \lim_{n\to\infty}\frac{1}{2^{n+1}}\|Qf_1(0,2^n x)\|$$

$$\begin{aligned}
&\le \lim_{n\to\infty} \frac{1}{2^{n+1}}\psi_1(0,2^n x) \\
&\le \lim_{n\to\infty} \frac{L^n}{2}\psi_1(0,x) \\
&= 0
\end{aligned}$$

and

$$\lim_{n\to\infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2\cdot 4^n} \right\| \le \lim_{n\to\infty} \frac{L^n}{2^{n+1}}\psi_1(0,x) = 0$$

for all $x \in V$. From this, we get (4.11). Notice that

$$\left\| \frac{Qf_1(2^n x, 2^n y)}{4^n} \right\| \le \frac{\psi_1(2^n x, 2^n y)}{4^n} \le \frac{L^n}{2^n}\psi_1(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_1(x,y) = 0$$

for all $x, y \in V$. If $0 < L < \frac{1}{2}$ and ψ_1 is continuous, then φ_1 is also continuous on $(V\backslash\{0\})^2$ and we can say that $f_1 - f_1(0) \equiv F_1$ by Theorem 3.2. On the other hand, since $L\psi_2(0,0) \ge 4\psi_2(0,0)$ and $\|2f_2(0)\| = \|Qf_2(0,0)\| \le \psi_2(0,0)$, we can show that $\psi_2(0,0) = 0$ and $f_2(0) = 0$. According to Theorem 3.3, there exists a unique mapping $F_2 : V \to Y$ satisfying (4.10), which is represented by (4.7). We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{4^n}{2} \left\| -f_2\left(\frac{x}{2^n}\right) + f_2\left(-\frac{x}{2^n}\right) \right\| &= \lim_{n\to\infty} \frac{4^n}{2} \left\| Qf_2\left(0, \frac{x}{2^n}\right) \right\| \\
&\le \lim_{n\to\infty} \frac{4^n}{2}\psi_2\left(0, \frac{x}{2^n}\right) \\
&\le \lim_{n\to\infty} \frac{L^n}{2}\psi_2(0,x) = 0
\end{aligned}$$

as well as

$$\lim_{n\to\infty} 2^{n-1} \left\| f_2\left(\frac{x}{2^n}\right) - f_2\left(-\frac{x}{2^n}\right) \right\| = 0$$

for all $x \in V$. From these and (4.7), we get (4.12). Notice that

$$\left\| 4^n Qf_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \le 4^n \psi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le L^n \psi_2(x,y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$, we have shown that

$$QF_2(x,y) = 0$$

for all $x, y \in V$. □

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