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A COUNTEREXAMPLE FOR IMPROVED SOBOLEV INEQUALITIES OVER THE 2-ADIC GROUP

Diego Chamorro

Abstract. On the framework of the 2-adic group $\mathbb{Z}_2$, we study a Sobolev-like inequality where we estimate the L^2 norm by a geometric mean of the BV norm and the $\dot{B}^{-1,\infty}_\infty$ norm. We first show, using the special topological properties of the p-adic groups, that the set of functions of bounded variations BV can be identified to the Besov space $\dot{B}^{1,\infty}_1$. This identification lead us to the construction of a counterexample to the improved Sobolev inequality.

1. Introduction

The general improved Sobolev inequalities were initially introduced by P. Grard, Y. Meyer and F. Oru in [6]. For a function f such that $f \in \dot{W}^{s_1,p}(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_\infty(\mathbb{R}^n)$, these inequalities read as follows:

$$\|f\|_{\dot{W}^{s,q}} \leq C\|f\|^{\theta}_{\dot{W}^{s_1,p}}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_\infty}, \tag{1}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1-\theta)\beta$ and $-\beta < s < s_1$. The method used for proving these estimates relies on the Littlewood-Paley decomposition and on a dyadic bloc manipulation and this explains the fact that the value $p = 1$ is forbidden here.

In order to study the case $p = 1$, it is necessary to develop other techniques. The case when $p = 1$, $s = 0$ and $s_1 = 1$ was treated by M. Ledoux in [11] using a special cut-off function; while the case $s_1 = 1$ and $p = 1$ was studied by A. Cohen, W. Dahmen, I. Daubechies and R. De Vore in [5]. In this last article, the authors give a BV-norm weak estimation using wavelet coefficients and isoperimetric inequalities and obtained, for a function f such that $f \in BV(\mathbb{R}^n)$ and $f \in \dot{B}^{-\beta,\infty}_\infty(\mathbb{R}^n)$, the estimation below:

$$\|f\|_{\dot{W}^{s,q}} \leq C\|f\|^{1/q}_{BV}\|f\|^{1-1/q}_{\dot{B}^{-\beta,\infty}_\infty}, \tag{2}$$

where $1 < q \leq 2$, $0 \leq s < 1/q$ and $\beta = (1-sq)/(q-1)$.

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In a previous work (see [3], [4]), we studied the possible generalizations of inequalities of type (1) and (2) to other frameworks than $\mathbb{R}^n$. In particular, we worked over stratified Lie groups and over polynomial volume growth Lie groups and we obtained some new weak-type estimates.

The aim of this paper is to study inequalities of type (1) and (2) in the setting of the 2-adic group $\mathbb{Z}_2$. The main reason for working in the framework of $\mathbb{Z}_2$ is that this group is completely different from $\mathbb{R}^n$ and from stratified or polynomial Lie groups. Indeed, since the 2-adic group is totally discontinuous, it is not absolutely trivial to give a definition for smoothness measuring spaces. Thus, the first step to do, in order to study these Sobolev-like inequalities, is to give an adapted characterization of such functional spaces. In the present article, this will be achieved using the Littlewood-Paley approach and, once this task is done, we will immediatly prove -following the classical path exposed in [6]- the inequalities (1) in the setting of the 2-adic group $\mathbb{Z}_2$.

For the estimate (2), we introduce the BV space in the following manner: we will say that $f \in BV(\mathbb{Z}_2)$ if there exists a constant $C > 0$ such that

$$\int_{\mathbb{Z}_2} |f(x+y) - f(x)|dx \leq C|y|_2 \quad (\forall y \in \mathbb{Z}_2).$$

As a surprising fact, we obtain the follwing.

Theorem 1. *We have the following relationship between the space of functions of bounded variation $BV(\mathbb{Z}_2)$ and the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$:*

$$BV(\mathbb{Z}_2) \simeq \dot{B}_1^{1,\infty}(\mathbb{Z}_2).$$

Of course, this identification is false in $\mathbb{R}^n$ and it is this special relationship in $\mathbb{Z}_2$ that give us our principal theorem which is the 2-adic counterpart of the inequality (2):

Theorem 2. *The following inequality is false in $\mathbb{Z}_2$. There is not an universal constant $C > 0$ such that we have*

$$\|f\|_{L^2}^2 \leq C\|f\|_{BV}\|f\|_{\dot{B}_\infty^{-1,\infty}}$$

for all $f \in BV \cap \dot{B}_\infty^{-1,\infty}(\mathbb{Z}_2)$.

This striking fact says that the improved Sobolev inequalities of type (2) depend on the group's structure and that they are no longer true for the 2-adic group $\mathbb{Z}_2$.

The plan of the article is the following: in Section 2 we recall some well known properties about p-adic groups, in Section 3 we define Sobolev and Besov spaces, in Section 4 we prove Theorem 1 and, finally, we prove Theorem 2 in Section 5.

Before finishing these preliminary remarks, it is important to say that the inequality (1) was generalized in [13] where the Besov space $\dot{B}_\infty^{-\beta,\infty}$ is replaced by the BMO space. Thus, with the study of p-adic BMO functions given in

[7] and [8] it would be interesting to investigate if such generalization is still valid in a p-adic setting.

2. p-adic groups

We write $a|b$ when a divide b or, equivalently, when b is a multiple of a. Let p be any prime number, for $0 \neq x \in \mathbb{Z}$, we define the p-adic valuation of x by $\gamma(x) = \max\{r : p^r|x\} \geq 0$ and, for any rational number $x = \frac{a}{b} \in \mathbb{Q}$, we write $\gamma(x) = \gamma(a) - \gamma(b)$. Furthermore if $x = 0$, we agree to write $\gamma(0) = +\infty$.

Let $x \in \mathbb{Q}$ and p be any prime number, with the p-adic valuation of x we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if} \quad x \neq 0 \\ \\ p^{-\infty} = 0 & \text{if} \quad x = 0. \end{cases} \tag{3}$$

This expression satisfy the following properties

a) $|x|_p \geq 0$, and $|x|_p = 0 \iff x = 0$;
b) $|xy|_p = |x|_p|y|_p$;
c) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

When a norm satisfy *c*) it is called a non-Archimedean norm and an interesting fact is that over $\mathbb{Q}$ *all* the possible norms are equivalent to $|\cdot|_p$ for some p: this is the so-called Ostrowski theorem (see [1] for a proof).

Definition 2.1. Let p be any prime number. We define the field of p-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ when using the norm $|\cdot|_p$.

We present in the following lines the algebraic structure of the set $\mathbb{Q}_p$. Every p-adic number $x \neq 0$ can be represented in a unique manner by the formula

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots), \tag{4}$$

where $\gamma = \gamma(x)$ is the p-adic valuation of x and x_j are integers such that $x_0 > 0$ and $0 \leq x_j \leq p - 1$ for $j = 1, 2, \ldots$. Remark that this canonical representation implies the identity $|x|_p = p^{-\gamma}$.

Let $x, y \in \mathbb{Q}_p$, using the formula (4) we define the sum of x and y by $x + y = p^{\gamma(x+y)}(c_0 + c_1 p + c_2 p^2 + \cdots)$ with $0 \leq c_j \leq p - 1$ and $c_0 > 0$, where $\gamma(x + y)$ and c_j are the unique solution of the equation

$$\begin{aligned} &p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots) + p^{\gamma(y)}(y_0 + y_1 p + y_2 p^2 + \cdots) \\ &= p^{\gamma(x+y)}(c_0 + c_1 p + c_2 p^2 + \cdots). \end{aligned}$$

Furthermore, for $a, x \in \mathbb{Q}_p$, the equation $a + x = 0$ has a unique solution in $\mathbb{Q}_p$ given by $x = -a$. In the same way, the equation $ax = 1$ has a unique solution in $\mathbb{Q}_p$: $x = 1/a$.

We take now a closer look at the topological structure of $\mathbb{Q}_p$. With the norm $|\cdot|_p$ we construct a distance over $\mathbb{Q}_p$ by writing

$$d(x, y) = |x - y|_p \tag{5}$$

and we define the balls $B_\gamma(x) = \{y \in \mathbb{Q}_p : d(x,y) \le p^\gamma\}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the p-adic valuation, this distance has the *ultra-metric* property (*i.e.,* $d(x,y) \le \max\{d(x,z), d(z,y)\} \le |x|_p + |y|_p$).

We gather with the next proposition some important facts concerning the balls in $\mathbb{Q}_p$.

Proposition 2.1. *Let γ be an integer, then we have*

1) *the ball $B_\gamma(x)$ is an open and a closed set for the distance* (5).
2) *every point of $B_\gamma(x)$ is its center.*
3) *$\mathbb{Q}_p$ endowed with this distance is a complete Hausdorff metric space.*
4) *$\mathbb{Q}_p$ is a locally compact set.*
5) *the p-adic group $\mathbb{Q}_p$ is a totally discontinuous space.*

For a proof of this proposition and more details see the books [1], [10] or [14].

3. Functional spaces

In this article, we will work with the subset $\mathbb{Z}_2$ of $\mathbb{Q}_2$ which is defined by $\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : |x|_2 \le 1\}$, and we will focus on real-valued functions over $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a locally compact commutative group, there exists a Haar measure dx which is translation invariant *i.e.*, $d(x+a) = dx$, furthermore we have the identity $d(xa) = |a|_2 dx$ for $a \in \mathbb{Z}_2^*$. We will normalize the measure dx by setting

$$\int_{\{|x|_2 \le 1\}} dx = 1.$$

This measure is then unique and we will note $|E|$ the measure for any subset E of $\mathbb{Z}_2$.

Another type of measures can be considered on the p-adic setting (see for example [9]). However, in this article we will only work with the previous one.

Lebesgue spaces $L^p(\mathbb{Z}_2)$ are thus defined in a natural way:
$\|f\|_{L^p} = (\int_{\mathbb{Z}_2} |f(x)|^p dx)^{1/p}$ for $1 \le p < +\infty$, with the usual modifications when $p = +\infty$.

Let us now introduce the Littlewood-Paley decomposition in $\mathbb{Z}_2$. We note $\mathcal{F}_j$ the Boole algebra formed by the equivalence classes $E \subset \mathbb{Z}_2$ modulo the subgroup $2^j\mathbb{Z}_2$. Then, for any function $f \in L^1(\mathbb{Z}_2)$, we call $S_j(f)$ the conditionnal expectation of f with respect to $\mathcal{F}_j$:

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y)dy.$$

The dyadic blocks are thus defined by the formula $\Delta_j(f) = S_{j+1}(f) - S_j(f)$ and the Littlewood-Paley decomposition of a function $f : \mathbb{Z}_2 \longrightarrow \mathbb{R}$ is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \qquad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x)dx. \tag{6}$$

We will need in the sequel some very special sets noted $Q_{j,k}$. Here is the definition and some properties:

Proposition 3.1. *Let $j \in \mathbb{N}$ and $k = \{0, 1, \dots, 2^j - 1\}$. Define the subset $Q_{j,k}$ of $\mathbb{Z}_2$ by*

$$Q_{j,k} = \left\{k + 2^j\mathbb{Z}_2\right\}. \tag{7}$$

Then

1) *We have the identity $\mathcal{F}_j = \bigcup_{0\leq k<2^j} Q_{j,k}$,*
2) *For $k = \{0, 1, \dots, 2^j - 1\}$ the sets $Q_{j,k}$ are mutually disjoint,*
3) *$|Q_{j,k}| = 2^{-j}$ for all k,*
4) *the 2-adic valuation is constant over $Q_{j,k}$.*

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (6), we obtain the following equivalence for the Lebesgue spaces $L^p(\mathbb{Z}_2)$ with $1 < p < +\infty$:

$$\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left(\sum_{j\in\mathbb{N}} |\Delta_j f|^2\right)^{1/2} \right\|_{L^p}.$$

See the book [12], Chapter IV, for a general proof.

Let us turn now to smoothness measuring spaces. As said in the introduction, it is not absolutely trivial to define Sobolev and Besov spaces over $\mathbb{Z}_2$ since we are working in a totally discontinuous setting. Here is an example of this situation with the Sobolev space $W^{1,2}$: one could try to define the quantity $|\nabla f|$ by the formula

$$|\nabla f| = \lim_{\delta\to 0} \sup_{d(x,y)<\delta} \frac{|f(x) - f(y)|}{d(x, y)}$$

and define the Sobolev space $W^{1,2}(\mathbb{Z}_2)$ by the norm

$$\|f\|_* = \|f\|_{L^2} + \left(\int_{\mathbb{Z}_2} |\nabla f|^2 dx\right)^{1/2}. \tag{8}$$

Now, using the Littlewood-Paley decomposition we can also write

$$\|f\|_{**} = \|S_0 f\|_{L^2} + \left\| \left(\sum_{j\in\mathbb{N}} 2^{2j} |\Delta_j f|^2\right)^{1/2} \right\|_2.$$

However, the quantities $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are not equivalent: in the case of (8) consider a function $f = c_k$ constant over each $Q_{j,k} = \{k + 2^j\mathbb{Z}_2\}$ for some fixed j. Then we have $|\nabla f| \equiv 0$ and for these functions the norm $\|\cdot\|_*$ would be equal to the L^2 norm.

This is the reason why we will use in this article the Littlewood-Paley approach to characterize Sobolev spaces:

$$\|f\|_{W^{s,p}} \simeq \|S_0 f\|_{L^p} + \left\| \left(\sum_{j\in\mathbb{N}} 2^{2js} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}, \tag{9}$$

with $1 < p < +\infty$ and $s > 0$. For Besov spaces we will define them by the norm

$$\|f\|_{B^{s,q}_p} \simeq \|S_0 f\|_{L^p} + \left(\sum_{j\in\mathbb{N}} 2^{jsq} \|\Delta_j f\|^q_{L^p} \right)^{1/q}, \tag{10}$$

where $s \in \mathbb{R}$, $1 \le p, q < +\infty$ with the necessary modifications when $p, q = +\infty$.

Remark 1. *For homogeneous functional spaces $\dot{W}^{s,p}$ and $\dot{B}^{s,q}_p$, we drop out the term $\|S_0 f\|_{L^p}$ in* (9) *and* (10).

Let us give some simple examples of function belonging to these functional spaces.

1) The function $f(x) = \log_2 |x|_2$ is in $\dot{B}^{1,\infty}_1(\mathbb{Z}_2)$. First note that $|x|_2 = 2^{-\gamma(x)}$ and thus $f(x) = -\gamma(x)$. Recall (cf. Proposition 3.1) that over each set $Q_{j,k}$, the quantity $\gamma(x)$ is constant, so the dyadic bloc $\Delta_j f$ is given by
$$\Delta_j f(x) = \begin{cases} -1 & \text{over} \quad Q_{j+1,0} \\ 0 & \text{elsewhere.} \end{cases}$$
Hence, taking the L^1 norm, we have $\|\Delta_j f\|_{L^1} = \frac{1}{2} 2^{-j}$ and then $f \in \dot{B}^{1,\infty}_1(\mathbb{Z}_2)$.
2) Set $h(x) = 1/|x|_2$, we have $h \in \dot{B}^{-1,\infty}_\infty$. For this, we must verify $\sup\limits_{j\ge 0} 2^{-j}\|\Delta_j h\|_{L^\infty} < +\infty$. By definition we obtain $h(x) = 2^{\gamma(x)}$ and then
$$\Delta_j h(x) = \begin{cases} 2^j & \text{over} \quad Q_{j+1,0} \\ 0 & \text{elsewhere.} \end{cases}$$
We finally obtain $\|\Delta_j h\|_{L^\infty} = 2^j$ and hence $2^{-j}\|\Delta_j h\|_{L^\infty} = 1$ for all j, so we write $h \in \dot{B}^{-1,\infty}_\infty$.

With the Littlewood-Paley characterisation of Sobolev spaces and Besov spaces given in (9) and (10) we have the following theorem:

Theorem 3. *In the framework of the* 2*-adic group* $\mathbb{Z}_2$ *we have, for a function* f *such that* $f \in \dot{W}^{s_1,p}(\mathbb{Z}_2)$ *and* $f \in \dot{B}^{-\beta,\infty}_\infty(\mathbb{Z}_2)$*, the inequality*

$$\|f\|_{\dot{W}^{s,q}} \le C\|f\|^\theta_{\dot{W}^{s_1,p}} \|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_\infty},$$

where $1 < p < q < +\infty$*,* $\theta = p/q$*,* $s = \theta s_1 - (1-\theta)\beta$ *and* $-\beta < s < s_1$*.*

Proof. We start with an interpolation result: let $(a_j)_{j\in\mathbb{N}}$ be a sequence, let $s = \theta s_1 - (1-\theta)\beta$ with $\theta = p/q$, then we have for $r, r_1, r_2 \in [1, +\infty]$ the estimate

$$\|2^{js} a_j\|_{\ell^r} \le C \|2^{js_1} a_j\|_{\ell^{r_1}}^{\theta} \|2^{-j\beta} a_j\|_{\ell^{r_2}}^{1-\theta}.$$

See [2] for a proof. Apply this estimate to the dyadic blocks $\Delta_j f$ to obtain

$$\left(\sum_{j\in\mathbb{Z}} 2^{2js} |\Delta_j f(x)|^2\right)^{1/2}$$

$$\le C \left(\sum_{j\in\mathbb{Z}} 2^{2js_1} |\Delta_j f(x)|^2\right)^{\theta/2} \left(\sup_{j\in\mathbb{Z}} 2^{-j\beta} |\Delta_j f(x)|\right)^{1-\theta}.$$

To finish, compute the L^q norm of the preceding quantities. □

4. The $BV(\mathbb{Z}_2)$ space and the proof of Theorem 1

We study in this section the space of functions of bounded variation BV and we will prove some surprising facts in the framework of 2-adic group $\mathbb{Z}_2$. Let us start recalling the definition of this space:

Definition 4.1. If f is a real-valued function over $\mathbb{Z}_2$, we will say that $f \in BV(\mathbb{Z}_2)$ if there exists a constant $C > 0$ such that

$$\int_{\mathbb{Z}_2} |f(x+y) - f(x)| dx \le C|y|_2, \quad (\forall y \in \mathbb{Z}_2). \tag{11}$$

We prove now Theorem 1 which asserts that in $\mathbb{Z}_2$, the BV space can be identified to the Besov space $\dot{B}_1^{1,\infty}$. For this, we will use two steps given by Propositions 4.1 and 4.2 below.

Proposition 4.1. *If f is a real-valued function over $\mathbb{Z}_2$ belonging to the Besov space $\dot{B}_1^{1,\infty}$, then $f \in BV$ and we have the inclusion $\dot{B}_1^{1,\infty} \subseteq BV$.*

Proof. Let $f \in \dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ and let us fix $|y|_2 = 2^{-m}$. We have to prove the following estimation for all $m > 0$

$$I = \int_{\mathbb{Z}_2} |f(x+y) - f(x)| dx \le C\, 2^{-m}.$$

Using the Littlewood-Paley decomposition given in (6), we will work on the formula below

$$I = \left\| \left(S_0 f(x+y) + \sum_{j\ge 0} \Delta_j f(x+y)\right) - \left(S_0 f(x) + \sum_{j\ge 0} \Delta_j f(x)\right) \right\|_{L^1}.$$

Then, by the dyadic block's properties we have to study

$$I \le \|S_m f(x+y) - S_m f(x)\|_{L^1} + \sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1}. \tag{12}$$

We estimate this inequality with the two following lemmas.

Lemma 4.1. *The first term in* (12) *is identically zero.*

Proof. Since we have fixed $|y|_2 = 2^{-m}$, then for $x \in Q_{m,k}$, we have $x+y \in Q_{m,k}$ with $k = \{0, \dots, 2^m - 1\}$. Applying the operators S_m to the functions $f(x+y)$ and $f(x)$ we get the desired result. □

The second term in (12) is treated by the next lemma.

Lemma 4.2. *Under the hypothesis of Proposition 4.1 and for* $|y|_2 = 2^{-m}$ *we have*

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq C\, 2^{-m}.$$

Proof. Indeed,

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq 2 \sum_{j=m+1}^{+\infty} \|\Delta_j f\|_{L^1}.$$

We use now the fact $\|\Delta_j f\|_{L^1} \leq C\, 2^{-j}$ for all j, since $f \in \dot{B}_1^{1,\infty}$, to get

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq C\, 2^{-m}.$$

□

With these two lemmas, and getting back to (12), we deduce the following inequality for all $y \in \mathbb{Z}_2$:

$$\int_{\mathbb{Z}_2} |f(x+y) - f(x)| dx \leq C\, |y|_2$$

and this concludes the proof of Proposition 4.1. □

Our second step in order to prove Theorem 1 is the next result.

Proposition 4.2. *In* $\mathbb{Z}_2$ *we have the inclusion* $BV(\mathbb{Z}_2) \subseteq \dot{B}_1^{1,\infty}(\mathbb{Z}_2)$.

Proof. Observe that we can characterize the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ by the condition

$$\|f(\cdot + y) + f(\cdot - y) - 2f(\cdot)\|_{L^1} \leq C\, |y|_2, \qquad \forall y \neq 0.$$

Let f be a function in $BV(\mathbb{Z}_2)$, then we have

$$\|f(\cdot + y) - f(\cdot)\|_{L^1} \leq C\ |y|_2.$$

Summing $\|f(\cdot - y) - f(\cdot)\|_{L^1}$ in both sides of the previous inequality we obtain

$$\|f(\cdot + y) - f(\cdot)\|_{L^1} + \|f(\cdot - y) - f(\cdot)\|_{L^1} \leq C\ |y|_2 + \|f(\cdot - y) - f(\cdot)\|_{L^1}$$

and by the triangular inequality we have

$$\|f(\cdot + y) + f(\cdot - y) - 2f(\cdot)\|_{L^1} \leq C\ |y|_2 + \|f(\cdot - y) - f(\cdot)\|_{L^1}.$$

We thus obtain

$$\|f(\cdot + y) + f(\cdot - y) - 2f(\cdot)\|_{L^1} \leq 2C\, |y|_2. \qquad \square$$

We have proved, in the setting of the 2-adic group $\mathbb{Z}_2$, the inequalities

$$C_1 \|f\|_{\dot{B}_1^{1,\infty}} \leq \|f\|_{BV} \leq C_2 \|f\|_{\dot{B}_1^{1,\infty}},$$

so Theorem 1 follows.

5. Improved Sobolev inequalities, *BV* space and proof of Theorem 2

We do not give here a global treatment of the family of inequalities of type (2); instead we focus on the next inequality

$$\|f\|_{L^2}^2 \leq C \|f\|_{BV} \|f\|_{\dot{B}_\infty^{-1,\infty}} \tag{13}$$

and we want to know if this estimation is true in a 2-adic framework. Since in the $\mathbb{Z}_2$ setting we have the identification $\|f\|_{BV} \simeq \|f\|_{\dot{B}_\infty^{1,\infty}}$, the estimation (13) becomes

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,\infty}} \|f\|_{\dot{B}_\infty^{-1,\infty}}. \tag{14}$$

This remark lead us to Theorem 2 which states that the previous inequalities are false.

Proof. We will construct a counterexample by means of the Littlewood-Paley decomposition, so it is worth to recall very briefly the dyadic bloc characterization of the norms involved in inequality (14). For the L^2 norm we have $\|f\|_{L^2}^2 = \sum_{j\in\mathbb{N}} \|\Delta_j f\|_{L^2}^2$, while for the Besov spaces $\dot{B}_1^{1,\infty}$ and $\dot{B}_\infty^{-1,\infty}$ we have

$$\begin{aligned} \|f\|_{\dot{B}_1^{1,\infty}} &= \sup_{j\in\mathbb{N}} 2^j \|\Delta_j f\|_{L^1} \quad \text{and} \\ \|f\|_{\dot{B}_\infty^{-1,\infty}} &= \sup_{j\in\mathbb{N}} 2^{-j} \|\Delta_j f\|_{L^\infty}. \end{aligned}$$

We construct a function $f : \mathbb{Z}_2 \longrightarrow \mathbb{R}$ by considering his values over the dyadic blocs and we will use for this the sets $Q_{j,k}$ defined in (7). First fix α and β two non negative real numbers and j_0, j_1 two integers such that $0 \leq j_0 \leq j_1$ with the condition

$$2^{2j_0} \leq \frac{\beta}{\alpha}.$$

Now define N_j as a function of α and β:

$$N_j = 2^j \quad \text{if } 0 \leq j \leq j_0 \quad \text{and} \quad N_j = \frac{\beta}{\alpha} 2^{-j} \leq 2^j \quad \text{if } j_0 < j \leq j_1. \tag{15}$$

and write

$$\Delta_j f(x) = \begin{cases} \alpha 2^j & \text{over} \quad Q_{j+1,0}, \\ -\alpha 2^j & \text{over} \quad Q_{j+1,1}, \\ \alpha 2^j & \text{over} \quad Q_{j+1,2}, \\ -\alpha 2^j & \text{over} \quad Q_{j+1,3}, \\ & \vdots \\ \alpha 2^j & \text{over} \quad Q_{j+1,2N_j-2}, \\ -\alpha 2^j & \text{over} \quad Q_{j+1,2N_j-1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Once this function is fixed, we compute the following norms

- $\|\Delta_j f\|_{L^1} = \sum_{k=0}^{N_j} \alpha 2^j 2^{-j} = \alpha N_j$,
- $\|\Delta_j f\|_{L^\infty} = \alpha 2^j$,
- $\|\Delta_j f\|_{L^2}^2 = \sum_{k=0}^{N_j} \alpha^2 2^{2j} 2^{-j} = \alpha^2 2^j N_j$,

and we build from these quantities the Besov and Lebesgue norms in the following manner:

1) For the Besov space $\dot{B}_\infty^{-1,\infty}$:
$\|f\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{0\leq j\leq j_1} 2^{-j}\alpha 2^j = \alpha$,
2) For the Besov space $\dot{B}_1^{1,\infty}$:
By the definition (15) of N_j we have $2^j\|\Delta_j f\|_{L^1} = 2^j\alpha N_j = 2^{2j}\alpha$ if $0 \leq j \leq j_0$ and $2^j\|\Delta_j f\|_{L^1} = \beta$ if $j_0 < j \leq j_1$. Since $2^{2j_0} \leq \frac{\beta}{\alpha}$ we have:
$$\|f\|_{\dot{B}_1^{1,\infty}} = \beta.$$
3) For the Lebesgue space L^2:

$$\|f\|_{L^2}^2 = \sum_{j=0}^{j_1} \alpha^2 2^j N_j = \sum_{j=0}^{j_0} \alpha^2 2^{2j} + \sum_{j>j_0}^{j_1} \alpha^2 2^j \frac{\beta}{\alpha} 2^{-j} = \sum_{j=0}^{j_0} \alpha^2 2^{2j} + (j_1 - j_0)\alpha\beta$$
$$= \alpha\beta\left(\frac{\alpha}{\beta}\sum_{j=0}^{j_0} 2^{2j} + (j_1 - j_0)\right).$$

With the condition $2^{2j_0} \leq \frac{\beta}{\alpha}$, we obtain from the previous formula that
$$\|f\|_{L^2}^2 \simeq \alpha\beta(j_1 - j_0) = \|f\|_{\dot{B}_1^{1,\infty}}\|f\|_{\dot{B}_\infty^{-1,\infty}}(j_1 - j_0).$$

Thus, getting back to (14) and therefore to (13), we have for an universal constant C the inequality

$$\begin{aligned} \|f\|_{\dot{B}_1^{1,\infty}}\|f\|_{\dot{B}_\infty^{-1,\infty}}(j_1 - j_0) &\leq C\|f\|_{\dot{B}_1^{1,\infty}}\|f\|_{\dot{B}_\infty^{-1,\infty}} \\ \iff (j_1 - j_0) &\leq C, \end{aligned}$$

which is false since we can freely choose the values of j_1 and j_0. Theorem 2 is proved. □

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Laboratoire d'Analyse et de Probabilités
Université d'Evry Val d'Essonne
Bâtiment I.B.G.B.I.
23 Bd. de France
91037 Evry Cedex-France, France
E-mail address: diego.chamorro@univ-evry.fr