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ON KERNEL OF ORDERED SEMIGROUPS – A CORRIGENDUM

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Abstract. According to the paper in [3], the kernel of an ordered semigroup S is a completely regular subsemigroup of S. In this note we show that the kernel of an ordered semigroup S is not a completely regular subsemigroup of S, in general.

1. Introduction

The intersection of the ideals of an ordered semigroup S, if it is nonempty, is called the kernel of S. The kernel of S is an ideal of S, and so a subsemigroup of S as well. According to Lemma 2.3 in [3] (or Theorem 2.6 in [4]), the kernel $K(S)$ of an ordered semigroup S is a completely regular simple subsemigroup of S. That is, it is a simple subsemigroup of S which is at the same time regular, left regular and right regular. The characterization of nil-extensions of left strongly simple ordered semigroups via some archimedean ordered semigroups and the characterizations of semilattices and chains of nil-extensions of left strongly simple ordered semigroups considered in Theorems 2.3, 2.4 and 3.2 in [3] are based on that lemma. However, this lemma is not correct. The kernel of an ordered semigroup S is not a completely regular subsemigroup of S, in general. We prove it in the example below. In addition, the proof that the kernel of an ordered semigroup S is a simple subsemigroup of S given in the Theorem 2.6 in [4] based on the completely regularity of the kernel is also not true.

For convenience, let us first give the necessary definitions: An *ordered semigroup* (or *po-semigroup*) is an ordered set $(S, \leq)$ with a multiplication "$\cdot$" which is compatible with the ordering (that is, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$). Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. It is called an *ideal* of S if it is both a left and right ideal of S. A *subsemigroup* of an ordered semigroup S is a nonempty

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subset T of S such that $T^2 \subseteq T$. A subsemigroup T of an ordered semigroup $(S, \cdot, \leq)$ is called *regular* if the set T with the multiplication "$\cdot$" on T and the order $\leq_T := \leq \cap (T \times T)$ on T (induced by the order of S) is regular. That is, for every $a \in T$ there exists $x \in T$ such that $a \leq axa$. It is called *left* (resp. *right*) *regular* if for every $a \in T$ there exists $x \in T$ such that $a \leq xa^2$ (resp. $a \leq a^2x$).

2. Main results

Here is an example of an ordered semigroup the kernel of which is not regular (or left regular) answering the question that the kernel of an ordered semigroup S cannot be a completely regular subsemigroup of S, in general. Some more information on the subject are also given.

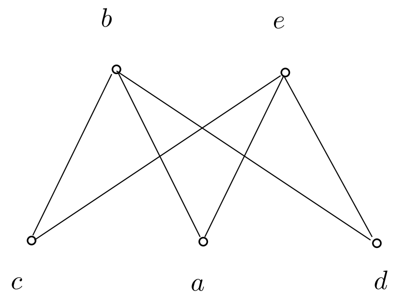
Example ([2, Example 2]). Consider the ordered semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication and the order below:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leq := \{(a,a), (a,b), (a,e), (b,b), (c,b), (c,c), (c,e), (d,b), (d,d), (d,e), (e,e)\}.$$

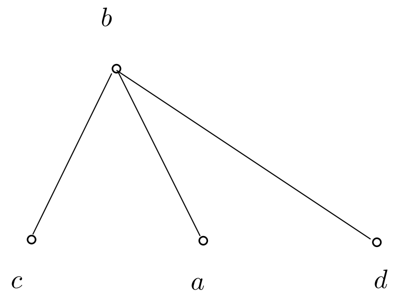
We give the covering relation and the figure of S.

$$\prec = \{(a,b), (a,e), (c,b), (c,e), (d,b), (d,e)\}.$$



The right ideals of S are the sets: $\{a, b, c, d\}$ and S.
The left ideals of S are the sets: $\{a\}$, $\{a, c\}$, $\{a, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and S.
The ideals of S are the sets: $\{a, b, c, d\}$ and S, so the kernel $K(S)$ of S is the subsemigroup $\{a, b, c, d\}$ of S with the multiplication and the figure below:

$\cdot$	a	b	c	d
a	a	b	a	a
b	a	b	a	a
c	a	b	a	a
d	a	b	a	a



The set $K(S)\,(=\{a,b,c,d\})$ is not a regular subsemigroup of S. This is because for every $x \in \{a,b,c,d\}$, we have $cxc = a$ but the elements c and a are incomparable.

It might be noted that the kernel $K(S)$ of S is not a left regular subsemigroup of S (there is no $x \in K(S)$ such that $c \le xc^2$) but it is right regular subsemigroup of S (for every $x \in K(S)$ there exists $y \in K(S)$ such that $x \le x^2y$). Since the subsemigroup $K(S)$ of S is not regular (or left regular), it cannot be a completely regular subsemigroup of S.

Note 1. This is an example of an ordered semigroup S the kernel of which is not a right regular subsemigroup of S. It might be noted that it is a left regular subsemigroup of S and it is not a regular subsemigroup of S.

Let us consider the ordered semigroup $S = \{a,b,c,d,e\}$ defined by the multiplication given in the table below and the same order as in the example above.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	a	a	a	a	a
d	a	a	a	a	a
e	a	a	a	a	e

$$\le := \{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,b),(d,d),(d,e),(e,e)\}.$$

The right ideals of S are the sets: $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,c,d,e\}$ and S.
The left ideals of S are the sets: $\{a,b,c,d\}$ and S.
The ideals of S are the sets: $\{a,b,c,d\}$ and S. The kernel $K(S)$ of S is the subsemigroup of $(S,\cdot,\le)$ defined by the table and the order below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	a	a	a
d	a	a	a	a

$$\leq = \{(a,a),(a,b),(b,b),(c,b),(c,c),(d,b),(d,d)\}.$$

This is not right regular as there is no $x \in K(S)$ such that $c \leq c^2x$. It is left regular, and not regular.

For the examples above we used computer programs.

Note 2. A subsemigroup K of an ordered semigroup S is called *simple* if for every ideal T of K, we have $T = K$. The proof that the kernel of an ordered semigroup S is a simple subsemigroup of S given in [4, Theorem 2.6] is not correct as it is based on the completely regularity of the kernel. However the kernel of S is a simple subsemigroup of S (cf. also [1, Lemma 3.2]). An independent proof based only on definitions is the following: Let K be the kernel of an ordered semigroup $(S,\cdot,\leq)$ and T an ideal of K. The (nonempty) set $M := \{a \in S \mid a \leq k_1tk_2 \text{ for some } k_1, k_2 \in K,\, t \in T\}$ is an ideal of S and $K \subseteq M$. Besides, $M \subseteq T$. Indeed, $a \in M$ implies $S \ni a \leq k_1tk_2 \in KTK \subseteq T \subseteq K$ for some k_1, $k_2 \in K$, $t \in T$. Since K is an ideal of S, we have $a \in K$. Since T is an ideal of K, we have $a \in T$. Thus we have $K \subseteq T$, and $T = K$. So K is simple.

Remark. The second example is obtained from the first by putting the rows of the first table as columns in the second. The following holds: If $(S,\cdot,\leq)$ is an ordered semigroup and " $\circ$ " an operation on S defined by $a \circ b := ba$, then $(S,\circ,\leq)$ is an ordered semigroup. In addition: $(S,\cdot,\leq)$ is left (resp. right) regular if and only if $(S,\circ,\leq)$ is right (resp. left) regular. $(S,\cdot,\leq)$ is regular if and only if $(S,\circ,\leq)$ is so. The set A is a left (resp. right) ideal of $(S,\cdot,\leq)$ if and only if A is a right (resp. left) ideal of $(S,\circ,\leq)$.

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