Commun. Korean Math. Soc. **28** (2013), No. 1, pp. 191–207
http://dx.doi.org/10.4134/CKMS.2013.28.1.191

THE SHRINKING PROJECTION METHODS FOR HEMI-RELATIVELY NONEXPANSIVE MAPPINGS, VARIATIONAL INEQUALITIES AND EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce the shrinking projection method for hemi-relatively nonexpansive mappings to find a common solution of variational inequality problems and equilibrium problems in uniformly convex and uniformly smooth Banach spaces and prove some strong convergence theorems to the common solution by using the proposed method.

1. Introduction

Let E be a Banach space and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle f, x\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

It is known that the duality mapping J has the following properties:

(1) If E is smooth, then J is single-valued;

(2) If E is strictly convex, then J is one-to-one;

(3) If E is reflexive, then J is surjective;

(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(5) If E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is singe-valued and also one-to-one; see [1-4, 17, 19].

Received February 14, 2012.
2010 *Mathematics Subject Classification.* 47H05, 47H09, 47J25.
Key words and phrases. variational inequality, equilibrium problem, hemi-relatively nonexpansive mapping, shrinking projection method.
The third author was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2011-0021821).

As is well-known to all, variational inequalities are being used as a mathematical programming tool in modeling a wide class of problems arising in several branches of pure and applied sciences; for example, operations research, economic equilibrium and engineering design.

In this paper, we consider the following variational inequality: Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

A point $x_0 \in C$ is called a solution of the variational inequality (1.1) if $\langle Ax_0, y - x_0\rangle \geq 0$. The solutions set of the variational inequality (1.1) is denoted by $VI(A, C)$. When A is provided with some monotonicity, many iterative methods for solving the variational inequality (1.1) have been developed; see [5-13].

Most recently, utilizing shrinking projection method, Ying Liu [11] established the following strong convergence theorem for relatively weak nonexpansive mapping and variational inequality in a uniformly convex and uniformly smooth Bananch space.

Theorem 1.1 ([11, Theorem 3.1]). *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Assume that A is a continuous operator of C into E^* satisfying the conditions* (1.2) *and* (1.3) *and $S : C \to C$ is a relatively weak nonexpansive mapping with $F := F(S) \cap VI(A, C) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the following iterative scheme*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle Jx_0 - Jx_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} Jx_0, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where the sequences $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

$$0 \leq \delta_n < 1, \quad \limsup_{n\to\infty} \delta < 1, \quad 0 < \alpha_n < 1, \quad \liminf_{n\to\infty} \alpha_n(1 - \alpha) > 0,$$

converges strongly to $\Pi_{F(S)\cap VI(A,C)} Jx_0$.

Here, we remark that Theorem 1.1 improved the relevant result of Jinlu Li [10] and Jianghua Fan [7], in detail that Theorem 1.1 removed the compactness of $J - \beta A$ instead by the continuity of A, and obtained a strong convergence result.

Recall that a mapping $A : D(A) \subset E \to E^*$ is said to be monotone if the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in D(A). \tag{1.3}$$

A is said to be λ-inverse strongly monotone if there exists a positive real number λ such that

$$\langle Ax - Ay, x - y\rangle \geq \lambda\|Ax - Ay\|^2, \quad \forall x, y \in D(A). \tag{1.4}$$

If A is λ-inverse strongly monotone, then it is *Lipschitz* continuous with constant $\frac{1}{\lambda}$, i.e., $\|Ax - Ay\| \leq \frac{1}{\lambda}\|x - y\|$, $\forall x,\ y \in D(A)$, and hence uniformly continuous.

For finding an element of a nonexpansive mapping and $VI(A, C)$, Takahashi and Toyoda [20] introduced the following iterative scheme in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \mu_n Ax_n), \quad \forall n \geq 1, \tag{1.5}$$

where $x_0 \in C$, P_C is a metric projection of H onto C, A is a λ-inverse strongly monotone operator. Furthermore they proved a weak convergence theorem:

Theorem 1.2 ([20, Theorem 3.1])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $\lambda > 0$. Let A be an λ-inverse strongly-monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by* (1.5) *for all $n \in \mathbb{N} \cup \{0\}$, where $\{\mu_n\} \subset [a, b]$ for some $a,\ b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset [c, d]$ for some $c,\ d \in (0, 1)$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(A,C)}x_n$.*

On the other hand, the equilibrium problem introduced in [3] in 1994, is always a hot topic of intensive research efforts, because it has a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provided a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization. Numerous problems in physics, optimization and economics reduce to finding a solution of equilibrium problem. Some methods have been proposed to solve the equilibrium problem; see [10-13].

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for f is as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall\ y \in C. \tag{1.6}$$

The set of solutions of the problem (1.6) is denoted by $EP(f)$. For solving the equilibrium problem, ones always assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) For all $x,\ y,\ z \in C$,

$$\limsup_{t\downarrow 0} f(tz+(1-t)x,y) \leq f(x,y);$$

(A4) For all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x,y) = \langle Ax, y-x\rangle, \quad \forall x,y \in C.$$

Then f satisfies (A1)-(A4).

In this paper, motivated and inspired by the results mentioned above, we introduce a new hybrid projection algorithm based on the shrinking projection method for a closed hemi-relatively nonexpansive mapping, variational inequality, equilibrium problem. Using the new algorithm, we prove some strong convergence theorem which approximate a common element in the fixed points set of the closed hemi-relatively nonexpansive mappings, the solutions set of a variational inequality and the solutions set of the equilibrium problem in a uniformly convex and uniformly smooth Banach space. Our results extend and improve the recent ones announced by Li [10], Fan [7], Liu [11], Takahashi and Toyoda [3] and many others.

2. Preliminaries

A Banach space E is said to be strictly convex if $\frac{x+y}{2} < 1$ for all $x,y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{\to\infty}\|\frac{x_n+y_n}{2}\| = 1$.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for each $x,y \in U_E$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x,y \in U_E$.

It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E and, if E is uniformly smooth if and only if E^* is uniformly convex.

Let C be a closed convex subset of E and T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Recall that an operator T in Banach space E is said to be closed if $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$.

A mapping T from C into itself is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x,y \in C.$$

The mapping T is said to be relatively nonexpansive [14-16] if

$$\widehat{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\, p \in F(T).$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [14-16]. A point $p \in C$ is called a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that

$$\lim_{n\to\infty} (x_n - Tx_n) = 0.$$

The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$.

A mapping T from C into itself is said to be relatively weak nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\, p \in F(T).$$

The mapping T is said to be hemi-relatively nonexpansive if

$$F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\, p \in F(T).$$

It is obvious that a relatively nonexpansive mapping is a relatively and weakly nonexpansive mapping and, further, a relatively and weakly nonexpansive mapping are all hemi-relatively nonexpansive, but the converses are not true as in the following example:

Example 2.1 ([18])**.** Let E be any smooth Banach space and $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ as follows: For all $n \geq 1$,

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0. \end{cases}$$

Then T is a hemi-relatively nonexpansive mapping, but it is not relatively nonexpansive mapping.

Next, we give an important example which is also hemi-relatively nonexpansive.

Example 2.2 ([15])**.** Let E be a strictly convex reflexive smooth Banach space. Let A be a maximal monotone operator of E into E^* and J_r be the resolvent for A with $r > 0$. Then $J_r = (J + rA)^{-1}J$ is a hemi-relatively nonexpansive mapping from E onto $D(A)$ with $F(J_r) = A^{-1}0$.

In [8, 2], Alber introduced the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2, \tag{2.2}$$

where $\phi \in E^*$ and $x \in E$. It is easy to see that

$$V(\phi, x) \geq (\|\phi\| - \|x\|)^2 \tag{2.3}$$

and so the functional $V : E^* \times E \to \mathbb{R}^+$ is nonnegative.

In order to prove our results in the next section, we present several necessary definitions and lemmas.

Definition 2.3 ([9])**.** If E be a uniformly convex and uniformly smooth Banach space, then the generalized projection $\Pi_C : E^* \to C$ is a mapping that assigns an arbitrary point $\phi \in E^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem

$$V(\phi, \Pi_C(\phi)) = \inf_{y \in C} V(\phi, y). \tag{2.4}$$

Li [10] proved that the generalized projection operator $\Pi_C : E^* \to C$ is continuous if E is a reflexive, strictly convex and smooth Banach space.

Consider the function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = V(Jy, x), \quad \forall x, y \in E.$$

The following properties of the operator Π_C and V are useful for our paper; see, for example, [1, 10].

(B1) $V : E^* \times E \to \mathbb{R}$ is continuous;

(B2) $V(\phi, x) = 0$ if and only if $\phi = Jx$;

(B3) $V(J\Pi_C(\phi), x) \leq V(\phi, x)$ for all $\phi \in E^*$ and $x \in E$;

(B4) The operator Π_C is J fixed at each point $x \in E^*$ and $x \in E$;

(B5) If E is smooth, then, for any given $\phi \in E^*$ and $x \in C$, $x \in \Pi_C(\phi)$ if and only if

$$\langle \phi - Jx, x - y \rangle \geq 0, \quad \forall y \in C;$$

(B6) The operator $\Pi_C : E^* \to c$ is single valued if and only if E is strictly convex;

(B7) If E is smooth, then, for any given point $\phi \in E^*$ and $x \in \Pi_C(\phi)$, the following inequality holds:

$$V(Jx, y) \leq V(\phi, y) - V(\phi, x), \quad \forall y \in C;$$

(B8) $v(\phi, X)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed;

(B9) If E is reflexive, then, for any point $\phi \in E^*$, $\Pi_C(\phi)$ is a nonempty closed convex and bounded subset of C.

Using some properties of the generalized projection operator Π_C, Alber [1] proved the following theorem:

Lemma 2.4 ([1])**.** *Let E be a strictly convex reflexive smooth Banach space. Let A be an arbitrary operator from a Banach space E to E^* and β be an arbitrary fixed positive number. Then $x \in C \subset E$ is a solution of the variational inequality* (1.1) *if and only if x is a solution of the following operator equation in E:*

$$x = \Pi_C(Jx - \beta Ax). \tag{2.5}$$

Lemma 2.5 ([9])**.** *Let E be a uniformly convex smooth Banach space and $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.6 ([5]). *Let E be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J(\phi + \Phi)\rangle, \quad \forall \phi, \Phi \in E^*. \tag{2.6}$$

From Qin et al. [16], the following lemma can be obtained immediately.

Lemma 2.7. *Let E be a uniformly convex Banach space, $s > 0$ be a positive number and $B_s(0)$ be a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\Sigma_{i=1}^N(\alpha_i x_i)\|^2 \leq \Sigma_{i=1}^N(\alpha_i\|x_i\|^2) - \alpha_i\alpha_j g(\|x_i - x_j\|) \tag{2.7}$$

for all $x_1, x_2, \ldots, x_N \in B_s(0) = \{x \in E : \|x\| \leq s\}$, $i \neq j$ for all $i, j \in \{1, 2, \ldots, N\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_N \in [0, 1]$ such that $\sum_{i=1}^N \alpha_i = 1$.

Lemma 2.8 ([3]). *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach spaces E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (B1)-(B4) *and let $r > 0$, $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle Jz - Jx, y - z\rangle \geq 0, \quad \forall\ y \in C. \tag{2.8}$$

Lemma 2.9 ([21]). *Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach spaces E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (B1)-(B4)*. For all $r > 0$ and $x \in E$, define the mapping*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle Jz - Jx, y - z\rangle \geq 0,\ \forall\ y \in C\}.$$

Then the following hold:

(C1) *T_r is single-valued*;

(C2) *T_r is a firmly nonexpansive-type mapping, that is, for all $x,\ y \in E$,*

$$\langle JT_r x - JT_r y, T_r x - T_r y\rangle \leq \langle Jx - Jy, T_r x - T_r y\rangle;$$

(C3) *$F(T_r) = \hat{F}(T_r) = EP(f)$;*

(C4) *$EP(f)$ is closed and convex.*

Lemma 2.10 ([21]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1)-(C4) *and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.9}$$

Lemma 2.11 ([11, Lemma 2.5]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of E. Suppose that there exists a positive number η such that*

$$\langle Ax, J^{-1}(Jx - \eta Ax)\rangle \geq 0, \quad \forall\ x \in C, \tag{2.10}$$

and

$$\langle Ax, y\rangle \leq 0, \quad \forall\ x \in K,\ y \in VI(A, C). \tag{2.11}$$

Then $VI(A,C)$ is closed and convex.

Lemma 2.12 ([11, Lemma 2.6]). *If E is a reflexive, strictly convex and smooth Banach space, then $\Pi_C = J^{-1}$.*

Lemma 2.13 ([11, Lemma 2.6]). *Let E be a strictly convex and smooth real Banach space, C be a closed convex subset of E and T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main results

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A$_1$)-(A$_4$). *Assume that A is a continuous operators of C into E^* satisfying the conditions* (2.10) *and* (2.11), *and $T : C \to C$ is a closed hemi-relatively nonexpansive mapping with $\digamma := F(T) \cap VI(A,C) \cap EP(f) \neq \emptyset$. For an arbitrary element $x_0 \in C$, put $C_0 = C$ and let $\{x_n\}$ be a sequence generated by the following iterative scheme*:

(3.1)

$$\begin{cases} z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTz_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle Ju_n - Jy_n, y - u_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad\qquad\qquad \leq \alpha_n \phi(z, x_0) + \beta_n \phi(z, x_n) + \gamma_n \phi(z, z_n) \\ \qquad\qquad\qquad\qquad \leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$ with the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma Jx_0$, where $\Pi_\digamma$ is the generalized projection from C onto $\digamma$.

Proof. We divide the proof into five steps.

Step 1. $\Pi_F Jx_0$ and $\Pi_{C_{n+1}} Jx_0$ are well defined.

From Lemma 2.9(C4), Lemma 2.11 and Lemma 2.13, one has that $\Pi_F Jx_0$ is well defined.

Now, we show that C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. From the definitions of C_n, it is obvious that C_n is closed for all $n \in \mathbb{N} \cup \{0\}$.

Next, we prove that C_n is convex for all $n \in \mathbb{N}\cup\{0\}$. Since $\phi(z, u_n) \leq \phi(z, y_n)$ is equivalent to

$$2\langle Jy_n - Ju_n, z\rangle \leq \|y_n\|^2 - \|u_n\|^2;$$

$\phi(z, y_n) \le \alpha_n \phi(z, x_0) + \beta_n \phi(z, x_n) + \gamma_n \phi(w, z_n)$ is equivalent to

$$2\langle \alpha_n Jx_0 + \beta_n Jx_n + \gamma_n Jz_n - Jy_n, z\rangle \le \alpha_n \|x_0\|^2 + \beta_n \|x_n\|^2 + \gamma_n \|z_n\|^2;$$

and $\alpha_n \phi(z, x_0) + \beta_n \phi(z, x_n) + \gamma_n \phi(z, z_n) \le \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n)\}$ is equivalent to

$$2\langle Jx_n - Jz_n, z\rangle \le \|x_n\|^2 - \|z_n\|^2.$$

It follows that C_n is convex for all $n \in \mathbb{N} \cup \{0\}$. Thus, for all $n \in \mathbb{N} \cup \{0\}$, C_n is closed and convex and so $\Pi_{C_{n+1}} Jx_0$ is well defined.

Step 2. $\digamma \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Observe that $\digamma \subset C_0 = C$ is obvious. Suppose that $\digamma \subset C_n$ for some $n \in \mathbb{N}$. Let $w \in \digamma \subset C_n$. Then, from the definition of ϕ and V, the property (B3) of V, Lemma 2.6, the conditions (2.10) and (2.11), it follows that

$$\begin{aligned}
\phi(w, \Pi_C(Jx_n - \eta Ax_n)) &= V(J\Pi_C(Jx_n - \eta Ax_n), w) \\
&\le V(Jx_n - \eta Ax_n, w) \\
&= \|Jx_n - \eta Ax_n\|^2 - 2\langle Jx_n - \eta Ax_n, w\rangle + \|w\|^2 \\
&\le \|Jx_n\|^2 - 2\eta\langle Ax_n, J^{-1}(Jx_n - \eta Ax_n)\rangle \\
&\quad - 2\langle Jx_n - \eta Az_n, w\rangle + \|w\|^2 \\
&\le \|Jx_n\|^2 - 2\langle Jx_n, w\rangle + \|w\|^2 \\
&= \phi(w, x_n).
\end{aligned} \tag{3.2}$$

Since $u_n = T_{r_n} y_n$, applying Lemma 2.10, the properties (B3) and (B8) of the operator V and (3.2), we obtain

$$\begin{aligned}
\phi(w, u_n) = \phi(w, T_{r_n} y_n) \le \phi(w, y_n) &= V(Jy_n, w) \\
&\le \alpha_n V(Jx_0, w) + \beta_n V(Jx_n, w) + \gamma_n V(JTz_n, w) \\
&\le \alpha_n \phi(w, x_0) + \beta_n \phi(w, x_n) + \gamma_n \phi(w, z_n) \\
&\le \alpha_n \phi(w, x_0) + \beta_n \phi(w, x_n) + \gamma_n \phi(w, x_n) \\
&= \alpha_n \phi(w, x_0) + (1 - \alpha_n)\phi(w, x_n),
\end{aligned} \tag{3.3}$$

which shows that $w \in C_{n+1}$. Depending on the randomicity of w, one can learn that $\digamma \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. $\{x_n\}$ is a Cauchy sequence.

Since $x_n = \Pi_{C_n} Jx_0$, and from (2.4), one has

$$V(Jx_0, x_n) \le V(Jx_0, y), \quad \forall\, y \in C,$$

and $\digamma \subset C_n \subset C$, one also has

$$V(Jx_0, x_n) \le V(Jx_0, w), \quad \forall\, w \in \digamma.$$

Therefore, $\{V(Jx_0, x_n)\}$ is bounded. Moreover, from the definition of V, it follows that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}} Jx_0 \in C_{n+1}$ and $x_n = \Pi_{C_n} Jx_0$, we have

$$V(Jx_0, x_n) \le V(Jx_0, x_{n+1}),\ \forall\, n \ge 0.$$

Hence $\{V(Jx_0, x_n)\}$ is nondecreasing and so $\lim_{n\to\infty} V(Jx_0, x_n)$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} Jx_0 \in C_n$ for any positive integer $m \geq n$. From the property (B3), we have

$$V(Jx_n, x_m) \leq V(Jx_0, x_m) - V(Jx_0, x_n)$$

for all $n \in \mathbb{N} \cup \{0\}$ and any positive integer $m \geq n$. This implies that

$$V(Jx_n, x_m) \to 0 \quad (n, m \to \infty).$$

The definition of ϕ implies that

$$\phi(x_m, x_n) \to 0 \quad (n, m \to \infty). \tag{3.4}$$

Applying Lemma 2.5, we obtain

$$\|x_m - x_n\| \to 0 \quad (n, m \to \infty). \tag{3.5}$$

Hence $\{x_n\}$ is a Cauchy sequence. In view of the completeness of a Banach space E and the closeness of C, it follows that

$$\lim_{n\to\infty} x_n = p \tag{3.6}$$

for some $p \in C$.

Step 4. $p \in F$.

First, we show that $p \in F(T)$. In fact, since $x_{n+1} \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).$$

Thus, by (3.4) and Lemma 2.5, we have

$$\|x_{n+1} - u_n\| \to 0 \quad (n \to \infty)$$

and hence

$$\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| \to 0 \quad (n \to \infty), \tag{3.7}$$

which implies that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} x_n = p. \tag{3.8}$$

On the other hand, since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, $\{Jx_n\}$, $\{JTx_n\}$ and $\{JSx_n\}$ are also bounded. Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|Jx_0\|, \|Jx_n\|, \|JTx_n\|\}$. Therefore, from Lemma 2.7, it follows that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and the inequality (2.7).

It follows from the property (B3) of the operator V, (3.2) and the definition of T that

(3.10)

$$\begin{aligned}\phi(w,y_n)&=V(Jy_n,w)\\&\le V(\alpha_nJx_0+\beta_nJx_n+\gamma_nJTz_n,w)\\&=\phi(w,J^{-1}(\alpha_nJx_0+\beta_nJx_n+\gamma_nJTz_n))\\&=\|w\|^2-2\alpha_n\langle Jx_0,w\rangle-2\beta_n\langle Jx_n,w\rangle-2\gamma_n\langle JTz_n,w\rangle\\&\quad+\|\alpha_nJx_0+\beta_nJx_n+\gamma_nJTz_n\|^2\\&\le\|w\|^2-2\alpha_n\langle Jx_0,w\rangle-2\beta_n\langle Jx_n,w\rangle-2\gamma_n\langle JTz_n,w\rangle\\&\quad+\alpha_n\|Jx_0\|^2+\beta_n\|Jx_n\|^2+\gamma_n\|JTz_n\|^2-\beta_n\gamma_ng(\|Jx_n-JTz_n\|)\\&=\alpha_n\phi(w,x_0)+\beta_n\phi(w,x_n)+\gamma_n\phi(w,Tz_n)-\beta_n\gamma_ng(\|Jx_n-JTz_n\|)\\&\le\alpha_n\phi(w,x_0)+\beta_n\phi(w,x_n)+\gamma_n\phi(w,x_n)-\beta_n\gamma_ng(\|Jx_n-JTz_n\|)\\&=\alpha_n\phi(w,x_0)+(1-\alpha_n)\phi(w,x_n)-\beta_n\gamma_ng(\|Jx_n-JTz_n\|).\end{aligned}$$

And from (3.3), we get

$$\phi(w,u_n)=\phi(w,T_{r_n}y_n)\le\phi(w,y_n). \tag{3.11}$$

Substituting (3.10) into (3.11), we obtain

$$\phi(w,u_n)\le\alpha_n\phi(w,x_0)+(1-\alpha_n)\phi(w,x_n)-\beta_n\gamma_ng(\|Jx_n-JTz_n\|).$$

The above inequality implies that

$$\beta_n\gamma_ng(\|Jx_n-JTz_n\|)\le\alpha_n\phi(w,x_0)+(1-\alpha_n)\phi(w,x_n)-\phi(w,u_n). \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\phi(w,x_n)-\phi(w,u_n)&=2\langle Ju_n-Jx_n,w\rangle+\|x_n\|^2-\|u_n\|^2\\&\le2\langle Ju_n-Jx_n,p\rangle+(\|x_n\|-\|u_n\|)(\|x_n\|+\|u_n\|)\\&\le2\|Ju_n-Jx_n\|\|w\|+\|x_n-u_n\|(\|x_n\|+\|u_n\|).\end{aligned}$$

It follows from (3.7) and (3.9) that

$$\lim_{n\to\infty}(\phi(w,x_n)-\phi(w,u_n))=0. \tag{3.13}$$

In view of $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty}\beta_n\gamma_n>0$, the inequality (3.12) implies that

$$g(\|Jx_n-JTz_n\|)\to0\quad(n\to\infty).$$

Therefore, from the property of g, we get

$$\|Jx_n-JTz_n\|\to0\quad(n\to\infty).$$

Furthermore, since J^{-1} is uniformly norm to norm continuous on bounded sets, we see that

$$\|x_n-Tz_n\|\to0\quad(n\to\infty). \tag{3.14}$$

On the other hand, by the construction of C_n, we know that

$$\begin{aligned}\phi(z,u_n) \le \phi(z,y_n) &\le \alpha_n\phi(z,x_0)+\beta_n\phi(z,x_n)+\gamma_n\phi(z,z_n)\\ &\le \alpha_n\phi(z,x_0)+(1-\alpha_n)\phi(z,x_n).\end{aligned}$$

From $x_{n+1}=\Pi_{C_{n+1}}Jx_0\in C_{n+1}$, we have

$$\begin{aligned}\phi(x_{n+1},u_n) \le \phi(x_{n+1},y_n) &\le \alpha_n\phi(x_{n+1},x_0)+\beta_n\phi(x_{n+1},x_n)+\gamma_n\phi(x_{n+1},z_n)\\ &\le \alpha_n\phi(x_{n+1},x_0)+(1-\alpha_n)\phi(x_{n+1},x_n).\end{aligned}$$

From (3.4) and $\lim_{n\to\infty}\alpha_n=0$, the above inequality implies that

$$\phi(x_{n+1},z_n)\to 0 \quad (n\to\infty).$$

Applying Lemma 2.5, one has

$$\|x_{n+1}-z_n\|\to 0 \quad (n\to\infty)$$

and, by (3.5), we obtain that

$$\|x_n-z_n\| \le \|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|\to 0 \quad (n\to\infty). \tag{3.15}$$

Thus, from (3.14) and (3.15), one sees that

$$\|z_n-Tz_n\| \le \|z_n-x_n\|+\|x_n-Tz_n\|\to 0 \quad (n\to\infty).$$

Thus, from the closedness of T, we obtain that $p\in F(T)$.

Secondly, we show that $p\in EP(f)$. From $u_n=T_{r_n}x_n$ and the construction of C_n, one has

$$\begin{aligned}\phi(u_n,y_n) &= \phi(T_{r_n}y_n,y_n)\\ &\le \phi(w,y_n)-\phi(w,T_{r_n}y_n)\\ &\le \phi(w,x_n)-\phi(w,T_{r_n}y_n)\\ &\le \phi(w,x_n)-\phi(w,u_n).\end{aligned}$$

And, by (3.13), it follows that

$$\phi(u_n,y_n)\to 0 \quad (n\to\infty).$$

Applying Lemma 2.5, we obtain

$$\|u_n-y_n\|\to 0 \quad (n\to\infty).$$

Since J is a uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty}\|Ju_n-Jy_n\|=0.$$

From the assumption that $r_n\ge a$, one has

$$\lim_{n\to\infty}\frac{\|Ju_n-Jy_n\|}{r_n}=0.$$

Observing that $u_n=T_{r_n}y_n$, one obtains

$$f(u_n,y)+\frac{1}{r_n}\langle Ju_n-Jy,y-u_n\rangle\ge 0, \quad \forall\, y\in C.$$

From (A2), one gets

$$\begin{aligned}\|y_n - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle Ju_n - Jy_n, y - u_n\rangle \\ &\leq -f(u_n, y) \\ &\leq f(y, u_n), \quad \forall\, y \in C.\end{aligned}$$

Taking $n \to \infty$ in above inequality, it follows from (A4) and (3.8) that

$$f(y, p) \leq 0, \quad \forall\, y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Note that $y,\ p \in C$ and so one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from $(A1)$ that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y),$$

that is,

$$f(y_t, y) \geq 0.$$

Let $t \downarrow 0$. From (A3), we obtain $f(p, y) \geq 0$ for all $y \in C$, which imply that $p \in EP(f)$.

Finally, we show that $p \in VI(A, C)$. In fact, by (3.15), we have

$$\|\Pi_C(Jx_n - \eta Ax_n) - x_n\| \to 0 \quad (n \to \infty).$$

Since $\lim_{n\to\infty} x_n = p$, we obtain

$$\lim_{n\to\infty} z_n = p.$$

By the continuity of the operator J, A, Π_C, we have

$$\lim_{n\to\infty} \|\Pi_C(Jx_n - \eta Ax_n) - \Pi_C(Jp - \eta Ap)\| = 0.$$

Note that

$$\begin{aligned}&\|\Pi_C(Jx_n - \eta Ax_n) - p)\| \\ \leq\ &\|\Pi_C(Jx_n - \eta Ax_n) - x_n\| + \|x_n - p\| \to 0 \ \ (n \to \infty).\end{aligned}$$

Hence it follows from the uniqueness of the limit that $p = \Pi_C(Jp - \eta Ap)$. From Lemma 2.4, we have $p \in VI(A, C)$. Therefore, we have $p \in F$.

Step 5. $p = \Pi_F Jx_0$.

Since $p \in F$, from the property (B3) of the operator Π_C, we have

$$V(J\Pi_F Jx_0, p) + V(Jx_0, \Pi_F Jx_0) \leq V(Jx_0, p). \tag{3.16}$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}} Jx_0$ and $F \subset C_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, it follows from the property (B7) of the operator Π_C that

$$V(Jx_{x+1}, \Pi_F Jx_0) + V(Jx_0, x_{n+1}) \leq V(Jx_0, \Pi_F Jx_0). \tag{3.17}$$

Furthermore, by the continuity of the operator V, we get

$$\lim_{n\to\infty} V(Jx_0, x_{n+1}) = V(Jx_0, p). \tag{3.18}$$

Combining (3.16), (3.17) with (3.18), we obtain

$$V(Jx_0, p) = V(Jx_0, \Pi_F Jx_0).$$

Therefore, it follows from the uniqueness of $\Pi_F Jx_0$ that $p = \Pi_F Jx_0$. This completes the proof. □

Remark 3.2. Theorem 3.1 improves Theorem 3.1 of Liu [11] in the following senses:

(1) The hemi-relatively nonexpansive mapping is more general than the relatively weak nonexpansive one in Liu [11].

(2) Our iterative algorithm (3.1) is completely different from the iterative algorithm of Theorem 3.1 of Liu [11].

(3) In contrast to Theorem 3.1 of Liu [11], our algorithm in Theorem 3.1 is related to three problems, that is, the fixed point, variational inequality and equilibrium problems.

When $\alpha_n \equiv 0$ in (3.1), we can obtain the new modified Mann iteration for the hemi-relatively nonexpansive mapping T, the variational inequality (1.1), the equilibrium problem (1.6) as follows:

Corollary 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A_1)-(A_4). *Assume that A is a continuous operators of C into E^* satisfying the conditions* (2.10) *and* (2.11), *and $T : C \to C$ is a closed hemi-relatively nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. For an arbitrary element $x_0 \in C$, put $C_0 = C$ and let $\{x_n\}$ be a sequence generated by the following iterative scheme*:

$$\begin{cases} z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle Ju_n - Jy_n, y - u_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \\ \qquad\qquad \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{\alpha_n\}$ and $\{r_n\}$ satisfies that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.*

If $T = I$ in the iteration algorithm of Corollary 3.3, it reduces to the new modified Mann iteration for the variational inequality (1.1) and the generalized equilibrium problem (1.6) as follows:

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction* *from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A_1)-(A_4). *Assume that A is a continuous operators of C into E^* satisfying the conditions* (2.10) *and* (2.11) *with $\digamma := VI(A,C) \cap EP(f) \neq \emptyset$. For an arbitrary element $x_0 \in C$, put $C_0 = C$ and let $\{x_n\}$ be a sequence generated by the following iterative scheme*:

$$\begin{cases} y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)J\Pi_C(Jx_n - \eta Ax_n)), \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle Ju_n - Jy_n, y - u_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \\ \qquad\qquad \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\alpha_n\}$ and $\{r_n\}$ satisfies that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma Jx_0$, where $\Pi_\digamma$ is the generalized projection from C onto $\digamma$.

Remark 3.5. See Remark 3.1 of Liu [11], Corollary 3.4 also does the corresponding promotions about Li [10] and Fan [7] as Liu [11]'s job.

If the mapping A is a λ-inverse strongly monotone mapping in Corollary 3.3, then the following result can be also obtained by Corollary 3.3 and Theorem 3.1.

Corollary 3.6. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A_1)-(A_4). *Assume that A is a $\lambda-$inverse strongly monotone mapping of C into E^* satisfying the conditions* (2.10) *and* (2.11) *and $T : C \to C$ is a closed hemi-relatively nonexpansive mapping with $\digamma := F(T) \cap VI(A,C) \cap EP(f) \neq \emptyset$. For an arbitrary element $x_0 \in C$, put $C_0 = C$ and let $\{x_n\}$ be a sequence generated by the following iterative scheme*:

$$\begin{cases} z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle Ju_n - Jy_n, y - u_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \\ \qquad\qquad \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma Jx_0$, where $\Pi_\digamma$ is the generalized projection from C onto $\digamma$.

Proof. Since A is λ-inverse strongly monotone, by (1.6), we have

$$\|Ax - Ay\| \leq \frac{1}{\lambda}\|x - y\|,$$

for all $x,\ y \in C$, then it is *Lipschitz* continuous with constant $\frac{1}{\lambda}$. By Corollary 3.5, we can directly obtain that the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$. □

Remark 3.7. Corollary 3.6 improves Theorem 3.1 of Takahashi and Toyoda [20] in the following senses:

(1) The hemi-relatively nonexpansive mapping is more general than a nonexpansive one in Takahashi and Toyoda [20].

(2) Our modified Mann iteration obtains strong convergence result about a λ-inverse strongly monotone mapping and a closed hemi-relatively nonexpansive mapping and generalized equilibrium problem (1.9) in a uniformly convex and uniformly smooth Banach space.

Acknowledgement. The authors thank the referee for the valuable comments and suggestions.

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