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SPECTRAL INEQUALITIES OF THE LAPLACIAN ON A CURVED TUBE WITH VARYING CROSS SECTION

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Abstract. In this note, we consider a curved tube with varying cross-section formed by rotating open bounded Euclidean domains with respect to a reference curve, and successfully give a lower bound to the threshold of the Laplacian on the tube, subject to Dirichlet boundary conditions on the surface and Neumann conditions at the ends of the tube. This generalizes the corresponding result in [1].

1. Introduction

A lower bound to the spectral threshold of the Laplacian of curved tubes with constant cross section has been given in [1]. In this note, we consider a new model, a curved tube with varying cross-section. More precisely, given a bounded or unbounded open interval I, let $\Gamma(s) : I \to \mathbb{R}^d$ ($d \geq 2$) be a unit-speed curve with curvatures

$$k_i : I \to \mathbb{R}, \qquad i \in \{1, \ldots, d-1\},$$

with respect to an appropriate Frenet frame $\{e_1, \ldots, e_d\}$ satisfying the assumption $\langle H1 \rangle$

$\langle H1 \rangle$. Γ possesses a positively oriented C^1-smooth Frenet frame $\{e_1, \ldots, e_d\}$ with the properties that $e_1 = \dot{\Gamma}$ and $\dot{e}_i(s)$ lies in the span of $e_1(s), \ldots, e_{i+1}(s)$ for any $i \in \{1, \ldots, d-1\}$ and any $s \in I$.

Given a bounded open connected set $w \in \mathbb{R}^{d-1}$ with the center of mass at the origin, we define the tube Ω by expanding or shrinking w in all directions $f(s)$ times with respect to its center of mass for each $s \in I$ and rotating $f(s)w$ with respect to the curve $\Gamma(s)$ possessing the Frenet frame above, i.e.

$$\Omega := \mathcal{L}(I \times w), \mathcal{L}(s, u_2, \ldots, u_d) := \Gamma(s) + f(s)e_\mu(s)u_\mu, \tag{1.1}$$

where $f \in L^\infty(I) \cap C^1(I)$ is a positive or negative real-valued function. Obviously, when $f(s) = 1$, our model is the same as the curved tube considered

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in [1]. The repeated indices invention is used in this note, that is, the Latin and Greek indices run over $1, 2, \ldots, d$ and $2, \ldots, d$, respectively. Without loss of generality, we assume that $f(s)$ is positive for any $s \in I$. If necessary, one can additionally require that $f(s)$ preserves the center of mass at the origin for each $s \in I$. This setting makes sense, since it can be easily seen when w is a circular region in $\mathbb{R}^{d-1}$ with the center of mass at the origin. We make the assumption $\langle H2\rangle$ as follows.

$\langle H2\rangle$. (1) $k_1 \in L^\infty(I)$ and $a\|k_1\|_\infty \|f\|_\infty < 1$,
(2) Ω does not overlap itself, where $a := \sup_{u\in w} |u| = \sup_{u\in w} \sqrt{u_\mu u_\mu}$.

The assumption $\langle H2\rangle$ makes sure that the mapping $\mathcal{L}$ is a diffeomorphism what we will see in the next section.

Denote by Δ and $\sigma(-\Delta)$ the Laplacian, subject to Dirichlet boundary conditions on surface $L(I \times \partial w)$ and, if ∂I is not empty, Neumann boundary conditions on the ends $L(\partial I \times w)$ of the tube, on the Hilbert space $L^2(\Omega)$ and the spectrum of the nonnegative Laplacian $-\triangle$, respectively. We can prove the following.

Theorem 1.1 (Main theorem). *Under the assumptions $\langle H1\rangle$ and $\langle H2\rangle$, we have*

$$\inf \sigma(-\Delta) \geq \|f\|_\infty^{-2} \inf_{s\in I} \lambda_0(k_1(s)), \tag{1.2}$$

where

$$\lambda_0(k) := \inf_{\psi \in W_0^{1,2}(w)} \frac{\int_w \overline{\psi_{,\mu}(u)} \psi_{,\mu}(u)(1 - ku_2 f)du}{\int_w |\psi(u)|^2 (1 - ku_2 f)du}. \tag{1.3}$$

Especially, when $f(t) = c > 0$ is a function of constant value, then

$$\inf \sigma(-\Delta) \geq c^{-2} \min \left\{ \lambda_0\left(\sup_{s\in I} k_1(s)\right), \lambda_0\left(\inf_{s\in I} k_1(s)\right) \right\},$$

and moreover, in this case we have

$$\lambda_0(k) \geq \left(\frac{|\mathbb{S}^{d-1}|}{cd|\mathbb{S}^1|a|w|}\right)^{\frac{2}{d}} j^2_{\frac{d-2}{2},1},$$

where $\lambda_0(k)$ denotes the spectral threshold of $-\triangle$ in the tube of cross section cw, a domain formed by expanding or shrinking the domain w in all directions c times with respect to its center of mass, built either over a circle of curvature k if $k \neq 0$ or over a straight line if $k = 0$, and moreover, $|\cdot|$ and $j_{\frac{d-2}{2},1}$ denote the volume of the prescribed domain and the first zero of the Bessel function $J_{\frac{d-2}{2}}$ respectively.

2. Proof of main theorem

Under the assumption $\langle H1 \rangle$, we have the Serret-Frenet formula

$$\dot{e}_i = \mathcal{K}_{ij} e_j, \tag{2.1}$$

where $\mathcal{K} \equiv (\mathcal{K}_{ij})$ is the skew-symmetric $d \times d$ matrix defined by

$$\mathcal{K} := \begin{pmatrix} 0 & k_1 & \dots & 0 & 0 \\ -k_1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & k_{d-1} \\ 0 & 0 & \dots & -k_{d-1} & 0 \end{pmatrix},$$

with k_i the ith curvature of Γ, a continuous function of the arc-length parameter $s \in I$.

Let $\Omega_0 := I \times w$ and $u := (u_2, \dots, u_d) \in w$. Then the tube Ω is the image of the mapping $\mathcal{L} : \Omega_0 \to \mathbb{R}^d$ defined by (1.1). Assuming that $\mathcal{L} : \Omega_0 \to \Omega$ is a C^1-diffeomorphism, then we can identify Ω with the Riemannian manifold (Ω_0, G), where $G \equiv (G_{ij})$ is the metric induced by the immersion $\mathcal{L}$. Hence, by applying (1.1) and the expression of the standard Euclidean metric of $\mathbb{R}^d$, we know that the metric matrix G satisfies

$$G := \begin{pmatrix} h_1 & h_2 & h_3 & \dots & h_{d-1} & h_d \\ h_2 & f^2 & 0 & \dots & 0 & 0 \\ h_3 & 0 & f^2 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ h_{d-1} & 0 & 0 & \dots & f^2 & 0 \\ h_d & 0 & 0 & \dots & 0 & f^2 \end{pmatrix},$$

with

$$h_1 := h^2 + f^2 \widetilde{h}_\mu \widetilde{h}_\mu + (f_s u_\mu)^2, \quad h(s,u) := 1 - k_1 u_2 f(s), \quad \widetilde{h}_\mu := -\mathcal{K}_{\mu\nu} u_\nu,$$

and

$$h_\mu := f^2 \widetilde{h}_\mu + u_\mu f f_s,$$

where f_s denotes the derivative of f with respect to s, and $\mathcal{K}_{\mu\nu}, 2 \le \mu, \upsilon \le d$ are the entries of the matrix K above.

By the skew-symmetric property of $\mathcal{K}$, we could easily get that $|G| := \det G = f^{2(d-1)} h^2$, which allows us to define the volume element of Ω by $d\Omega := f^{d-1} h(s,u) ds du$, with $du = du_2 \cdots du_d$ the $(d-1)$-dimensional Lebesgue measure of w.

So, under the first condition (1) of the assumption $\langle H2 \rangle$, the induced metric matrix G is degenerated, which implies that the mapping $\mathcal{L}$ is an immersion.

In this case, we can easily obtain the inverse matrix G^{-1} of G given as follows
(2.2)

$$G^{-1} := \frac{1}{f^2h^2}\begin{pmatrix} f^2 & -h_2 & -h_3 & \cdots & -h_{d-1} & -h_d \\ -h_2 & h^2+\frac{h_2^2}{f^2} & \frac{h_2h_3}{f^2} & \cdots & \frac{h_2h_{d-1}}{f^2} & \frac{h_2h_d}{f^2} \\ -h_3 & \frac{h_3h_2}{f^2} & h^2+\frac{h_3^2}{f^2} & \cdots & \frac{h_3h_{d-1}}{f^2} & \frac{h_3h_d}{f^2} \\ \vdots & & & \ddots & & \vdots \\ -h_{d-1} & \frac{h_{d-1}h_2}{f^2} & \frac{h_{d-1}h_3}{f^2} & \cdots & h^2+\frac{h_{n-1}^2}{f^2} & \frac{h_{d-1}h_d}{f^2} \\ -h_d & \frac{h_dh_2}{f^2} & \frac{h_dh_3}{f^2} & \cdots & \frac{h_dh_{d-1}}{f^2} & h^2+\frac{h_d^2}{f^2} \end{pmatrix}.$$

Naturally, we know that the mapping $\mathcal{L}$ is a C^1-diffeomorphism under the assumption $\langle H2\rangle$. This indicates that our assumption makes sense, therefore we could investigate the spectrum of the Laplacian on the Riemannian manifold (Ω_0, G) if we want to know the spectrum of the Laplacian on the curved tube Ω. Actually, by introducing the unitary transformation $\psi \to \psi\mathcal{L}$, we may identify the Hilbert space $L^2(\Omega)$ with $\mathcal{H} := L^2(\Omega_0, d\Omega)$ and the Laplacian $\Delta = \Delta_\Omega$ with the self adjoint operator H associated with the quadratic form Q on $\mathcal{H}$ defined by

$$Q(\psi,\psi) := \int_{\Omega_0} \overline{\psi_{,i}} G^{ij}\psi_{,j} d\Omega \tag{2.3}$$

and
(2.4)

$$\psi \in DomQ := \left\{\psi \in W^{1,2}(\Omega_0, d\Omega) | \psi(q,u) = 0 \text{ for a.e. } (s,u) \in I \times \partial w\right\},$$

where $\psi(x)$ for $x \in \partial\Omega_0$ means the corresponding trace of the function ψ on the boundary.

As in [1], we can prove the following lemma.

Lemma 2.1. *In matrix-inequality sense, we have*

$$G^{-1} \geq diag\left(0, \frac{1}{f^2(s)}, \ldots, \frac{1}{f^2(s)}\right)$$

for each $s \in I$.

Proof. By (2.2), we have

$$G^{-1} - diag\left(0, \frac{1}{f^2(s)}, \ldots, \frac{1}{f^2(s)}\right) = (hf)^{-2}A,$$

where $A := diag(f^2, 0, \ldots, 0) + \tau$, and τ is a matrix depending on the entries of G^{-1}. However, A is positive definite, since for any $\xi \in \mathcal{R}^d$ we have

$$\xi_i A_{ij}\xi_j \equiv f^2\xi_1^2 - 2\xi_1 h_\mu \xi_\mu + (f^{-1}h_\mu\xi_\mu)^2 = (-f\xi_1 + f^{-1}h_\mu\xi_\mu)^2 \geq 0,$$

then our conclusion follows. □

Then, by using this lemma, we could give the proof of our main theorem as follows.

Proof of Theorem 1.1. By (1.3), Lemma 2.1, (2.3) and (2.4), for any $\psi \in DomQ$ we have

$$\begin{aligned} Q(\psi,\psi) &\geq \int_I ds \int_w f^{-2}\overline{\psi_{,\mu}(s,u)}\psi_{,\mu}(s,u)\left(1-k_s f(s)u_2\right)f^{d-1}du \\ &\geq \|f\|_\infty^{-2}\int_I \lambda_0(k_1(s))f^{d-1}ds\int_w |\psi(s,u)|^2\left(1-k_s f(s)u_2\right)du \\ &\geq \|f\|_\infty^{-2}\inf_{s\in I}\lambda_0(k_1(s))\int_I ds\int_w |\psi(s,u)|^2\left(1-k_s f(s)u_2\right)f^{d-1}du, \end{aligned}$$

which implies our conclusion (1.2). Especially, when $f(s) \equiv c$ is a function of constant value, by Lemma 4.1, Proposition 4.2 and Proposition 4.5 in [1], the rest part of our main theorem follows since there is no essential difference between the case $f(s) \equiv c$ and the case $f(s) \equiv 1$. □

References

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