Commun. Korean Math. Soc. **28** (2013), No. 1, pp. 163–175
http://dx.doi.org/10.4134/CKMS.2013.28.1.163

EINSTEIN LIGHTLIKE HYPERSURFACES OF A LORENTZIAN SPACE FORM WITH A SEMI-SYMMETRIC METRIC CONNECTION

Dae Ho Jin

Abstract. In this paper, we prove a classification theorem for Einstein lightlike hypersurfaces M of a Lorentzian space form $(\bar{M}(c), \bar{g})$ with a semi-symmetric metric connection subject such that the second fundamental forms of M and its screen distribution $S(TM)$ are conformally related by some non-zero constant.

1. Introduction

In the classical theory of spacetime, the Riemannian curvature tensor will affect the rate of change of separation of null and timelike curves (see Sections 4.1 and 4.2 in [8]). Null curves can represent the histories of photons, the effect of the Riemannian curvature tensor will be to distort or focus small bundles of light rays. While the rest spaces of timelike curves are spacelike subspaces of the tangent spaces, the rest spaces of null curves are lightlike subspaces of the tangent spaces [12]. To investigate this, Hawking and Ellis introduced the notion of so-called screen spaces in Section 4.2 of their book [8]. Since for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, in a 1996 book [4] Duggal-Bejancu published their work on the general theory of degenerate (lightlike) submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [5, 7]).

The classification of Einstein hypersurfaces M in Euclidean spaces $\mathbf{R}^{n+1}$ was first studied by Fialkow [10] and Thomas [13] in the middle of 1930's. It was proved that if M is a connected Einstein hypersurface ($n \geq 3$), that is $Ric = \kappa g$, for some constant κ, then κ is non-negative. Moreover,

- if $\kappa = 0$, then M is locally isometric to $\mathbf{R}^n$ and
- if $\kappa > 0$, then M is contained in an n-sphere.

Received July 28, 2011; Revised January 5, 2012.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. screen homothetic, Einstein manifold, semi-symmetric metric connection.

Motivated by the rich existing Riemannian geometry endow with a semi-symmetric metric connection (see two papers of Hayden [9] and Yano [15]), we study lightlike hypersurfaces M of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric metric connection. The objective of this paper is the study of lightlike version of above classical results. We focus on the geometry of Einstein lightlike hypersurfaces (M, g) of a Lorentzian space form $(\bar{M}(c), \bar{g})$ with a semi-symmetric metric connection subject such that whose shape operator is homothetic to the shape operator of its screen distribution. The reason for this geometric condition on M is due to the fact that such a class admits a canonical integrable screen distribution and a symmetric induced Ricci tensor of M [1]. These both conditions are required to recover an induced scalar curvature of M of a Lorentzian manifold [3]. The paper contains several new results which are related to the symmetric Ricci tensor. Calling such a class by screen homothetic lightlike hypersurfaces (M, g), we prove that M is locally a product manifold $L \times M_\rho \times M_\sigma$, where L is a null curve, and M_ρ and M_σ are leaves of some distributions of M (Theorem 4.1). Using this theorem we prove a characterization theorem for Einstein screen homothetic lightlike hypersurfaces M of a Lorentzian space form $\bar{M}(c)$ with a semi-symmetric metric connection (Theorem 4.2).

2. Semi-symmetric metric connection

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. A connection $\bar{\nabla}$ on $\bar{M}$ is called a *semi-symmetric metric connection* [9, 14, 15] if it is metric, i.e., $\bar{\nabla}_X \bar{g} = 0$ and its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.1}$$

for any vector fields X and Y of $\bar{M}$, where π is a 1-form on $\bar{M}$.

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. It is well known that the normal bundle $TM^\perp$ of the lightlike hypersurfaces M is a vector subbundle of TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM), \tag{2.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [4] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \; \forall X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution respectively. Then the tangent bunde $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.3}$$

Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.2). From the decompositions (2.2) and (2.3), the local Gauss and Weingartan formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.7}$$

for any $X,\, Y \in \Gamma(TM)$, where the symbols ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.8}$$

for any $X,\, Y,\, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \ \forall\, X \in \Gamma(TM). \tag{2.9}$$

But the connection ∇^* is metric. Using (2.1) and (2.4), we show that

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \quad \forall\, X,\, Y \in \Gamma(TM) \tag{2.10}$$

and B is symmetric, where T is the torsion tensor with respect to ∇. From (2.8) and (2.10), we show that the induced connection ∇ of M is a semi-symmetric non-metric connection of M. From the fact $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.11}$$

The above second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.12}$$

$$g(A_N X, PY) = C(X,PY), \quad \bar{g}(A_N X, N) = 0, \tag{2.13}$$

for all $X,\, Y \in \Gamma(TM)$. By (2.12), we show that A^*_ξ is $\Gamma(S(TM))$-valued self-adjoint shape operators related to B and satisfies

$$A^*_\xi \xi = 0. \tag{2.14}$$

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 2.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ admitting a semi-symmetric metric connection. The following assertions are equivalent*:

(1) *The screen distribution $S(TM)$ is an integrable distribution.*
(2) *C is symmetric, i.e., $C(X,Y) = C(Y,X)$ for all $X, Y \in \Gamma(S(TM))$.*
(3) *The shape operator A_N is self-adjoint with respect to g, i.e.,*

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall\, X,\, Y \in \Gamma(S(TM)).$$

Proof. First, note that a vector field X on M belongs to $S(TM)$ if and only if we have $\eta(X) = 0$. Next, by using (2.6) and (2.10), we have

$$C(X,Y) - C(Y,X) = \eta([X,Y]), \quad \forall\, X,\, Y \in \Gamma(S(TM)),$$

which implies the equivalence of (1) and (2). Finally, the equivalence of (2) and (3) follows from the first equation of (2.13) [denote $(2.13)_1$]. □

Note 1. In case $S(TM)$ is integrable, M is locally a product manifold $L \times M^*$ where L is a null curve tangent to the normal bundle $TM^\perp$ and M^* is a leaf of the screen distribution $S(TM)$ [4, 5].

Denote by $\bar{R}$, R and R^* the curvature tensors of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations (2.4)∼(2.7) for M and $S(TM)$, for any $X, Y, Z, W \in \Gamma(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} &\bar{g}(\bar{R}(X,Y)Z,\, PW) \\ &= g(R(X,Y)Z,\, PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\bar{g}(\bar{R}(X,Y)Z,\, \xi) \\ &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + [\tau(X) - \pi(X)]B(Y,Z) - [\tau(Y) - \pi(Y)]B(X,Z), \end{aligned} \tag{2.16}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, N) = g(R(X,Y)Z,\, N), \tag{2.17}$$

$$\begin{aligned} &\bar{g}(\bar{R}(X,Y)\xi,\, N) \\ &= \bar{g}(R(X,Y)\xi,\, N) \\ &= g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X,Y), \end{aligned} \tag{2.18}$$

$$\begin{aligned} &g(R(X,Y)PZ,\, PW) \\ &= g(R^*(X,Y)PZ,\, PW) \\ &\quad + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \end{aligned} \tag{2.19}$$

$$\begin{aligned} &g(R(X,Y)PZ,\, N) \\ &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad + [\tau(Y) + \pi(Y)]C(X,PZ) - [\tau(X) + \pi(X)]C(Y,PZ). \end{aligned} \tag{2.20}$$

The *Ricci tensor*, denoted by $\bar{Ric}$, of $\bar{M}$ is defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(Z,X)Y\} \tag{2.21}$$

for any X, $Y \in \Gamma(T\bar{M})$. Locally, $\bar{Ric}$ is given by

$$\bar{Ric}(X,Y) = \sum_{i=1}^{m+2} \epsilon_i \, \bar{g}(\bar{R}(E_i, X)Y,\ E_i),$$

where $\{E_1, \dots, E_{m+2}\}$ is an orthonormal frame field of $T\bar{M}$. If

$$\bar{Ric} = \bar{\kappa} g, \tag{2.22}$$

then we say that $\bar{M}$ is an *Einstein manifold*. If $\dim(\bar{M}) > 2$, then $\bar{\kappa}$ is a constant. For $\dim(\bar{M}) = 2$, any $\bar{M}$ is Einstein but $\bar{\kappa}$ in (2.22) is not necessarily constant. The *scalar curvature* $\bar{r}$ is defined by

$$\bar{r} = \sum_{i=1}^{m+2} \epsilon_i \, \bar{Ric}(E_i,\ E_i). \tag{2.23}$$

Putting (2.22) in (2.23) implies that $\bar{M}$ is Einstein if and only if

$$\bar{Ric} = \frac{\bar{r}}{m+2} \, \bar{g}.$$

A semi-Riemannian manifold $\bar{M}$ of constant curvature c is called a *space form* and denote it by $\bar{M}(c)$. In this case, the curvature $\bar{R}$ of $\bar{M}$ is given by

$$\bar{R}(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}, \quad \forall\, X,\ Y,\ Z \in \Gamma(T\bar{M}). \tag{2.24}$$

3. Induced Ricci and scalar curvatures

Consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $TM^{\perp} = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\bar{M}$. Then, by using (2.21), we obtain

$$\begin{aligned} \bar{Ric}(X,Y) \ &= \sum_{a=1}^{m} \epsilon_a \, \bar{g}(\bar{R}(W_a, X)Y,\ W_a) \\ &\quad + \bar{g}(\bar{R}(\xi, X)Y,\ N) + \bar{g}(\bar{R}(N, X)Y,\ \xi), \end{aligned} \tag{3.1}$$

where ϵ_a denotes the causal character (± 1) of respective vector field W_a. Let $R^{(0,\,2)}$ denote the induced Ricci type tensor of type (0, 2) on M given by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}\,, \quad \forall\, X,\ Y \in \Gamma(TM). \tag{3.2}$$

Using the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, we obtain

$$R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a \, g(R(W_a, X)Y,\ W_a) + \bar{g}(R(\xi, X)Y,\ N).$$

Substituting (2.15) and (2.17) in (3.1) an using (2.12) and (2.13), we obtain

$$\begin{aligned} R^{(0,2)}(X,Y) \;&=\; \bar{Ric}(X,Y) + B(X,Y)trA_N \\ &\quad - g(A_N X, A^*_\xi Y) - \bar{g}(R(\xi,Y)X,\, N) \end{aligned} \tag{3.3}$$

for any $X,\, Y \in \Gamma(TM)$. This shows that $R^{(0,\,2)}$ is not symmetric. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor*, denoted by Ric, if it is symmetric. Using (2.18), (3.3) and the first Bianchi's identity, we obtain

$$R^{(0,\,2)}(X,\,Y) - R^{(0,\,2)}(Y,\,X) = 2d\tau(X,\,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Theorem 3.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric metric connection. Then the Ricci type tensor $R^{(0,\,2)}$ of M is an induced symmetric Ricci tensor if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

Note 2. If $R^{(0,\,2)}$ is symmetric, then the 1-form τ is closed on TM. Therefore there exists a smooth function f such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\bar{\xi} = \alpha\,\xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\text{In}\,\alpha)$. Setting $\alpha = \exp(f)$ in this equation, we get $\bar{\tau}(X) = 0$ for any $X \in \Gamma(TM)$. We call the pair $\{\xi,\, N\}$ such that the corresponding 1-form τ vanishes the *canonical null pair* of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ considered by Kupeli [11]. Thus all $S(TM)$ are mutually isomorphic. For this reason, we consider only lightlike hypersurfaces M with the canonical null pair $\{\xi, N\}$ of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric metric connection.

The scalar curvature $\bar{r}$ of $\bar{M}$, defined by (2.23), and the scalar quantity r of M, obtained from $R^{(0,\,2)}$ by the method of (2.23) are given by

$$\bar{r} = \bar{Ric}(\xi,\xi) + \bar{Ric}(N,N) + \sum_{a=1}^{m} \epsilon_a\, \bar{Ric}(W_a, W_a),$$

$$r = R^{(0,\,2)}(\xi,\xi) + \sum_{a=1}^{m} \epsilon_a\, R^{(0,\,2)}(W_a, W_a),$$

respectively. Using these relations and (3.3), we obtain

$$\begin{aligned} R^{(0,\,2)}(\xi,\xi) \;&=\; \bar{Ric}(\xi,\xi) \\ R^{(0,\,2)}(W_a,W_a) \;&=\; \bar{Ric}(W_a,W_a) + g(A^*_\xi W_a, W_a)trA_N \\ &\quad - g(A_N W_a, A^*_\xi W_a) - \bar{g}(R(\xi,\, W_a)W_a,\, N). \end{aligned}$$

Thus we have

$$\begin{aligned} r \;&=\; \bar{r} + trA^*_\xi\, trA_N - tr(A^*_\xi A_N) \\ &\quad - \sum_{a=1}^{m} \epsilon_a \{\bar{g}(R(\xi, W_a)W_a, N) + \bar{g}(\bar{R}(N, W_a)W_a, N)\}. \end{aligned} \tag{3.4}$$

For any semi-Riemannian space form $\bar{M}(c)$, we have

$$\bar{R}(\xi, Y)X = c\bar{g}(X, Y)\xi, \quad \bar{R}ic(X, Y) = (m+1)c\,\bar{g}(X, Y)$$

and $\bar{r} = c\,m(m+1)$, $\bar{g}(\bar{R}(N, W_a)W_a, N) = 0$. Thus

$$R^{(0,2)}(X,Y) \;=\; mc\,g(X,Y) + B(X,Y)trA_N - g(A_N X, A^*_\xi Y); \tag{3.5}$$

$$r \;=\; m^2 c + trA^*_\xi\, trA_N - tr(A^*_\xi A_N). \tag{3.6}$$

Definition 1. A lightlike hypersurface M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *screen homothetic* [1] if there exist a non-zero constant φ such that the shape operators A_N and A^*_ξ of M and its screen distribution $S(TM)$ respectively are related by $A_N = \varphi\, A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, PY), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.7}$$

Theorem 3.2. *Let M be a screen homothetic lightlike hypersurface of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric metric connection. Then the Ricci type tensor $R^{(0,2)}$ is symmetric, the screen distribution $S(TM)$ is integrable and M is a locally product manifold $L \times M^*$ where L is a null curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$.*

Proof. By using (2.18), (2.24) and the fact $A_N = \varphi\, A^*_\xi$, we show that τ is closed, i.e., $d\tau = 0$ on TM. Thus $R^{(0,2)}$ is symmetric. By using (3.7), we show that C is symmetric on $S(TM)$. Thus, by Theorem 2.1, $S(TM)$ is integrable. From Note 1, M is locally a product manifold $L \times M^*$, where L is a null curve and M^* is a leaf of the screen distribution $S(TM)$. □

Theorem 3.3. *Let M be a screen homothetic lightlike hypersurface of a semi-Riemannian manifold $\bar{M}(c)$ admitting a semi-symmetric metric connection. Then we have $c = 0$.*

Proof. Using (2.16), (2.24) and the facts $\bar{M} = \bar{M}(c)$ and $\tau = 0$, we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \pi(X)B(Y, Z) - \pi(Y)B(X, Z) \tag{3.8}$$

for all $X, Y, Z \in \Gamma(TM)$. Using (2.17), (2.20), (3.7) and (3.8), we have

$$(X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ) = c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \tag{3.9}$$

for all $X, Y, Z \in \Gamma(TM)$. Replace Y by ξ in this equation, we obtain

$$(\xi\varphi)B(X, PZ) = cg(X, PZ). \tag{3.10}$$

Assume that M is screen homothetic. By (3.10), we have $c = 0$. □

4. Einstein lightlike hypersurfaces

In this section, let M be a screen homothetic Einstein lightlike hypersurface equipped with the canonical null pair $\{\xi, N\}$ of a Lorentzian space form

$(\bar{M}^{m+2}(c), \bar{g})$ admitting a semi-symmetric metric connection. Under this hypothesis, we show that $S(TM)$ is integrable by Theorem 3.2 and $c = 0$ by Theorem 3.3. Using (3.5), (3.6) and M is Einstein, we obtain

$$r = Ric(\xi, \xi) + \sum_{a=1}^{m} \epsilon_a \, Ric(W_a, W_a) = \kappa\{g(\xi, \xi) + \sum_{a=1}^{m} \epsilon_a \, g(W_a, W_a)\} = \kappa m.$$

Thus we have

$$Ric(X, Y) = (r/m)g(X, Y)$$

which provides a geometric interpretation of lightlike Einstein hypersurfaces (same as in Riemannian case) as we have shown that the constant $\kappa = r/m$. Since ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0 due to (2.14) and A^*_ξ is $S(TM)$-valued real self-adjoint operator, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then we have

$$A^*_\xi E_i = \lambda_i E_i, \quad 1 \le i \le m.$$

Since M is screen homothetic and $Ric = \kappa g$, the equation (3.5) reduce to

$$g(A^*_\xi X, A^*_\xi Y) - s \, g(A^*_\xi X, Y) + \varphi^{-1}\kappa \, g(X, Y) = 0, \tag{4.1}$$

where $s = trA^*_\xi$. Put $X = Y = E_i$ in (4.1), each λ_i is a solution of the equation

$$x^2 - s\,x + \varphi^{-1}\kappa = 0. \tag{4.2}$$

The equation (4.2) has at most two distinct solutions which are smooth real valued functions on $\mathcal{U}$. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \rho$ and $\lambda_{p+1} = \cdots = \lambda_m = \sigma$, by renumbering if necessary. From (4.2), we have

$$s = \rho + \sigma = p\rho + (m - p)\sigma, \qquad \rho\sigma = \varphi^{-1}\kappa. \tag{4.3}$$

Since φ and κ are constants, $\rho\sigma$ is a constant. From this result and the equation

$$(p - 1)\rho = -(m - p - 1)\sigma, \tag{4.4}$$

we show that the functions ρ and σ are constants.

Theorem 4.1. *Let M be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form $(\bar{M}(c), \bar{g})$ admitting a semi-symmetric metric connection. Then M is locally a product manifold $L \times M_\rho \times M_\sigma$, where L is a null curve tangent to $TM^\perp$, and M_ρ and M_σ are totally umbilical leaves of some integerable distributions of M.*

Proof. If (4.2) has only one solution ρ, by Theorem 3.2, we show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, where $M^* = M_\rho$. Since $B(X, Y) = g(A^*_\xi X, Y) = \rho g(X, Y)$ for all $X, Y \in \Gamma(TM)$, M is totally umbilical. By (3.7), we get $C(X, PY) = \varphi\rho \, g(X, PY)$ for all $X, Y \in \Gamma(TM)$. Thus M^* is also totally umbilical. Thus this theorem is true. □

Assume that (4.2) has exactly two distinct solutions ρ and σ. In case $p = 0$ or $p = m$: We also show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$ and $M^* = M_\rho$ or M_σ. M and M^* are also totally umbilical. In this case, this theorem is also true. In case $0 < p < m$: Consider the following four distributions D_ρ, D_σ, D^s_ρ and D^s_σ on M;

$$D_\rho = \{X \in \Gamma(TM) \mid A^*_\xi X = \rho PX\}, \quad D^s_\rho = PD_\rho,$$
$$D_\sigma = \{U \in \Gamma(TM) \mid A^*_\xi U = \sigma PU\}, \quad D^s_\sigma = PD_\sigma.$$

Clearly we show that $D_\rho \cap D_\sigma = TM^\perp$ and $D^s_\rho \cap D^s_\sigma = \{0\}$.

Step 1. If $D_\rho \neq D_\sigma$, then $D_\rho \perp_g D_\sigma$ and $D_\rho \perp_B D_\sigma$.

For any $X \in \Gamma(D_\rho)$ and $U \in \Gamma(D_\sigma)$, we get $A^*_\xi PX = A^*_\xi X = \rho PX$ and $A^*_\xi PU = A^*_\xi U = \sigma PU$. This imply that the projection morphism P maps $\Gamma(D_\rho)$ onto $\Gamma(D^s_\rho)$ and $\Gamma(D_\sigma)$ onto $\Gamma(D^s_\sigma)$. Since PX and PU are eigenvector fields of the real self-adjoint operator A^*_ξ corresponding to the different eigenvalues ρ and σ respectively, we have $g(PX, PU) = 0$. From the facts $g(X,U) = g(PX,PU) = 0$ and $B(X,U) = g(A^*_\xi X, U) = \rho g(PX,PU) = 0$, we show that $D_\rho \perp_g D_\sigma$ and $D_\rho \perp_B D_\sigma$ respectively.

Step 2. If $D_\rho \neq D_\sigma$, then $S(TM) = D^s_\rho \oplus_{orth} D^s_\sigma$.

Since $\{E_i\}_{1 \leq i \leq p}$ and $\{E_a\}_{p+1 \leq a \leq m}$ are vector fields of D^s_ρ and D^s_σ respectively and D^s_ρ and D^s_σ are mutually orthogonal vector subbundle of $S(TM)$, D^s_ρ and D^s_σ are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus $S(TM) = D^s_\rho \oplus_{orth} D^s_\sigma$.

Step 3. $Im(A^*_\xi - \rho P) \subset \Gamma(D^s_\sigma)$ and $Im(A^*_\xi - \sigma P) \subset \Gamma(D^s_\rho)$.

From (4.1), we show that $(A^*_\xi)^2 - (\rho+\sigma)A^*_\xi + \rho\sigma P = 0$. Let $Y \in Im(A^*_\xi - \rho P)$, then there exists $X \in \Gamma(TM)$ such that $Y = (A^*_\xi - \rho P)X$. Then $(A^*_\xi - \sigma P)Y = 0$ and $Y \in \Gamma(D_\sigma)$. Thus $Im(A^*_\xi - \rho P) \subset \Gamma(D_\sigma)$. Since the morphism $A^*_\xi - \rho P$ maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have $Im(A^*_\xi - \rho P) \subset \Gamma(D^s_\sigma)$. By duality, we also have $Im(A^*_\xi - \sigma P) \subset \Gamma(D^s_\rho)$.

Step 4. D^s_ρ and D^s_σ are integrable distributions.

For X, $Y \in \Gamma(D_\rho)$ and $U \in \Gamma(D_\sigma)$, using (2.8), (2.12) and Step 1, we have

$$(\nabla_X B)(Y,U) = -\, g((A^*_\xi - \rho P)\nabla_X Y,\, U) + \rho B(X,Y)\eta(U).$$

Replacing Z by U to (3.8) and using Step 1, we have $(\nabla_X B)(Y,U) = (\nabla_Y B)(X,U)$. From this results, (2.10) and Step 1, we have $g((A^*_\xi - \rho P)[X,Y],\, U) = 0$. Since D^s_σ is non-degenerate and $Im(A^*_\xi - \rho P) \subset \Gamma(D^s_\sigma)$, we have $(A^*_\xi - \rho P)[X,Y] = 0$. Thus $[X,Y] \in \Gamma(D_\rho)$ and D_ρ is integrable. By duality, D_σ is also integrable. For any X, $Y \in \Gamma(D^s_\rho)$, since D_ρ is integrable, we have $[X,Y] \in \Gamma(D_\rho)$. Also since $S(TM)$ is integrable, $[X,Y] \in \Gamma(S(TM))$. This results imply $[X,Y] \in \Gamma(D^s_\rho)$. Thus D^s_ρ is integrable. By duality, so is D^s_σ.

Step 5. $\rho\pi(X) = \sigma\pi(X) = 0$ for all $X \in \Gamma(S(TM))$.

For $X,\, Y \in \Gamma(D^s_\rho)$, using (2.8), (2.12) and the fact ρ is a constant, we have

$$(\nabla_X B)(Y, Z) = -g((A^*_\xi - \rho P)\nabla_X Y, Z) + \rho B(X, Y)\eta(Z)$$

for any $Z \in \Gamma(TM)$. Using this equation, (2.10), (3.8) and the facts $(A^*_\xi - \rho P)[X,\, Y] = 0$ and $(A^*_\xi - \rho P)X = 0$ for any $X \in \Gamma(D_\rho)$, we obtain

$$\rho\pi(X)g(Y, Z) = \rho\pi(Y)g(X, Z).$$

Taking $Z \in \Gamma(S(TM))$ and using $S(TM)$ is non-degenerate, we have $\rho\pi(X)Y = \rho\pi(Y)X$. Suppose there exists a vector field $X_o \in \Gamma(D^s_\rho)$ such that $\rho\pi(X_o)_x \neq 0$ at each point $x \in M$, then $X = fX_o$ for any $X \in \Gamma(D^s_\rho)$, where f is a smooth function. It follows that all vectors from the fiber $(D^s_\rho)_x$ are colinear with $(X_o)_x$. It is a contradiction as $\dim\,(D^s_\rho)_x = p > 1$. Thus we have $\rho\pi|_{D^s_\rho} = 0$. Using this result and (4.4), we have $\sigma\pi|_{D^s_\rho} = 0$. By duality, we also have $\sigma\pi|_{D^s_\sigma} = 0$. By using (4.4), we have $\rho\pi|_{D^s_\sigma} = 0$. Thus we have our assertion.

Step 6. D^s_ρ and D^s_σ are auto-parallel distributions.

For all $X \in \Gamma(D^s_\rho)$ and $U \in \Gamma(D^s_\sigma)$, using (2.8), (2.12), the definition of D_ρ and D_σ and the fact that ρ and σ are constants, we have

$$(\nabla_X B)(U, Z) = -g((A^*_\xi - \sigma P)\nabla_X U,\, Z),$$
$$(\nabla_U B)(X, Z) = -g((A^*_\xi - \rho P)\nabla_U X,\, Z),$$

for all $Z \in \Gamma(S(TM))$. Using this results, (3.8) and Step 5, we obtain

$$g(\{(A^*_\xi - \sigma P)\nabla_X U - (A^*_\xi - \rho P)\nabla_U X\},\, Z) \;=\; 0.$$

Using Step 3 and the fact $S(TM)$ is non-degenerate, we get

$$(A^*_\xi - \sigma P)\nabla_X U = (A^*_\xi - \rho P)\nabla_U X.$$

As the left term of this equation is in $\Gamma(D^s_\rho)$ and the right term is in $\Gamma(D^s_\sigma)$, and $D^s_\rho \cap D^s_\sigma = \{0\}$, we have $(A^*_\xi - \sigma P)\nabla_X U = 0$ and $(A^*_\xi - \rho P)\nabla_U X = 0$. This imply $\nabla_X U \in \Gamma(D_\sigma)$ and $\nabla_U X \in \Gamma(D_\rho)$. From the facts $\nabla_X U = \nabla^*_X U$ and $\nabla_U X = \nabla^*_U X$ by Step 1, we have

$$\nabla_X U \in \Gamma(D^s_\sigma), \quad \nabla_U X \in \Gamma(D^s_\rho), \quad \forall\, X \in \Gamma(D^s_\rho),\ \forall\, U \in \Gamma(D^s_\sigma). \tag{4.5}$$

For $X,\, Y \in \Gamma(D^s_\rho)$ and $U,\, V \in \Gamma(D^s_\sigma)$, since $g(X, U) = 0$, we have

$$g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0, \quad g(\nabla_V U, X) + g(U, \nabla_V X) = 0.$$

Using (4.5) and Step 1, we have $g(X, \nabla_Y U) = g(U, \nabla_V X) = 0$. Thus we get

$$g(\nabla_Y X,\, U) = 0, \quad g(X,\, \nabla_V U) = 0. \tag{4.6}$$

This results imply that D^s_ρ and D^s_σ are auto-parallel distributions.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D^s_\rho \oplus_{orth} D^s_\sigma$, where D^s_ρ and D^s_σ are auto-parallel distributions with respect to the induced connection ∇^* on M^* by Step 6, by the decomposition theorem of de Rham [2], we have $M^* = M_\rho \times M_\sigma$, where M_ρ and M_σ are leaves of D^s_ρ and D^s_σ respectively. Thus we have our theorem.

Theorem 4.2. *Let M be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form $(\bar{M}(c), \bar{g})$ admitting a semi-symmetric metric connection. Then $c = 0$ and M is locally a product manifold $L \times M_\rho \times M_\sigma$, where L is a null curve tangent to $TM^\perp$, and M_ρ and M_σ are leaves of some integerable distributions of M such that*

(1) *If $\kappa \neq 0$, M_ρ or M_σ is an m-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of κ, and the other is a point.*

(2) *If $\kappa = 0$, M_ρ is an $(m-1)$-dimensional or an m-dimensional totally geodesic Euclidean space and M_σ is a spacelike curve or a point.*

Proof. First of all, we prove if $0 < p < m$, then $\kappa = 0$. Moreover $\rho\sigma = 0$: For $X \in \Gamma(D^s_\rho)$ and $U \in \Gamma(D^s_\sigma)$, using (2.10), (4.5), (4.6) and Step 5, we have

$$g(R(X,U)U,\, X) = g(\nabla_X \nabla_U U,\, X).$$

From the second equation of (4.6), we know that $\nabla_U U$ has no component of D_ρ. Since P maps $\Gamma(D_\sigma)$ onto $\Gamma(D^s_\sigma)$ and $S(TM) = D^s_\rho \oplus_{orth} D^s_\sigma$, we have

$$\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi, \quad P(\nabla_U U) \in \Gamma(D^s_\sigma).$$

Using (2.6), (2.12), (3.7) and the facts $A^*_\xi X = \rho X$ and $A^*_\xi U = \sigma U$, we have

$$\nabla_X \nabla_U U = \nabla_X P(\nabla_U U) + X(\eta(\nabla_U U))\xi - \varphi\rho\sigma g(U,U)PX.$$

Due to $(4.5)_1$, we show that $g(\nabla_X P(\nabla_U U), X) = 0$. From the above results we deduce

$$g(R(X,U)U,\, X) = -\,\varphi\rho\sigma g(X,X)g(U,U).$$

On the other hand, from the Gauss equation (2.15), we have

$$g(R(X,U)U,\, X) = \varphi\rho\sigma g(X,X)g(U,U).$$

From the last two equations, we show that if $0 < p < m$, then $\kappa = \varphi\rho\sigma = 0$.

(1) Let $\kappa \neq 0$: In case $(trA^*_\xi)^2 \neq 4\varphi^{-1}\kappa$. The equation (4.2) has two non-vanishing distinct solutions ρ and σ. If $0 < p < m$, then we have $\kappa = 0$. Thus $p = 0$ or $p = m$. If $p = 0$, then $D^s_\rho = \{0\}$ and $D^s_\sigma = S(TM)$. If $p = m$, then $D^s_\rho = S(TM)$ and $D^s_\sigma = \{0\}$. From (2.15) and (2.19), we have

$$R^*(X,Y)Z = 2\varphi\rho^2\{g(Y,\, Z)X - g(X,\, Z)Y\}, \quad \forall\ X,\, Y,\, Z \in \Gamma(D_\rho);$$
$$R^*(U,V)W = 2\varphi\sigma^2\{g(V,\, W)U - g(U,\, W)V\}, \quad \forall\ U,\, V,\, W \in \Gamma(D_\sigma).$$

Thus either M_ρ or M_σ (which are leaves of D_ρ and D_σ respectively) is a Riemannian manifold M^* of constant curvature either $2\varphi\rho^2$ or $2\varphi\sigma^2$, and the other leaf is a point $\{x\}$. If $M^* = M_\rho$, for all $X, Y \in \Gamma(S(TM))$, since $B(X,Y) = \rho g(X,Y)$, we have $C(X,Y) = \varphi\rho g(X,Y)$ and $Ric^*(X,Y) = 2(m-1)\varphi\rho g(X,Y)$. If $M^* = M_\sigma$, for all $U, V \in \Gamma(S(TM))$, since $B(U,V) = \sigma g(U,V)$, we have $C(U,V) = \varphi\sigma g(U,V)$ and $Ric^*(U,V) = 2(m-1)\varphi\sigma g(U,V)$. Thus M^* is a totally umbilical and M is locally a product manifold $L \times M^* \times \{x\}$ or $L \times \{x\} \times M^*$, where M^* is an m-dimensional totally umbilical Einstein Riemannian space form of constant curvature $2\varphi\sigma^2$ or $2\varphi\rho^2$ which is isometric to a sphere or a hyperbolic space, $\{x\}$ is a point.

In case $(trA^*_\xi)^2 = 4\varphi^{-1}\kappa$. The equation (4.2) has only one non-zero constant solution, named by ρ, and ρ is only one eigenvalue of A^*_ξ. In this case, the equations (4.3) reduce to $s = 2\rho = m\rho$; $\rho^2 = \varphi^{-1}\kappa$. Thus we have $m = 2$. From (2.15) and (2.19), we have

$$R^*(X,Y)Z = 2\kappa\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(S(TM)).$$

Thus M^* is a Riemannian 2-surface of constant curvature 2κ. Since $B(X,Y) = \rho g(X,Y)$ for all X, $Y \in \Gamma(TM)$, we have $C(X,Y) = \varphi\rho g(X,Y)$ for all X, $Y \in \Gamma(S(TM))$. Thus M^* is a totally umbilical and M is a locally product $L \times M^* \times \{x\}$ where L is a null curve and M^* is a Riemannian 2-surface of constant curvature 2κ which is isometric to a 2-sphere or a 2-hyperbolic space.

(2) Let $\kappa = 0$: The equation (4.2) reduces to $x(x - s) = 0$. In case $trA^*_\xi \neq 0$. Let $\rho = 0$ and $\sigma = s$. From (4.4), we have $(m - p - 1)s = 0$. So $p = m - 1$. Thus the leaf M_ρ of D^s_ρ is totally geodesic $(m-1)$-dimensional Riemannian manifold and the leaf M_σ of D^s_σ is a spacelike curve. In the sequel, let $X, Y, Z \in \Gamma(D^s_\rho)$ and $U \in \Gamma(D^s_\sigma)$. From (2.15), (2.19) and $c = 0$, we have $R^*(X,Y)Z = R(X,Y)Z = \bar{R}(X,Y)Z = 0$. Using (4.6) and the fact ∇^* is metric, we have

$$g(\nabla^*_X Y,\, U) = -g(Y,\, \nabla^*_X U) = -g(Y,\, \nabla_X U) = 0.$$

Thus $\nabla^*_X Y \in \Gamma(D^s_\rho)$. From this result, (2.6), (4.5) and the integrable property of D^s_ρ, we have $g(R^*(X,Y)Z, U) = 0$. This implies $Q_\rho R^*(X,Y)Z = R^*(X,Y)Z = 0$, where Q_ρ is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(D^s_\rho)$ and $Q_\rho R^*$ is the curvature tensor of D^s_ρ. Thus M_ρ is a Euclidean manifold and M is locally a product $L \times M_\rho \times M_\sigma$, where M_ρ is an $(m-1)$-dimensional totally geodesic Euclidean space and M_σ is a spacelike curve in M.

In case $trA^*_\xi = 0$. Then we have $\rho = \sigma = 0$ and $A^*_\xi = 0$ or equivalently $B = 0$ and $D^s_\rho = D^s_\sigma = S(TM)$. Thus M is totally totally geodesic in $\bar{M}$. Since M is screen homothetic, we also have $C = A_N = 0$. Thus the leaf M^* of $S(TM)$ is also totally geodesic. Thus we have $\bar{\nabla}_X Y = \nabla^*_X Y$ for any tangent vector fields X and Y to the leaf M^*. This implies that M^* is a Euclidean m-space. Thus M is locally a product $L \times M^* \times \{x\}$ where L is a null curve and $\{x\}$ is a point. □

References

[1] C. Atindogbe and K. L. Duggal, *Conformal screen on lightlike hypersurfaces*, Int. J. Pure Appl. Math. **11** (2004), no. 4, 421–442.

[2] G. de Rham, *Sur la réductibilité d'un espace de Riemannian*, Comm. Math. Helv. **26** (1952), 328–344.

[3] K. L. Duggal, *On scalar curvature in lightlike geometry*, J. Geom. Phys. **57** (2007), no. 2, 473–481.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[5] K. L. Duggal and D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[6] ———, *A Classification of Einstein lightlike hypersurfaces of a Lorentzian space form*, J. Geom. Phys. **60** (2010), no. 12, 1881–1889.

[7] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Frontiers in Mathematics, Birkhäuser, 2010.

[8] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge, 1973.

[9] H. A. Hayden, *Subspace of a space with torsion*, Proc. London Math. Soc. **34** (1932), 27–50.

[10] A. Fialkow, *Hypersurfaces of a space of constant curvature*, Ann. of Math. **39** (1938), no. 4, 762–785.

[11] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Mathematics and Its Applications, vol. 366, Kluwer Acad. Publishers, Dordrecht, 1996.

[12] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[13] T. Y. Thomas, *On closed spaces of constant mean curvature*, Amer. J. Math. **58** (1936), no. 4, 702–704.

[14] M. M. Tripathi, *A new connection in a Riemannian manifold*, Int. Electron. J. Geom. **1** (2008), no. 1, 15–24.

[15] K. Yano, *On semi-symmetric metric connections*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579–1586.

Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr