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THE SNOWFLAKE CURVE AS AN ATTRACTOR OF AN ITERATED FUNCTION SYSTEM

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Abstract. Although the snowflake curve, the boundary of the Koch snowflake, is one of the well-known fractals, there is no iterated function system (IFS) for it in the literature. In this study, we give an IFS for this familiar closed curve.

1. Introduction

Many fractals can be obtained as attractors of iterated function systems (IFS) ([1], [2], [3]). The Koch curve is a well-known example. It is the attractor of the IFS consisting of the similarity contractions f_1, f_2, f_3, f_4 on the plane such that

$$\begin{aligned}
f_1(x,y) &= \left(\frac{x}{3}, \frac{y}{3}\right)\\
f_2(x,y) &= \left(\frac{x}{6} - \frac{\sqrt{3}y}{6} + \frac{1}{3}, \frac{\sqrt{3}x}{6} + \frac{y}{6}\right)\\
f_3(x,y) &= \left(\frac{x}{6} + \frac{\sqrt{3}y}{6} + \frac{1}{2}, -\frac{\sqrt{3}x}{6} + \frac{y}{6} + \frac{\sqrt{3}}{6}\right)\\
f_4(x,y) &= \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right)
\end{aligned}$$

(see Fig. 1) [4]. It is known that there is another IFS for the Koch curve consisting of only two contractions with contractivity factor $\frac{1}{\sqrt{3}}$ which map the triangle ABC to the triangles DAB and EAC respectively as shown in Fig. 2. The Koch snowflake can also be obtained as the attractor of the IFS consisting of seven contractions which can be immediately written from Fig. 3. These contractions are also similarities.

Since the images of an attractor under the contractions of an IFS are usually small copies of the attractor, the notion of self-similarity is geared to the notion of iterated function system. One can easily exhibit some IFS for the Koch curve

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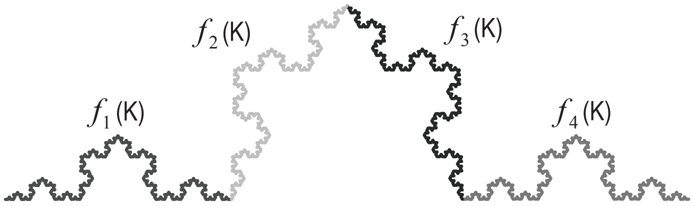

FIGURE 1. The Koch curve K as the union of 4 similar copies of itself.

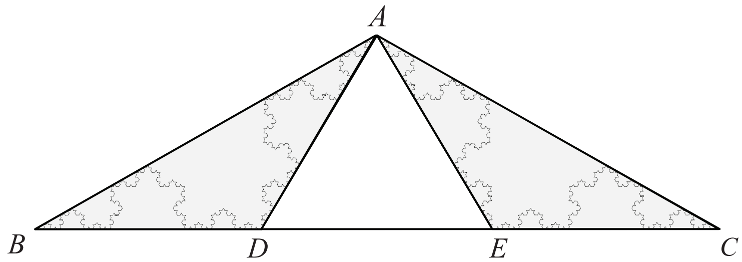

FIGURE 2. The Koch curve K as the union of 2 similar copies of itself.

and the Koch snowflake. However, it is clear that there is no IFS consisting only of similarity contractions for the snowflake curve.

In the next section, we give an IFS consisting of 8 contractions each of which is the composition of some similarities, symmetries and foldings.

2. An IFS for the snowflake curve

Consider the contractions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_1(x,y) = \left(-\frac{1}{3}|x|, \frac{1}{3}y\right) + \left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$$

$$f_2(x,y) = \left(\frac{1}{\sqrt{3}}|y|, -\frac{1}{\sqrt{3}}|x|\right) + \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Let f_1^1 be the folding defined by $f_1^1(x,y) = (|x|, y)$ for all $x, y \in \mathbb{R}$. Let f_1^2 be the projective similarity with center $C = \left(-\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$ and contractivity factor $\frac{1}{3}$ which maps the polygonal path $ABCDEFG$ onto the path $A'B'C'D'E'F'G'$ as shown in Fig. 4. Then the contraction f_1 can be written as the composition of the folding f_1^1 and the projective similarity f_1^2 as shown in Fig. 5: $f_1 = f_1^2 \circ f_1^1$.

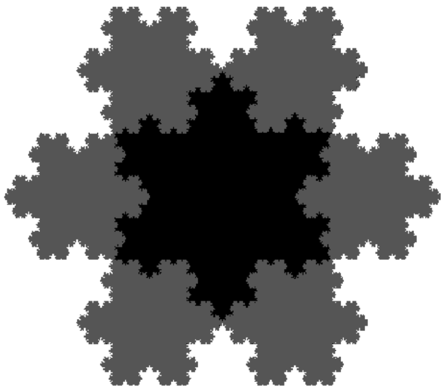

FIGURE 3. The Koch snowflake as the union of 7 similar copies of itself.

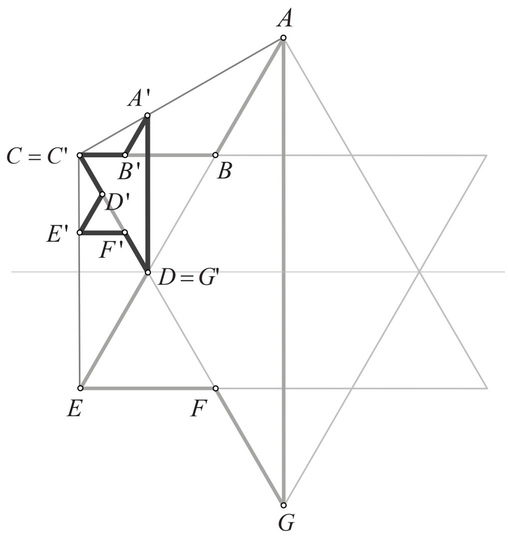

FIGURE 4. f_1^2 maps the path $ABCDEFG$ onto the path $A'B'C'D'E'F'G'$.

Let f_2^1 and f_2^2 be the foldings defined on the plane by $f_2^1(x,y) = (x,|y|)$ and $f_2^2(x,y) = (-|x|,y)$ respectively. Let f_2^3 be the projective similarity with center $C = \left(0, \frac{\sqrt{3}}{2}\right)$ and contractivity factor $\frac{1}{\sqrt{3}}$ which maps the polygonal path $ABCD$ onto the polygonal path $A'B'C'D'$ as shown in Fig. 6. Let f_2^4 be the

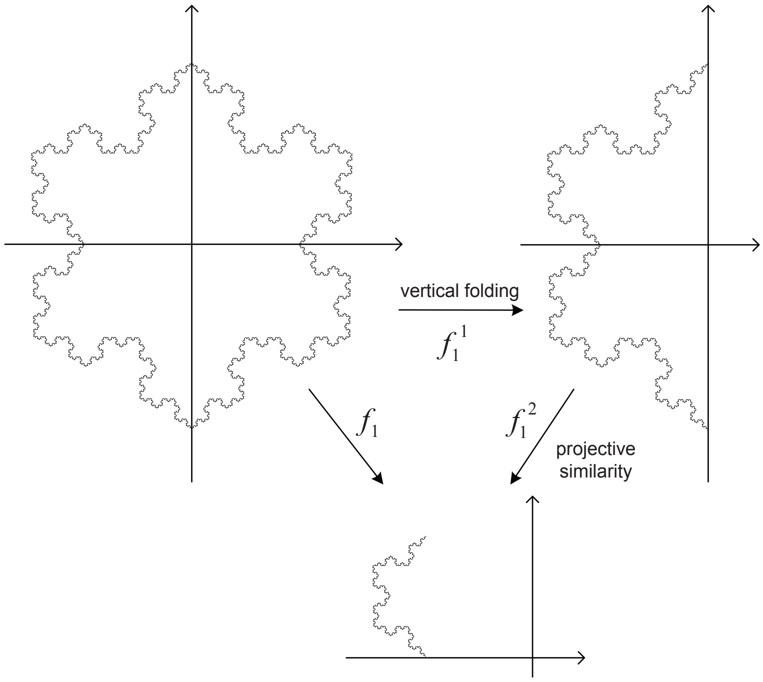

FIGURE 5. The mapping $f_1 = f_1^2 \circ f_1^1$.

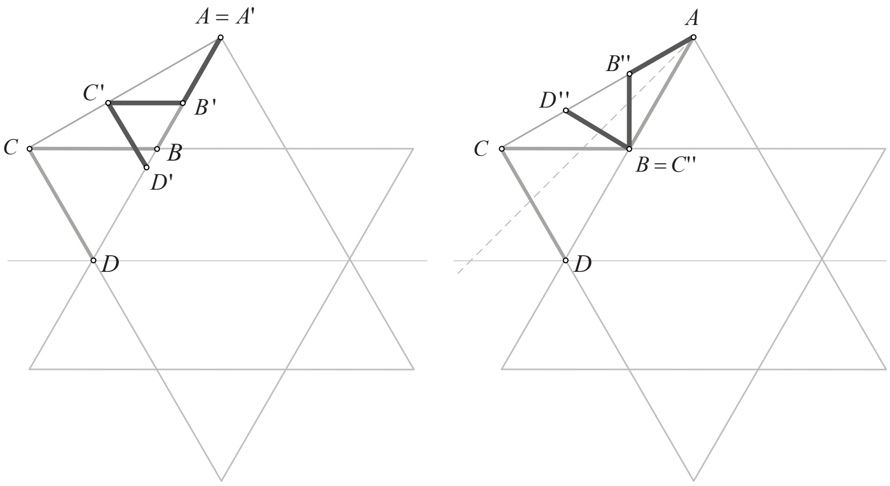

FIGURE 6. The projective similarity f_2^3 (left) and the symmetry f_2^4 (right).

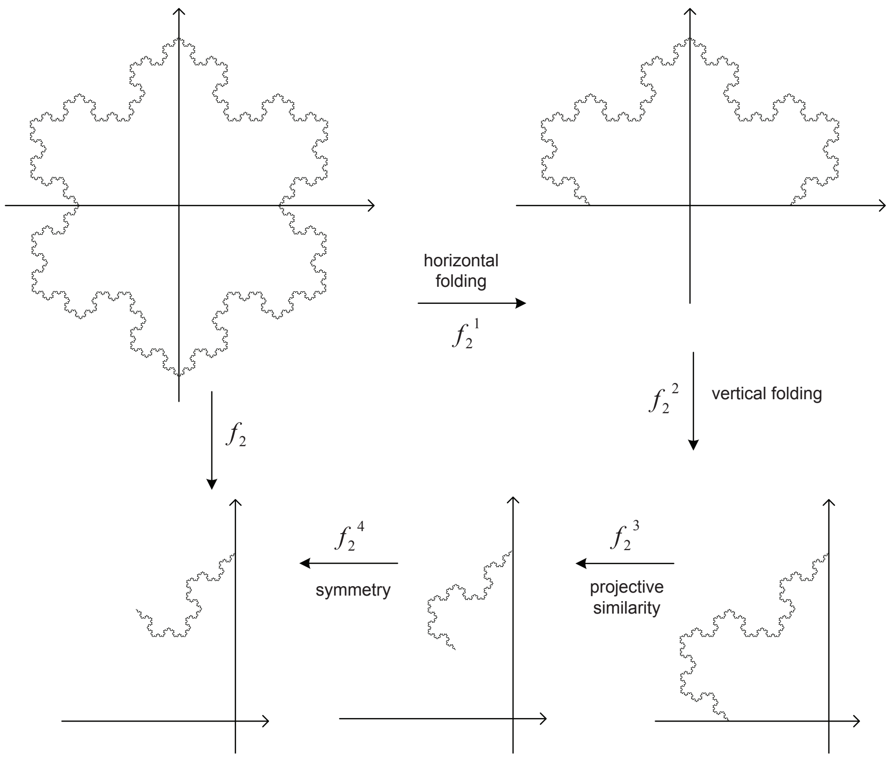

FIGURE 7. The map $f_2 = f_2^4 \circ f_2^3 \circ f_2^2 \circ f_2^1$.

symmetry with respect to the angle bisector between AC and AD as shown in Fig 6. Then the contraction f_2 can be written as the composition of the foldings f_2^1,f_2^2 and the projective similarity f_2^3 and the symmetry f_2^4 as shown in Fig. 7: $f_2 = f_2^4 \circ f_2^3 \circ f_2^2 \circ f_2^1$.

Note that f_1 and f_2 map the snowflake curve, S, onto its quadrant (see Fig. 8).

The mappings f_1 and f_2 may be adapted as below (f_3 to f_8) to map the snowflake curve onto its parts in the other three quadrants.

$$f_3(x,y) = \left(-\frac{1}{3}|x|, \frac{1}{3}y\right) + \left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}\right)$$

$$f_4(x,y) = \left(\frac{1}{\sqrt{3}}|y|, \frac{1}{\sqrt{3}}|x|\right) + \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$f_5(x,y) = \left(\frac{1}{3}|x|, \frac{1}{3}y\right) + \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right)$$

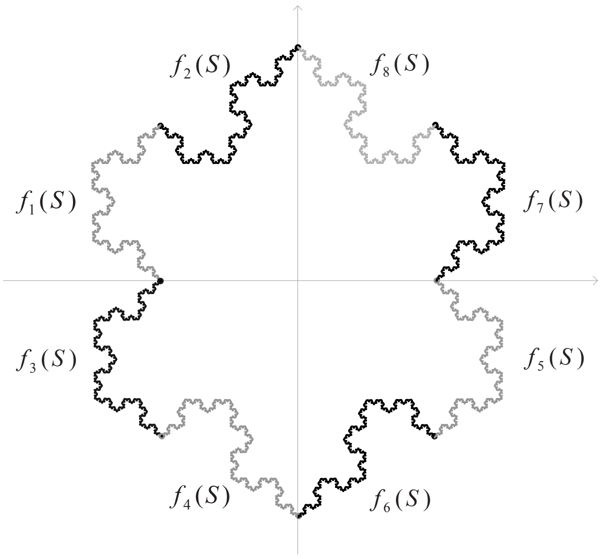

FIGURE 8. The snowflake curve S as an attractor of the IFS.

$$\begin{aligned}
f_6(x,y) &= \left(-\frac{1}{\sqrt{3}}|y|, \frac{1}{\sqrt{3}}|x|\right) + \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \\
f_7(x,y) &= \left(\frac{1}{3}|x|, \frac{1}{3}y\right) + \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right) \\
f_8(x,y) &= \left(-\frac{1}{\sqrt{3}}|y|, -\frac{1}{\sqrt{3}}|x|\right) + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).
\end{aligned}$$

The attractor of the IFS consisting of $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ is the desired fractal, the snowflake curve (see Fig. 8).

There are alternative iterated function systems for the snowflake curve each of which is formed by 6 contractions. Let g_1 be the mapping that folds the right part of the line AG onto the left part of it (as done by f_1^1 before) and g_2 be the mapping that folds the below part of the line EH onto the above part of it (see Fig. 9 (left)). $g_2 \circ g_1$ maps the snowflake curve to the Koch curve. As done in the previous section (see Fig. 2), the Koch curve can be obtained as the union of the images of itself under two similarities h_1 and h_2. So that the Koch curve is the union of the images of the snowflake curve under the contractions $f_1 = h_1 \circ g_2 \circ g_1$ and $f_2 = h_2 \circ g_2 \circ g_1$.

One can apply similar procedure to obtain the other parts (the other two Koch curves) of the snowflake curve (see Fig. 9).

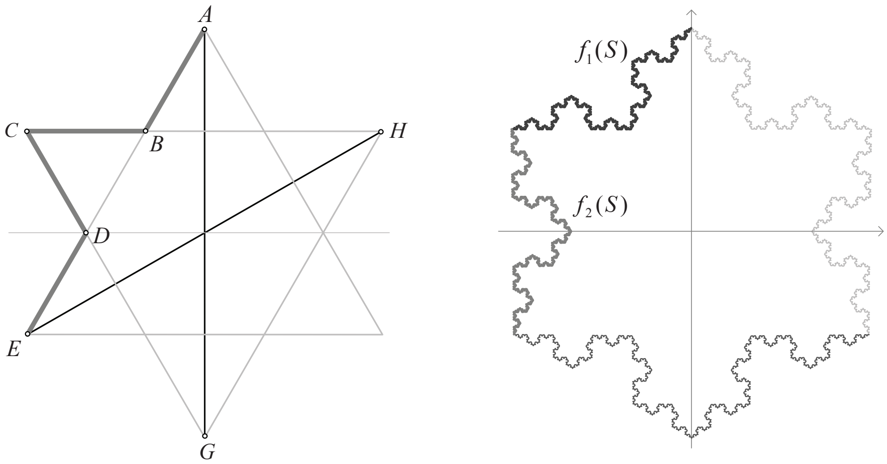

FIGURE 9. The lines AG and EH (left) and the images of S under f_1 and f_2 (right).

Note: In general, a folding map g that maps the half plane $ax + by + c > 0$ onto the other half plane can be defined by

$$g(x, y) = \left(\frac{b^2x - aby - ac - a|ax + by + c|}{a^2 + b^2}, \frac{a^2y - abx - bc - b|ax + by + c|}{a^2 + b^2} \right).$$

Note that in our case $a = 1$, $b = -\sqrt{3}$ and $c = 0$.

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