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A CERTAIN SUBCLASS OF JANOWSKI TYPE FUNCTIONS ASSOCIATED WITH k-SYMMETRIC POINTS

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Abstract. We introduce a subclass $\mathcal{S}_s^{(k)}(A,B)$ $(-1 \leq B < A \leq 1)$ of functions which are analytic in the open unit disk and close-to-convex with respect to k-symmetric points. We give some coefficient inequalities, integral representations and invariance properties of functions belonging to this class.

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are analytic in the open unit disk $\mathbb{U}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$.

Let $f(z)$ and $F(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leq 1$ and $f(z) = F(w(z))$, denote by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Now, we denote by $\mathcal{S}^*(A,B)$ and $\mathcal{C}(A,B)$ the subclasses of $\mathcal{A}$ as follows:

$$\mathcal{S}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U} \right\} \tag{1}$$

and

$$\mathcal{C}(A,B) = \left\{ f \in \mathcal{A} : \exists g \in \mathcal{S}^*(A,B) \text{ such that } \frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U} \right\}, \tag{2}$$

respectively. For $A = 1 - 2\alpha$ and $B = -1$ in (1) and (2), we can obtain the classes $\mathcal{S}^*(1-2\alpha,-1) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}(1-2\alpha,-1) = \mathcal{C}(\alpha)$, consisting of functions which are starlike of order α and close-to-convex of order α, respectively. Especially, we can obtain the classes $\mathcal{S}^*(1,-1) = \mathcal{S}^*$ and $\mathcal{C}(1,-1) = \mathcal{C}$ which

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are the classes of starlike functions and close-to-convex functions, respectively, for $A = 1$ and $B = -1$.

Sakaguchi [6] once introduced a classes $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{3}$$

Following him, many authors discussed this class and its subclasses (see [4], [5], [7], [8], [10], [11], [12] and [13]). In the present paper, we introduced the following class of analytic functions with respect to k-symmetric points, and obtain some interesting results.

Definition. Let $\mathcal{S}_s^{(k)}(A, B)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left|\frac{zf'(z)}{f_k(z)} - 1\right| < \left|A - B\frac{zf'(z)}{f_k(z)}\right| \quad (z \in \mathbb{U}), \tag{4}$$

where $-1 \le A < B \le 1, k \ge 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} \varepsilon^{-\mu} f(\varepsilon^{\mu} z), \tag{5}$$

where $\varepsilon = \exp(\frac{2\pi i}{k})$ with $k \in \mathbb{Z}$.

By the definition of $f_k(z)$, we can easily obtain the expansion of $f_k(z)$. That is, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$f_k(z) = z + \sum_{n=2}^{\infty} \sigma_k(n) a_n z^n,$$

where $\sigma_k(n) = \begin{cases} 1, n = lk+1 \\ 0, n \neq lk+1 \end{cases}$ $(l \in \mathbb{N}_0)$. And we note that $f_1(z) = f(z)$ and $f_2(z) = \frac{1}{2}(f(z) - f(-z))$.

Now the following identities follow directly from the above definition [3]:

$$f_k(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_k(z), \tag{6}$$

$$f_k'(\varepsilon^{\mu} z) = f_k(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} f'(\varepsilon^{\mu} z). \tag{7}$$

Remark 1.1. Using the definition of the subordination, we can easily obtain that the equivalent condition of belonging to the class $\mathcal{S}_s^{(k)}(A, B)$ $(-1 \le B < A \le 1)$ is

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

It is easy to know that $\mathcal{S}_s^{(2)}(1,-1)=\mathcal{S}_s^*$ and $\mathcal{S}_s^{(1)}(1,-1)=\mathcal{S}^*$, so $\mathcal{S}_s^{(k)}(A,B)$ has a meaning as the generalization of $\mathcal{S}_s^*$ and $\mathcal{S}^*$, respectively.

In this paper, we will discuss the coefficient inequalities, integral representations and some invariance properties of functions belonging to the class $\mathcal{S}_s^{(k)}(A,B)$.

2. Coefficient inequalities

Theorem 2.1. *Let* $f(z)\in\mathcal{S}_s^{(k)}(A,B)$. *Then* $f_k(z)\in\mathcal{S}^*(A,B)\subset\mathcal{S}$.

Proof. For $f(z)\in\mathcal{S}_s^{(k)}(A,B)$, we can obtain $\frac{zf'(z)}{f_k(z)}\prec\frac{1+Az}{1+Bz}$. Substituting z by $\varepsilon^\mu z$ respectively $(\mu=0,1,2,\dots,k-1)$, then

$$\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)}{f_k(\varepsilon^\mu z)}\prec\frac{1+A\varepsilon^\mu z}{1+B\varepsilon^\mu z}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).\tag{8}$$

According to the definition of $f_k(z)$ and $\varepsilon=\exp(\frac{2\pi i}{k})$, we know $\varepsilon^{-\mu}f_k(\varepsilon^\mu z)=f_k(z)$. Then the equation (8) becomes

$$\frac{zf'(\varepsilon^\mu z)}{f_k(z)}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).\tag{9}$$

Let $\mu=0,1,2,\dots,k-1$ in (9) respectively, and sum them we can get

$$\frac{zf_k'(z)}{f_k(z)}=\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{zf'(\varepsilon^\mu z)}{f_k(z)}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).\tag{10}$$

That is, $f_k(z)\in\mathcal{S}^*(A,B)\subset\mathcal{S}$. □

Putting $A=1,B=-1$ and $k=2$ in Theorem 2.1, we can obtain the following corollary.

Corollary 2.2. *Let* $f(z)\in\mathcal{S}_s^*$, *defined as* (3). *Then the odd function* $\frac{1}{2}(f(z)-f(-z))$ *is a starlike function.*

Remark 2.3. Let $f(z)\in\mathcal{S}_s^{(k)}(A,B)$. Then $f(z)$ is a close-to-convex function.

We need the following lemma to give the coefficient estimate of functions in the class $\mathcal{S}_s^{(k)}(A,B)$.

Lemma 2.4. *Let* $f(z)=z+\sum_{n=2}^\infty a_nz^n$, $g(z)=z+\sum_{n=2}^\infty b_nz^n$ *and satisfy the inequality*

$$\left|\frac{zf'(z)}{g(z)}-1\right|\le\left|A-B\frac{zf'(z)}{g(z)}\right|\quad(z\in\mathbb{U}),$$

where $-1\le B<A\le1$. *Then for* $n\ge2$, *we have*

$$|na_n-b_n|^2\le2(1+|AB|)\sum_{j=1}^{n-1}j|a_j||b_j|.\tag{11}$$

Proof. Let $f(z)$ and $g(z)$ satisfy the inequality

$$\left|\frac{zf'(z)}{g(z)} - 1\right| \leq \left|A - B\frac{zf'(z)}{g(z)}\right| \quad (z \in \mathbb{U}). \tag{12}$$

Then (12) is equivalent to

$$\frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}.$$

By the definition of subordination, there exists a Schwarz function $w(z)$ which satisfies $w(0) = 0, |w(z)| < |z|$ and

$$\frac{zf'(z)}{g(z)} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U})$$

or

$$g(z) - zf'(z) = (Bzf'(z) - Ag(z))w(z) \quad (z \in \mathbb{U}).$$

Now if $w(z) = \sum_{n=1}^{\infty} c_n z^n$, then

$$\sum_{n=2}^{\infty}(b_n - na_n)z^n = \left((B-A)z + \sum_{n=2}^{\infty}(Bna_n - Ab_n)z^n\right)\left(\sum_{n=1}^{\infty} c_n z^n\right). \tag{13}$$

Comparing the coefficient of z^n in (13), we have

$$\begin{aligned} &b_n - na_n \\ &= (B-A)c_{n-1} + (2Ba_2 - Ab_2)c_{n-2} + \cdots + ((n-1)Ba_{n-1} - Ab_{n-1})c_1. \end{aligned} \tag{14}$$

Thus the coefficient combination on the right-hand side of (14) depends only on the coefficients combination $Ba_1 - Ab_1, \ldots, (n-1)Ba_{n-1} - Ab_{n-1}$ on the left-hand side. Hence, for $n \geq 2$, we can write

$$\sum_{j=2}^{n}(b_j - ja_j)z^j + \sum_{j=n+1}^{\infty} d_j z^j = \left(\sum_{j=1}^{n-1}(jBa_j - Ab_j)z^j\right) w(z), \tag{15}$$

with $a_1 = b_1 = 1$. Squaring the modulus of the both sides of (15) and integrating along $|z| = r < 1$, and using the fact that $|w(z)| < 1$, we obtain

$$\sum_{j=2}^{n}|b_j - ja_j|^2 r^{2j} + \sum_{j=n+1}^{\infty}|d_j|^2 r^{2j} < \sum_{j=1}^{n-1}|jBa_j - Ab_j|^2 r^{2j}.$$

Letting $r \to 1$ on the left-hand side of this inequality, we obtain

$$\sum_{j=2}^{n}|b_j - ja_j|^2 \leq \sum_{j=1}^{n-1}|jBa_j - Ab_j|^2.$$

This implies that

$$\begin{aligned}|na_n - b_n|^2 \leq& \sum_{j=1}^{n-1}\left(|jBa_j - Ab_j|^2 - |b_j - ja_j|^2\right)\\ \leq& \sum_{j=1}^{n-1}\left((B^2-1)j^2|a_j|^2 + (A^2-1)|b_j|^2 + 2j(1+|AB|)|a_j||b_j|\right)\\ \leq& 2(1+|AB|)\sum_{j=1}^{n-1} j|a_j||b_j|,\end{aligned}$$

since $-1 \leq B < A \leq 1$, hence the proof of Lemma 2.4 is complete. □

Applying the above Lemma 2.4, we give the following theorem about to the coefficient estimate of functions in $\mathcal{S}_s^{(k)}(A,B)$.

Theorem 2.5. *Let $f(z) \in \mathcal{S}_s^{(k)}(A,B)$. Then we have*

(i) *For $n = lk+1$ $(l \in \mathbb{N}_0)$,*

$$(n-1)^2|a_n|^2 \leq 2(1+|AB|)\sum_{j=0}^{l-1}(jk+1)|a_{jk+1}|^2. \tag{16}$$

(ii) *For $n \neq lk+1$ $(l \in \mathbb{N}_0)$,*

$$n^2|a_n|^2 \leq 2(1+|AB|)\sum_{j=0}^{[\frac{n-1}{k}]}(jk+1)|a_{jk+1}|^2, \tag{17}$$

where $[\frac{n-1}{k}]$ denotes the biggest integer among the integers smaller than $\frac{n-1}{k}$.

Proof. We note that $zf'(z)$ and $f_k(z)$ satisfy the condition of Lemma 2.4. And, at the same time, by the definition of $f_k(z)$ we have

$$\begin{aligned}f_k(z) =& z + \sum_{n=2}^{\infty}\sigma_k(n)a_n z^n\\ =& z + \sum_{l=1}^{\infty} a_{lk+1}z^{lk+1}.\end{aligned}$$

Using Lemma 2.4, let $n = lk+1$ in (11), we can get (16). And if $n \neq lk+1$, from (11), we can get (17). □

Next, we give that sufficient condition for functions belonging to the class $\mathcal{S}_s^{(k)}(A,B)$.

Theorem 2.6. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{U}$. If for $-1 \leq B < A \leq 1$, we have*

$$\sum_{n=2, n\neq lk+1}^{\infty}(1+|B|)n|a_n| + \sum_{l=1}^{\infty}(lk + (A-B)(lk+1))|a_{lk+1}| \leq A - B.$$

Then $f(z) \in \mathcal{S}_s^{(k)}(A,B)$.

Proof. At first, we note that $f_k(z) = z + \sum_{n=2}^{\infty} \sigma_k(n) a_n z^n$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For the proof of Theorem 2.6, it suffices to show that the values for zf'/f_k satisfy

$$\left| \frac{zf'(z) - f_k(z)}{Af_k(z) - Bzf'(z)} \right| \leq 1.$$

And we have

$$\begin{aligned} \left| \frac{zf'(z) - f_k(z)}{Af_k(z) - Bzf'(z)} \right| &= \left| \frac{\sum_{n=2}^{\infty}(n - \sigma_k(n)) a_n z^n}{(A-B)z + \sum_{n=2}^{\infty}(A\sigma_k(n) - Bn) a_n z^n} \right| \\ &\leq \frac{\sum_{n=2}^{\infty}(n - \sigma_k(n))|a_n||z|^{n-1}}{(A-B) - \sum_{n=2}^{\infty}|A\sigma_k(n) - Bn||a_n||z|^{n-1}} \\ &\leq \frac{\sum_{n=2}^{\infty}(n - \sigma_k(n))|a_n|}{(A-B) - \sum_{n=2}^{\infty}|A\sigma_k(n) - Bn||a_n|}. \end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}(n - \sigma_k(n))|a_n| \leq (A-B) - \sum_{n=2}^{\infty}|A\sigma_k(n) - Bn||a_n|,$$

which is equivalent to

$$\sum_{n=2}^{\infty}(n - \sigma_k(n) + |A\sigma_k(n) - Bn|)|a_n| \leq A - B. \tag{18}$$

Hence $\left| \frac{zf'(z) - f_k(z)}{Af_k(z) - Bzf'(z)} \right| \leq 1$, and Theorem 2.6 is proved. □

Corollary 2.7. *For* $k = 2, A = 1 - 2\alpha$ *and* $B = -1$ *in Theorem 2.6, we can obtain the result in* [1].

3. Integral representations and invariance properties

We give the integral representation of functions in the class $\mathcal{S}_s^{(k)}(A,B)$ and investigate the invariance properties of the following operators:

$$F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt$$

and

$$f_\lambda(z) = (1-\lambda)z + \lambda f(z),$$

where $m \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. And we introduce some lemmas we need.

Lemma 3.1 ([6]). *Let* $N(z)$ *be regular and* $D(z)$ *starlike in* $\mathbb{U}$ *and* $N(0) = D(0) = 0$. *Then for* $-1 \leq B < A \leq 1$,

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz}$$

implies that

$$\frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Lemma 3.2 ([2])**.** *If $g(z) \in \mathcal{S}^*(A,B)$, then*

$$G(z) = \frac{m+1}{z^m}\int_0^z t^{m-1}g(t)dt \in \mathcal{S}^*(A,B).$$

In Theorems 3.3 and 3.4, we give the integral representations of functions in $\mathcal{S}_s^{(k)}(A,B)$.

Theorem 3.3. *Let $f(z) \in \mathcal{S}_s^{(k)}(A,B)$. Then we have*

$$f_k(z) = z \cdot \exp\left\{(A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{w(\zeta)}{\zeta(1+Bw(\zeta))}d\zeta\right\}, \tag{19}$$

where $f_k(z)$ is defined by equality (5)*, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0, |w(z)| < 1$.*

Proof. Let $f(z) \in \mathcal{S}_s^{(k)}(A,B)$, from the definition of the subordination, we have

$$\frac{zf'(z)}{f_k(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \tag{20}$$

where $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$. Substituting z by $\varepsilon^\mu z$ respectively ($\mu = 0, 1, 2, \ldots, k-1$), we have

$$\frac{zf'(\varepsilon^\mu z)}{\varepsilon^{-\mu} f_k(\varepsilon^\mu z)} = \frac{1+Aw(\varepsilon^\mu z)}{1+Bw(\varepsilon^\mu z)} \tag{21}$$

for $\mu = 0, 1, 2, \ldots, k-1$, and $z \in \mathbb{U}$. Using the equality (6) and (7), sum (21) we can obtain

$$\frac{zf_k'(z)}{f_k(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1}\frac{1+Aw(\varepsilon^\mu z)}{1+Bw(\varepsilon^\mu z)},$$

and equivalently,

$$\frac{f_k'(z)}{f_k(z)} - \frac{1}{z} = (A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{w(\varepsilon^\mu z)}{z(1+Bw(\varepsilon^\mu z))}. \tag{22}$$

Integrating equality (22), we have

$$\begin{aligned}\log\frac{f_k(z)}{z} &= (A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z\frac{w(\varepsilon^\mu \zeta)}{\zeta(1+Bw(\varepsilon^\mu \zeta))}d\zeta \\ &= (A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{w(\zeta)}{\zeta(1+Bw(\zeta))}d\zeta.\end{aligned}$$

Therefore, arrange the above equality for $f_k(z)$, we can obtain

$$f_k(z) = z \cdot \exp\left\{(A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{w(\zeta)}{\zeta(1+Bw(\zeta))}d\zeta\right\},$$

and so the proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let $f(z) \in \mathcal{S}_s^{(k)}(A,B)$. Then we have*

$$f(z) = \int_0^z \exp\left\{(A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{w(t)}{t(1+Bw(t))}dt\right\} \cdot \left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right) d\zeta,$$

where $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$.

Proof. Let $f(z) \in \mathcal{S}_s^{(k)}(A,B)$, from equalities (19) and (20) we have

$$\begin{aligned} f'(z) &= \frac{f_k(z)}{z} \cdot \left(\frac{1+Aw(z)}{1+Bw(z)}\right) \\ &= \exp\left\{(A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{w(\zeta)}{\zeta(1+Bw(\zeta))}d\zeta\right\} \cdot \left(\frac{1+Aw(z)}{1+Bw(z)}\right). \end{aligned}$$

Integrating the above equality, we can obtain

$$f(z) = \int_0^z \exp\left\{(A-B)\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{w(t)}{t(1+Bw(t))}dt\right\} \cdot \left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right) d\zeta.$$

□

Next, we investigate two invariance properties for the functions in $\mathcal{S}_s^{(k)}(A,B)$.

Theorem 3.5. *If $f(z) \in \mathcal{S}_s^{(k)}(A,B)$, then so does*

$$F(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} f(t)dt \tag{23}$$

for $m = 1, 2, \ldots$.

Proof. By using the equation (23), we have

$$F_k(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} f_k(t)dt$$

and

$$\frac{zF'(z)}{F(z)} = -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)dt}.$$

Hence

$$
\begin{aligned}
\frac{zF'(z)}{F_k(z)} &= \left(-m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)dt}\right)\frac{F(z)}{F_k(z)} \\
&= \frac{z^m f(z) - m\int_0^z t^{m-1} f(t)dt}{\int_0^z t^{m-1} f(t)dt} \qquad (24)\\
&:= \frac{N(z)}{D(z)}.
\end{aligned}
$$

Since $f_k \in \mathcal{S}^*(A,B)$, by Lemma 3.2, we note that $F_k(z) \in \mathcal{S}^*(A,B)$. Differentiating (24), we have

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz}.$$

By Lemma 3.1, we conclude that

$$\frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Hence we have $F(z) \in \mathcal{S}_s^{(k)}(A,B)$. □

Theorem 3.6. *If $f(z) \in \mathcal{S}_s^{(k)}(A,B)$ and $f_\lambda(z) = (1-\lambda)z + \lambda f(z)$, $0 \le \lambda \le 1$, then*

(i) *for $B = 0$, $f_\lambda(z) \in \mathcal{S}_s^{(k)}(A,0)$.*
(ii) *for $|z| < \frac{1}{B}\sin(\frac{B}{2A}\pi)$, $B > 0$, $f_\lambda(z) \in \mathcal{S}_s^{(k)}(A,B)$.*
(iii) *for $|z| < \frac{1}{B}\sin(\frac{B}{2B-A}\frac{\pi}{2})$, $B < 0$, $f_\lambda(z) \in \mathcal{S}_s^{(k)}(A,B)$.*

Proof. Since $f(z) \in \mathcal{S}_s^{(k)}(A,B)$,

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz}.$$

Put

$$f_{\lambda,k}(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} \varepsilon^{-\mu} f_\lambda(\varepsilon^\mu z).$$

Then $f_{\lambda,k}(z) = (1-\lambda)z + \lambda f_k(z)$ and $zf'_\lambda(z) = (1-\lambda)z + \lambda zf'(z)$. Hence

$$\frac{zf'_\lambda(z)}{f_{\lambda,k}(z)} = \frac{(1-\lambda)\frac{z}{f_k(z)} + \lambda\frac{zf'(z)}{f_k(z)}}{(1-\lambda)\frac{z}{f_k(z)} + \lambda}.$$

Since $f_k \in \mathcal{S}^*(A,B)$,

$$\frac{tf_k(sz)}{sf_k(tz)} \prec \begin{cases} (\frac{1+Bsz}{1+Btz})^{\frac{A-B}{B}}, B \neq 0, \\ \exp(A(s-t)z), B = 0. \end{cases} \qquad (25)$$

Put $s = 1$ and $t = 0$ into (25), then we can obtain

$$\frac{f_k(z)}{z} \prec \begin{cases} (1+Bz)^{\frac{A-B}{B}}, B \neq 0, \\ \exp(Az), B = 0. \end{cases} \tag{26}$$

For the case $B = 0$, it suffices to show that

$$\left| \frac{(1-\lambda)\frac{z}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)}}{(1-\lambda)\frac{z}{f_k(z)} + \lambda} - 1 \right| < A. \tag{27}$$

Since $\frac{zf'(z)}{f_k(z)} \prec 1 + Az$, $\left|\frac{zf'(z)}{f_k(z)} - 1\right| < A$. Since $\frac{f_k(z)}{z} \prec \exp(Az)$, there exists a Schwarz function w_2 which satisfies $w_2(0) = 0$ and $|w_2| < 1$ in $\mathbb{U}$ such that

$$\frac{f_k(z)}{z} = \exp(Aw_2(z)).$$

Hence

$$\begin{aligned} \left| \frac{(1-\lambda)\frac{z}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)}}{(1-\lambda)\frac{z}{f_k(z)} + \lambda} - 1 \right| &= \lambda \left| \frac{\frac{zf'(z)}{f_k(z)} - 1}{(1-\lambda)\frac{z}{f_k(z)} + \lambda} \right| \\ &< \frac{A\lambda}{|(1-\lambda)\exp(-Aw_2(z)) + \lambda|}. \end{aligned}$$

Using the fact that $|w_2(z)| < 1$ in $\mathbb{U}$, we can obtain

$$|(1-\lambda)\exp(-Aw_2(z)) + \lambda| > \lambda,$$

by simple calculations. And this implies that

$$\frac{zf'(z)}{f_{\lambda,k}(z)} \prec 1 + Az$$

in $\mathbb{U}$. For the case $B \neq 0$, we need to show that

$$\left| \frac{(1-\lambda)\frac{z}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)}}{(1-\lambda)\frac{z}{f_k(z)} + \lambda} - 1 \right| < \left| A - B \frac{(1-\lambda)\frac{z}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)}}{(1-\lambda)\frac{z}{f_k(z)} + \lambda} \right|. \tag{28}$$

And (28) is equivalent to

$$\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| (A-B)(\frac{1}{\lambda} - 1)\frac{z}{f_k(z)} + A - B\frac{zf'(z)}{f_k(z)} \right|.$$

Since $\frac{zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz}$,

$$\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| A - B\frac{zf'(z)}{f_k(z)} \right|.$$

We note that

$$\left| \arg\left(\frac{z}{f_k(z)}\right) - \arg\left(A - B\frac{zf'(z)}{f_k(z)}\right) \right| < \frac{\pi}{2} \tag{29}$$

implies that

$$\left|A - B\frac{zf'(z)}{f_k(z)}\right| < \left|(A-B)\left(\frac{1}{\lambda}-1\right)\frac{z}{f_k(z)} + A - B\frac{zf'(z)}{f_k(z)}\right|.$$

Hence it suffices to show that (29) holds. Since $\frac{zf'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz}$,

$$\left|\arg\left(A - B\frac{zf'(z)}{f_k(z)}\right)\right| \leq \arcsin(|B|r). \tag{30}$$

and

$$\left|\arg\left(\frac{z}{f_k(z)}\right)\right| = \left|\arg\left(\frac{f_k(z)}{z}\right)\right| \leq \frac{A-B}{B}\arcsin(Br). \tag{31}$$

Hence, by (30), (31) and the hypotheses of Theorem 3.6, we can easily show that

$$\begin{aligned}
&\left|\arg\left(\frac{z}{f_k(z)}\right) - \arg\left(A - B\frac{zf'(z)}{f_k(z)}\right)\right| \\
\leq\ &\left|\arg\left(\frac{z}{f_k(z)}\right)\right| + \left|\arg\left(A - B\frac{zf'(z)}{f_k(z)}\right)\right| \\
\leq\ &\arcsin(|B|r) + \frac{A-B}{B}\arcsin(Br) \\
<\ &\frac{\pi}{2}
\end{aligned}$$

and this completes the proof of Theorem 3.6. □

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