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CONVERGENCE OF ISHIKAWA'METHOD FOR GENERALIZED HYBRID MAPPINGS

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Abstract. In this paper, we first talk about a more wide class of nonlinear mappings, Then, we deal with weak convergence theorems for generalized hybrid mappings in a Hilbert space.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see [10].

A important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see for instance, [3, 5]. It is known that a mapping $F : C \to C$ is firmly nonexpansive if and only if

$$\|Fx - Fy\|^2 + \|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2$$

for all $x, y \in C$, where I is the identity mapping on H. It is also known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [2, 4].

Recently, Kohsaka and Takahashi [11] introduced the following nonlinear mapping: Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

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for all $x, y \in C$, where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They consider such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in E$. So, a nonspreading mapping [11] $S : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \le \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. A mapping $T : C \to C$ is called hybrid mapping [18] if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also [12] and [8]. Aoyama et al. [1] introduced a class of nonlinear mappings called λ-hybrid and obtained a generalization of Baillons nonlinear ergodic theorem; see also [19]. Kocourek et al. [13] introduced a more wide class of nonlinear mappings containing the class of hybrid mappings. They called such mappings generalized hybrid mappings. Let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is called generalized hybrid [22] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. It is well known that the (α, β)-generalized hybrid mapping is quasi-nonexpansive if the set of fixed points of it is nonempty. They proved fixed point theorems for such mappings; see [13]. We also observe that the mappings above generalize several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$.

There classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Halpern [6] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and defined $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n)Tx_n, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.2}$$

where $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval $[0, 1]$. The second iteration process is now known as Mann's iteration process [14] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is in the interval $[0, 1]$. The third iteration process is referred to as Ishikawa's iteration process [9] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases} \tag{1.4}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in the interval $[0, 1]$.

In this paper, we also use these iteration processes or it's evolvement to finish our proof.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner produce $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$$

for all $x, y \in H$. Further $\langle P_C x - P_C y, x - y \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see, for instance, [16]. A Hilbert space satisfies Opials condition [15], that is,

$$\liminf_{n \to \infty} \|x_n - u\| < \liminf_{n \to \infty} \|x_n - v\|$$

if $x_n \to u$ and $u \neq v$; see [15].

We give the crucial lemmas in order to prove the main theorem.

Lemma 2.1 ([7])**.** *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0, 1]$ the following inequality hold*:

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2. \tag{2.1}$$

Lemma 2.2 ([18])**.** *Let H be a Hilbert space and let S be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in S$, then $\{P_S(x_n)\}$ converges strongly to some $z \in S$, where P_S stands for the metric projection on H onto S.*

Using Opials theorem [15], we can also prove the following lemma; see, for instance, [17].

Lemma 2.3. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a nonempty subset $S \subset H$ satisfying* (i) *and* (ii):

(i) *For every $x^* \in S$, $\lim_{n \to \infty} \|x_n - x^*\|$ exists*:

(ii) *If a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to x^*, then $x^* \in S$.*

Then there exists $x_0 \in S$ such that $x_n \rightharpoonup x_0$.

3. Weakly convergence theorem

We are now in a position to prove our main theorem for weakly convergence of generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let C be a nonempty, closed convex subset of a real Hlibert space H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_0 = x \in C$, $u \in C$ and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of $[0,1]$ with $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $0 < a < \alpha_n < b < 1$. Then $\{x_n\}$ converges weakly to $x_0 \in F(T)$, where $x_0 = \lim_{n\to\infty} P_{F(T)}(x_n)$.

Proof. Since $F(T) \neq \emptyset$, T is quasi-nonexpansive. So, we have that for all $q \in F(T)$ and $n \in \mathbb{N}$ We have

$$\begin{aligned} & \| y_n - q \|^2 \\ &= (1-\beta_n)\|x_n - q\|^2 + \beta_n\|Tx_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\leq (1-\beta_n)\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned} & \| x_{n+1} - q \| \\ &= \| (1-\alpha_n)x_n + \alpha_n T y_n - q \| \\ &= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|Ty_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|y_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 \\ &\leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\leq \|x_n - q\|^2. \end{aligned} \tag{3.3}$$

Hence, we obtain that $\lim_{n\to\infty} \|x_n - z\|$ exists. This implies that $\{x_n\}, \{y_n\}$, $\{Ty_n\}$ are bounded. From 3.3, we know that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Tx_n\|^2.$$

Since $0 < a < \alpha_n < b < 1$, we have that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - a\beta_n(1-\beta_n)\|x_n - Tx_n\|^2$$

we also know from $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ that

$$0 \ \leq a\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$$

as $n \to \infty$. This means that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.4}$$

Since $\{x_n\}$ is bounded, then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$. Since T is a generalized hybrid mapping, then

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

hence

(3.5)

$$\begin{aligned}
0 &\le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 - \alpha\|Tx - Ty\|^2 - (1-\alpha)\|x - Ty\|^2 \\
&\le \beta(\|Tx_n\|^2 - 2\langle Tx_n, x^*\rangle + \|x^*\|^2) + (1-\beta)(\|x_n\|^2 - 2\langle x_n, x^*\rangle + \|x^*\|^2) \\
&\quad - \alpha(\|Tx_n\|^2 - 2\langle Tx_n, Tx^*\rangle + \|Tx^*\|^2) - (1-\alpha)(\|x_n\|^2 - 2\langle x_n, Tx^*\rangle \\
&\quad + \|Tx^*\|^2) \\
&= \|x^*\|^2 - \|Tx^*\|^2 + (\beta-\alpha)(\|Tx_n\|^2 - \|x_n\|^2) \\
&\quad + 2\alpha\langle Tx_n - x_n, Tx^*\rangle - 2\beta\langle Tx_n - x_n, x^*\rangle + 2\langle x_n, Tx^* - x^*\rangle \\
&\le \|x^*\|^2 - \|Tx^*\|^2 + (\beta-\alpha)(\|Tx_n\| + \|x_n\|)(\|Tx_n - x_n\|) \\
&\quad + 2\alpha\langle Tx_n - x_n, Tx^*\rangle - 2\beta\langle Tx_n - x_n, x^*\rangle + 2\langle x_n, Tx^* - x^*\rangle.
\end{aligned}$$

Thus, we have that for all $i \in \mathbb{N}$,

$$\begin{aligned}
0 &\le \|x^*\|^2 - \|Tx^*\|^2 + (\beta-\alpha)(\|Tx_{n_i}\| + \|x_{n_i}\|)(\|Tx_{n_i} - x_{n_i}\|) \\
&\quad + 2\alpha\langle Tx_{n_i} - x_{n_i}, Tx^*\rangle - 2\beta\langle Tx_{n_i} - x_{n_i}, x^*\rangle + 2\langle x_{n_i}, Tx^* - x^*\rangle.
\end{aligned} \tag{3.6}$$

From 3.4, we have that

$$\lim_{n\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0,$$

and $x_{n_i} \rightharpoonup x^*$ as $i \to \infty$, the above inequality implies that

$$\begin{aligned}
0 &\le \|x^*\|^2 - \|Tx^*\|^2 + 2\langle x^*, Tx^* - x^*\rangle \\
&= 2\langle x^*, Tx^*\rangle - \|x^*\|^2 - \|Tx^*\|^2 \\
&= -\|x^* - Tx^*\|^2.
\end{aligned} \tag{3.7}$$

So, we have $Tx^* = x^*$, i.e., $x^* \in F(T)$. Therefore we obtain that

$$x^* \in F(T).$$

This implies that the condition (ii) of Lemma 2.3 holds for $S = F(T)$. We also know that $\lim_{n\to\infty} \|x_n - q\|$ exists for $q \in S = F(T)$. So, we have from Lemma 2.3 that there exists $x_0 \in S$ such that $x_n \rightharpoonup x_0$ as $n \to \infty$. Moreover, since for any $q \in S = F(T)$,

$$\|x_{n+1} - q\| \le \|x_n - q\|, \quad \forall n \in \mathbb{N},$$

by Lemma 2.2 there exists some $x \in S$ such that $P_S(x_n) \to x$. The property of metric projection implies that

$$\langle x_0 - P_S(x_n), x_n - P_S(x_n)\rangle \le 0.$$

Therefore, we have

$$\langle x_0 - x, x_0 - x\rangle = \|x_0 - x\|^2 \le 0.$$

This means that $x_0 = x$, i.e., $x_n \rightharpoonup x_0 = \lim_{n\to\infty} P_{F(T)}(x_n)$. □

Using Theorem 3.1 we prove the following weakly convergence theorem for nonspreading mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty, closed convex subset of a real Hlibert space H. Let $T : C \to C$ be a nonspreading mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_0 = x \in C$, $u \in C$ and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of $[0,1]$ with $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $0 < a < \alpha_n < b < 1$. Then $\{x_n\}$ converges weakly to $x_0 \in F(T)$, where $x_0 = \lim_{n\to\infty} P_{F(T)}(x_n)$.

Proof. Since $T : C \to C$ is an nonspreading mapping, then

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2.$$

The mapping T is a (2,1)-generalized hybrid mapping. The result follows immediately from Theorem 3.1. □

Remark. We also can prove the weakly convergence theorem for hybrid mappings as $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$ and nonexpansive mappings as α= 1, β= 0 in a Hilbert space.

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