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ON SOME GENERATING FUNCTIONS OF GENERALIZED BATEMAN'S AND PASTERNAK'S POLYNOMIALS OF TWO VARIABLES

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Abstract. The aim of the present paper is to study some generating functions of generalized Bateman's and Pasternak's polynomials of two variables.

1. Introduction

In our earlier papers [1] and [2] we obtained some generating functions for Jacobi and generalized Rice polynomials of two variables respectively. In the present paper some generating functions for generalized Bateman's and Pasternak's polynomials of two variables have been obtained.

The following definitions are required in this paper:
The generalized Bateman's polynomial $Z_n^{(\alpha,\,\beta)}(b,x)$ is defined by Khan and Shukla [3]

$$Z_n^{(\alpha,\,\beta)}(b,x) = {}_2F_2\left[\begin{array}{ll} -n,\ 1+\alpha+\beta+n & ; \\ 1+\alpha,\ 1+b & ; \end{array} x\right]. \tag{1.1}$$

Another generalized Bateman's polynomial $F_n^{(\alpha,\,\beta)}(p,z)$ is defined by Khan and Shukla [3]

$$F_n^{(\alpha,\,\beta)}(p,z) = {}_3F_2\left[\begin{array}{ll} -n,\ 1+\alpha+\beta+n,\ \frac{1}{2}(1+z) & ; \\ 1+\alpha,\ p & ; \end{array} 1\right]. \tag{1.2}$$

The generalized Pasternak's polynomial $F_{n,m}^{(\alpha,\,\beta)}(z)$ is defined by Khan and Shukla [3]

$$F_{n,m}^{(\alpha,\,\beta)}(z) = {}_3F_2\left[\begin{array}{ll} -n,\ 1+\alpha+\beta+n,\ \frac{1}{2}(1+z+m) & ; \\ 1+\alpha,\ 1+m & ; \end{array} 1\right]. \tag{1.3}$$

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Kampé de Fériet series in the generalized form is defined by [6, p. 63, Eq. (16)]

(1.4)
$$F^{p:q;k}_{l:m;n}\left[\begin{array}{lll} (a_p) & :(b_q) & ;(c_k)\ ; \\ (\alpha_l) & :(\beta_m) & ;(\gamma_n)\ ; \end{array} x, y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}.$$

The generalized hypergeometric functions of three variables $F^{(3)}[x, y, z]$ are defined as (see [5]):

(1.5)
$$F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') : (c); (c'); (c''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} x, y, z\right]$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{m+p}((c))_m((c'))_n((c''))_p}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{m+p}((h))_m((h'))_n((h''))_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

2. Generalized Bateman's polynomials of two variables $Z_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(b_1, x; b_2, y)$

The generalized Bateman's polynomials of two variables $Z_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(b_1, x; b_2, y)$ are defined by

(2.1)
$$Z_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(b_1, x; b_2, y) = \sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(1+\alpha_2+\beta_2+n)_s}{r!s!(1+\alpha_1)_r(1+b_1)_r(1+\alpha_2)_s(1+b_2)_s}x^r y^s.$$

The definition (2.1) can also be expressed in terms of double hypergeometric function as follows:

(2.2)
$$Z_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(b_1, x; b_2, y) = F^{1:1;1}_{0:2;2}\left[\begin{array}{lll} -n & : 1+\alpha_1+\beta_1+n & ; 1+\alpha_2+\beta_2+n\ ; \\ - & : 1+\alpha_1, 1+b_1 & ; 1+\alpha_2, 1+b_2\ ; \end{array} x, y\right],$$

where we have used a special case of the double hypergeometric function defined by (1.4).

The definition (2.1) can also be represented as follows:

(2.3)
$$Z_n^{(\alpha_1,\, \beta_1;\, \alpha_2,\, \beta_2)}(b_1, x; b_2, y) = \sum_{r=0}^{n}\frac{(-n)_r(1+\alpha_1+\beta_1+n)_r x^r}{r!(1+\alpha_1)_r(1+b_1)_r}Z_{n-r}^{(\alpha_2,\, \beta_2+r)}(b_2, y),$$

where $Z_n^{(\alpha,\, \beta)}(b_2, y)$ is the well known generalized Bateman's polynomial of one variable defined by (1.1).

The relationships between generalized Bateman's polynomials of two variables and generalized Bateman's polynomials of single variable are as follows:

$$Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,0) = Z_n^{(\alpha_1,\,\beta_1)}(b_1,x) \tag{2.4}$$

and

$$Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,0;b_2,y) = Z_n^{(\alpha_2,\,\beta_2)}(b_2,y). \tag{2.5}$$

3. Generating functions for generalized Bateman's polynomials $Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)$

The polynomials $Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)$ admit the following generating functions:

(3.1)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$
$$= F^{3:0;0;0}_{2:0;2;2}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{c}-\\(1+\alpha_1:1),(1+b_1:1)\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}-\\(1+\alpha_2:1),(1+b_2:1)\end{array}\begin{array}{c};\\;\end{array}\ t,-xt,-yt\right],$$

where we have used a special case of generalized Lauricella function (see, for example, [5, p. 37]).

(3.2)

$$\sum_{n=0}^{\infty}\frac{1}{n!(\mu)_n}Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$
$$= F^{2:0;0;0}_{3:0;2;2}\left[\begin{array}{l}(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{c}-\\(1+\alpha_1:1),(1+b_1:1)\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}-\\(1+\alpha_2:1),(1+b_2:1)\end{array}\begin{array}{c};\\;\end{array}\ t,-xt,-yt\right].$$

(3.3)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!(\mu)_n}Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$
$$= F^{3:0;0;0}_{3:0;2;2}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{c}-\\(1+\alpha_1:1),(1+b_1:1)\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}-\\(1+\alpha_2:1),(1+b_2:1)\end{array}\begin{array}{c};\\;\end{array}\ t,-xt,-yt\right].$$

(3.4)

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu)_n} Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$

$$= F^{4:0;0;0}_{3:0;2;2}\left[\begin{array}{l} (\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2): \\ (\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1) \qquad : \end{array}\right.$$

$$\left.\begin{array}{l} -\,;\qquad - \qquad ;\qquad - \qquad ; \\ -\,;(1+\alpha_1:1),(1+b_1:1);(1+\alpha_2:1),(1+b_2:1); \end{array} t,-xt,-yt\right].$$

(3.5)

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu_1)_n(\mu_2)_n} Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$

$$= F^{4:0;0;0}_{4:0;2;2}\left[\begin{array}{l} (\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2): \\ (\mu_1:1,1,1),(\mu_2:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1): \end{array}\right.$$

$$\left.\begin{array}{l} -\,;\qquad - \qquad ;\qquad - \qquad ; \\ -\,;(1+\alpha_1:1),(1+b_1:1);(1+\alpha_2:1),(1+b_2:1); \end{array} t,-xt,-yt\right].$$

Special cases

If we put $\lambda=\mu$ in the generating function (3.3), we get

(3.6)

$$\sum_{n=0}^{\infty} \frac{1}{n!} Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$

$$= F^{2:0;0;0}_{2:0;2;2}\left[\begin{array}{l} (1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):-\,; \\ (1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1):-\,; \end{array}\right.$$

$$\left.\begin{array}{l} \qquad - \qquad ;\qquad - \qquad ; \\ (1+\alpha_1:1),(1+b_1:1);(1+\alpha_2:1),(1+b_2:1); \end{array} t,-xt,-yt\right].$$

When $\lambda_1=1+\alpha_1+\beta_1$ and $\lambda_2=1+\alpha_2+\beta_2$, then the generating function (3.4) reduces immediately to the form:

(3.7)

$$\sum_{n=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_n(1+\alpha_2+\beta_2)_n}{n!(\mu)_n} Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)t^n$$

$$= F^{2:0;0;0}_{1:0;2;2}\left[\begin{array}{l} (1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):-\,; \\ (\mu:1,1,1) \qquad :-\,; \end{array}\right.$$

$$\left.\begin{array}{l} \qquad - \qquad ;\qquad - \qquad ; \\ (1+\alpha_1:1),(1+b_1:1);(1+\alpha_2:1),(1+b_2:1); \end{array} t,-xt,-yt\right].$$

Further, the sequence $\{Z_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(b_1,x;b_2,y)\}_{n\in\mathbb{N}}$ admits the following generating functions:

$$\sum_{n=0}^{\infty}\frac{1}{n!}Z_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(b_1,x;b_2,y)t^n = e^t\ {}_1F_2\left[\begin{matrix}1+\alpha_1+\beta_1 & ; \\ 1+\alpha_1,\ 1+b_1 & ;\end{matrix}\ -xt\right]{}_1F_2\left[\begin{matrix}1+\alpha_2+\beta_2 & ; \\ 1+\alpha_2,\ 1+b_2 & ;\end{matrix}\ -yt\right]; \tag{3.8}$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!}Z_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2,\,\beta_2-n)}(b_1,x;b_2,y)t^n = (1-t)^{\alpha_1}\ {}_1F_1\left[\begin{matrix}1+\alpha_1+\beta_1 & ; \\ 1+b_1 & ;\end{matrix}\ xt\right]{}_2F_2\left[\begin{matrix}-\alpha_1,1+\alpha_2+\beta_2 & ; \\ 1+\alpha_2,\ 1+b_2 & ;\end{matrix}\ \tfrac{-yt}{1-t}\right]; \tag{3.9}$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_2)_n}{n!}Z_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2-n,\,\beta_2)}(b_1,x;b_2,y)t^n = (1-t)^{\alpha_2}\ {}_1F_1\left[\begin{matrix}1+\alpha_2+\beta_2 & ; \\ 1+b_2 & ;\end{matrix}\ yt\right]{}_2F_2\left[\begin{matrix}-\alpha_2,1+\alpha_1+\beta_1 & ; \\ 1+\alpha_1,\ 1+b_1 & ;\end{matrix}\ \tfrac{-xt}{1-t}\right]; \tag{3.10}$$

$$\sum_{n=0}^{\infty}\frac{(-b_1)_n}{n!}Z_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(b_1-n,x;b_2,y)t^n = (1-t)^{b_1}\ {}_1F_1\left[\begin{matrix}1+\alpha_1+\beta_1 & ; \\ 1+\alpha_1 & ;\end{matrix}\ xt\right]{}_2F_2\left[\begin{matrix}1+\alpha_2+\beta_2,-b_1 & ; \\ 1+\alpha_2,\ 1+b_2 & ;\end{matrix}\ \tfrac{-yt}{1-t}\right]; \tag{3.11}$$

$$\sum_{n=0}^{\infty}\frac{(-b_2)_n}{n!}Z_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(b_1,x;b_2-n,y)t^n = (1-t)^{b_2}\ {}_1F_1\left[\begin{matrix}1+\alpha_2+\beta_2 & ; \\ 1+\alpha_2 & ;\end{matrix}\ yt\right]{}_2F_2\left[\begin{matrix}1+\alpha_1+\beta_1,-b_2 & ; \\ 1+\alpha_1,\ 1+b_1 & ;\end{matrix}\ \tfrac{-xt}{1-t}\right]; \tag{3.12}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}Z_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(b_1,x;b_2,y)t^n = (1-t)^{-\lambda}F^{1:1;1}_{0:2;2}\left[\begin{matrix}\lambda & :1+\alpha_1+\beta_1 & ;1+\alpha_2+\beta_2 & ; \\ 0 & :1+\alpha_1,1+b_1 & ;1+\alpha_2,1+b_2 & ;\end{matrix}\ \tfrac{-xt}{1-t},\tfrac{-yt}{1-t}\right]; \tag{3.13}$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!(-\alpha_1-\beta_1)_n}Z_n^{(\alpha_1-n,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(b_1,x;b_2,y)t^n = {}_0F_1\left[\begin{matrix}- & ; \\ 1+b_1 & ;\end{matrix}\ -xt\right]F^{1:0;1}_{1:0;2}\left[\begin{matrix}-\alpha_1 & :-;1+\alpha_2+\beta_2 & ; \\ -\alpha_1-\beta_1 & :-;1+\alpha_2,1+b_2 & ;\end{matrix}\ t,-yt\right]; \tag{3.14}$$

(3.15)
$$\sum_{n=0}^{\infty} \frac{(-\alpha_2)_n}{n!(-\alpha_2-\beta_2)_n} Z_n^{(\alpha_1,\, \beta_1-n;\, \alpha_2-n,\, \beta_2-n)}(b_1,x;b_2,y)t^n$$
$$= {}_0F_1\left[\begin{array}{c;c} - & ; \\ 1+b_2 & ; \end{array} -yt\right] F^{1:0;1}_{1:0;2}\left[\begin{array}{l} -\alpha_2 \quad :- \ ;1+\alpha_1+\beta_1 \quad ; \\ -\alpha_2-\beta_2 \ :- \ ;1+\alpha_1, 1+b_1 \ ; \end{array} t,-xt\right];$$

(3.16)
$$\sum_{n=0}^{\infty} \frac{(-b_1)_n}{n!(-\alpha_1-\beta_1)_n} Z_n^{(\alpha_1,\, \beta_1-2n;\, \alpha_2,\, \beta_2-n)}(b_1-n,x;b_2,y)t^n$$
$$= {}_0F_1\left[\begin{array}{c} - \ ; \\ 1+\alpha_1 \ ; \end{array} -xt\right] F^{1:0;1}_{1:0;2}\left[\begin{array}{l} -b_1 \quad :- \ ;1+\alpha_2+\beta_2 \quad ; \\ -\alpha_1-\beta_1 \ :- \ ;1+\alpha_2, 1+b_2 \ ; \end{array} t,-yt\right];$$

(3.17)
$$\sum_{n=0}^{\infty} \frac{(-b_2)_n}{n!(-\alpha_2-\beta_2)_n} Z_n^{(\alpha_1,\, \beta_1-n;\, \alpha_2,\, \beta_2-2n)}(b_1,x;b_2-n,y)t^n$$
$$= {}_0F_1\left[\begin{array}{c} - \ ; \\ 1+\alpha_2 \ ; \end{array} -yt\right] F^{1:0;1}_{1:0;2}\left[\begin{array}{l} -b_2 \quad :- \ ;1+\alpha_1+\beta_1 \quad ; \\ -\alpha_2-\beta_2 \ :- \ ;1+\alpha_1, 1+b_1 \ ; \end{array} t,-xt\right].$$

Also, we have the following generating functions:

(3.18)
$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!(\mu)_n} Z_n^{(\alpha_1,\, \beta_1-n;\, \alpha_2,\, \beta_2-n)}(b_1,x;b_2,y)t^n$$
$$= F^{(3)}\left[\begin{array}{l} \lambda :: - \ ;- \ ;- \ :- \ ;1+\alpha_1+\beta_1 \quad ;1+\alpha_2+\beta_2 \quad ; \\ \mu :: - \ ;- \ ;- \ :- \ ;1+\alpha_1, 1+b_1 \ ;1+\alpha_2, 1+b_2 \ ; \end{array} t,-xt,-yt\right];$$

(3.19)
$$\sum_{n=0}^{\infty} \frac{(-\alpha_1)_n(-\alpha_2)_n}{n!} Z_n^{(\alpha_1-n,\, \beta_1;\, \alpha_2-n,\, \beta_2)}(b_1,x;b_2,y)t^n$$
$$= F^{(3)}\left[\begin{array}{l} - :: -\alpha_2 \ ;- \ ;-\alpha_1 \ :- \ ;1+\alpha_1+\beta_1 \ ;1+\alpha_2+\beta_2 \ ; \\ - :: - \quad ;- \ ;- \quad :- \ ;1+b_1 \quad ;1+b_2 \quad ; \end{array} t,xt,yt\right];$$

(3.20)
$$\sum_{n=0}^{\infty} \frac{(-b_1)_n(-b_2)_n}{n!} Z_n^{(\alpha_1,\, \beta_1-n;\, \alpha_2,\, \beta_2-n)}(b_1-n,x;b_2-n,y)t^n$$
$$= F^{(3)}\left[\begin{array}{l} - :: -b_2 \ ;- \ ;-b_1 \ :- \ ;1+\alpha_1+\beta_1 \ ;1+\alpha_2+\beta_2 \ ; \\ - :: - \quad ;- \ ;- \quad :- \ ;1+\alpha_1 \quad ;1+\alpha_2 \quad ; \end{array} t,xt,yt\right];$$

(3.21)
$$\sum_{n=0}^{\infty} \frac{(-\alpha_1)_n(-\alpha_2)_n}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n} Z_n^{(\alpha_1-n,\, \beta_1-n;\, \alpha_2-n,\, \beta_2-n)}(b_1,x;b_2,y)t^n$$
$$= F^{(3)}\left[\begin{array}{l} - :: -\alpha_2 \quad ;- \ ;-\alpha_1 \quad :- \ ;- \quad ;- \quad ; \\ - :: -\alpha_2-\beta_2 \ ;- \ ;-\alpha_1-\beta_1 \ :- \ ;1+b_1 \ ;1+b_2 \ ; \end{array} t,-xt,-yt\right];$$

(3.22)

$$\sum_{n=0}^{\infty} \frac{(-b_1)_n(-b_2)_n}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n} Z_n^{(\alpha_1,\,\beta_1-2n;\,\alpha_2,\,\beta_2-2n)}(b_1-n,x;b_2-n,y)t^n$$

$$= F^{(3)}\left[\begin{array}{l} -::-b_2\;;-\;;-b_1\;:-\;;-\;;-\;; \\ -::-\alpha_2-\beta_2\;;-\;;-\alpha_1-\beta_1\;:-\;;1+\alpha_1\;;1+\alpha_2\;; \end{array} t,-xt,-yt\right].$$

4. Generalized Bateman's polynomials of two variables $F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$

Another generalized Bateman's polynomials of two variables $F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}$ $(p_1,z_1;p_2,z_2)$ are defined by

(4.1)

$$F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$$
$$= \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(\frac{1}{2}(1+z_1))_r(1+\alpha_2+\beta_2+n)_s(\frac{1}{2}(1+z_2))_s}{r!s!(1+\alpha_1)_r(p_1)_r(1+\alpha_2)_s(p_2)_s}.$$

The relation (4.1) can also be expressed in terms of double hypergeometric function as follows:

(4.2)

$$F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$$
$$= F_{0:2;2}^{1:2;2}\left[\begin{array}{l} -n: 1+\alpha_1+\beta_1+n, \frac{1}{2}(1+z_1)\,; 1+\alpha_2+\beta_2+n, \frac{1}{2}(1+z_2)\;; \\ -\;: 1+\alpha_1,\; p_1\;; 1+\alpha_2,\; p_2\;; \end{array} 1,1\right],$$

where we have used a special case of the double hypergeometric function defined by (1.4).

The definition (4.1) can also be represented as follows:

$$F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$$
$$= \sum_{r=0}^{n} \frac{(-n)_r(1+\alpha_1+\beta_1+n)_r(\frac{1}{2}(1+z_1))_r}{r!(1+\alpha_1)_r(p_1)_r} F_{n-r}^{(\alpha_2,\,\beta_2+r)}(p_2,z_2), \tag{4.3}$$

where $F_n^{(\alpha,\,\beta)}(p_2,z)$ is the well known generalized Bateman's polynomial of one variable defined by (1.2).

The relationships between generalized Bateman's polynomials of two variables and generalized Bateman's polynomials of single variable are as follows:

$$F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,-1) = F_n^{(\alpha_1,\,\beta_1)}(p_1,z_1) \tag{4.4}$$

and

$$F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,-1;p_2,z_2) = F_n^{(\alpha_2,\,\beta_2)}(p_2,z_2). \tag{4.5}$$

5. Generating functions for generalized Bateman's polynomials $F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$

The polynomials $F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)$ admit the following generating functions:

(5.1)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{3:0;1;1}_{2:0;2;2}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1):1)\ ;(\tfrac{1}{2}(1+z_2):1)\ ;\\ (1+\alpha_1:1),(p_1:1);(1+\alpha_2:1),(p_2:1);\end{array}\ t,-t,-t\right],$$

where we have used a special case of generalized Lauricella function (see, for example, [5, p. 37].

(5.2)
$$\sum_{n=0}^{\infty}\frac{1}{n!(\mu)_n}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{2:0;1;1}_{3:0;2;2}\left[\begin{array}{l}(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1):1)\ ;(\tfrac{1}{2}(1+z_2):1)\ ;\\ (1+\alpha_1:1),(p_1:1)\ ;(1+\alpha_2:1),(p_2:1)\ ;\end{array}\ t,-t,-t\right];$$

(5.3)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!(\mu)_n}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{3:0;1;1}_{3:0;2;2}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\ :-\ ;\\ (\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :-\ ;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1):1)\ ;(\tfrac{1}{2}(1+z_2):1)\ ;\\ (1+\alpha_1:1),(p_1:1)\ ;(1+\alpha_2:1),(p_2:1)\ ;\end{array}\ t,-t,-t\right];$$

(5.4)
$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu)_n}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{4:0;1;1}_{3:0;2;2}\left[\begin{array}{l}(\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):\\ \qquad(\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\ :\end{array}\right.$$
$$\left.\begin{array}{l}-\ ;(\tfrac{1}{2}(1+z_1):1)\ ;(\tfrac{1}{2}(1+z_2):1)\ ;\\ -\ ;(1+\alpha_1:1),(p_1:1)\ ;(1+\alpha_2:1),(p_2:1)\ ;\end{array}\ t,-t,-t\right];$$

(5.5)

$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu_1)_n(\mu_2)_n}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$

$$= F^{4:0;1;1}_{4:0;2;2}\left[\begin{array}{l}(\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):\\(\mu_1:1,1,1),(\mu_2:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1):\end{array}\right.$$

$$\left.\begin{array}{llll}-\;;(\frac{1}{2}(1+z_1):1) & ;(\frac{1}{2}(1+z_2):1) & ; & \\ -\;;(1+\alpha_1:1),(p_1:1) & ;(1+\alpha_2:1),(p_2:1) & ; & t,-t,-t\end{array}\right].$$

Special cases

If we put $\lambda=\mu$ in the generating function (5.3), we get

$$\sum_{n=0}^{\infty}\frac{1}{n!}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$

(5.6)
$$= F^{2:0;1;1}_{2:0;2;2}\left[\begin{array}{l}(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\;:-\;;\\(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\;:-\;;\end{array}\right.$$

$$\left.\begin{array}{lll}(\frac{1}{2}(1+z_1):1) & ;(\frac{1}{2}(1+z_2):1) & ;\\(1+\alpha_1:1),(p_1:1) & ;(1+\alpha_2:1),(p_2:1) & ;\end{array} t,-t,-t\right].$$

When $\lambda_1=1+\alpha_1+\beta_1$ and $\lambda_2=1+\alpha_2+\beta_2$, then the generating function (5.4) reduces immediately to the form :

$$\sum_{n=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_n(1+\alpha_2+\beta_2)_n}{n!(\mu)_n}F_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$

(5.7)
$$= F^{2:0;1;1}_{1:0;2;2}\left[\begin{array}{ll}(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2) & :-\;;\\(\mu:1,1,1) & :-\;;\end{array}\right.$$

$$\left.\begin{array}{lll}(\frac{1}{2}(1+z_1):1) & ;(\frac{1}{2}(1+z_2):1) & ;\\(1+\alpha_1:1),(p_1:1) & ;(1+\alpha_2:1),(p_2:1) & ;\end{array} t,-t,-t\right].$$

Further, we have the following generating functions:

(5.8)

$$\sum_{n=0}^{\infty}\frac{1}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$

$$= e^t\;{}_2F_2\left[\begin{array}{cc}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1)\,; & \\ 1+\alpha_1,\;p_1\qquad ; & -t\end{array}\right]{}_2F_2\left[\begin{array}{cc}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2)\,; & \\ 1+\alpha_2,\;p_2\qquad ; & -t\end{array}\right];$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!}F_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$

(5.9)
$$= (1-t)^{\alpha_1}{}_2F_1\left[\begin{array}{cc}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1)\;; & \\ p_1\qquad ; & t\end{array}\right]$$

$$\times\;{}_3F_2\left[\begin{array}{cc}-\alpha_1,1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2)\;; & \\ 1+\alpha_2,\;p_2\qquad ; & \frac{-t}{1-t}\end{array}\right];$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_2)_n}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2-n,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$

(5.10)
$$=(1-t)^{\alpha_2}{}_2F_1\left[\begin{array}{c}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2)\\ p_2\end{array};t\right]$$
$$\times\,{}_3F_2\left[\begin{array}{c}-\alpha_2,1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1)\\ 1+\alpha_1,\ p_1\end{array};\frac{-t}{1-t}\right];$$

$$\sum_{n=0}^{\infty}\frac{(1-p_1)_n}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1-n,z_1;p_2,z_2)t^n$$

(5.11)
$$=(1-t)^{p_1-1}{}_2F_1\left[\begin{array}{c}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1)\\ 1+\alpha_1\end{array};t\right]$$
$$\times\,{}_3F_2\left[\begin{array}{c}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2),1-p_1\\ 1+\alpha_2,\ p_2\end{array};\frac{-t}{1-t}\right];$$

$$\sum_{n=0}^{\infty}\frac{(1-p_2)_n}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2-n,z_2)t^n$$

(5.12)
$$=(1-t)^{p_2-1}{}_2F_1\left[\begin{array}{c}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2)\\ 1+\alpha_2\end{array};t\right]$$
$$\times\,{}_3F_2\left[\begin{array}{c}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1),1-p_2\\ 1+\alpha_1,\ p_1\end{array};\frac{-t}{1-t}\right];$$

(5.13)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$
$$=(1-t)^{-\lambda}$$
$$\times\,F_{0:2;2}^{1:2;2}\left[\begin{array}{l}\lambda:1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1);1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2);\\ -:1+\alpha_1,\ p_1\quad;1+\alpha_2,\ p_2\quad;\end{array}\frac{-t}{1-t},\ \frac{-t}{1-t}\right];$$

(5.14)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!(-\alpha_1-\beta_1)_n}F_n^{(\alpha_1-n,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$
$$={}_1F_1\left[\begin{array}{c}\frac{1}{2}(1+z_1)\\ p_1\end{array};-t\right]F_{1:0;2}^{1:0;2}\left[\begin{array}{l}-\alpha_1:-;1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2);\\ -\alpha_1-\beta_1:-;1+\alpha_2,\ p_2\quad;\end{array}t,-t\right];$$

(5.15)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_2)_n}{n!(-\alpha_2-\beta_2)_n}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2-n,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$
$$={}_1F_1\left[\begin{array}{c}\frac{1}{2}(1+z_2)\\ p_2\end{array};-t\right]F_{1:0;2}^{1:0;2}\left[\begin{array}{l}-\alpha_2:-;1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1);\\ -\alpha_2-\beta_2:-;1+\alpha_1,\ p_1\quad;\end{array}t,-t\right];$$

(5.16)
$$\sum_{n=0}^{\infty}\frac{(1-p_1)_n}{n!(-\alpha_1-\beta_1)_n}F_n^{(\alpha_1,\,\beta_1-2n;\,\alpha_2,\,\beta_2-n)}(p_1-n,z_1;p_2,z_2)t^n$$
$$= {}_1F_1\left[\begin{array}{l}\tfrac12(1+z_1)\,;\\ 1+\alpha_1\ \ ;\end{array} -t\right]F^{1:0;2}_{1:0;2}\left[\begin{array}{l}1-p_1\qquad :-;1+\alpha_2+\beta_2,\tfrac12(1+z_2)\,;\\ -\alpha_1-\beta_1:-;1+\alpha_2,\ p_2\qquad\qquad ;\end{array} t,-t\right];$$

(5.17)
$$\sum_{n=0}^{\infty}\frac{(1-p_2)_n}{n!(-\alpha_2-\beta_2)_n}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-2n)}(p_1,z_1;p_2-n,z_2)t^n$$
$$= {}_1F_1\left[\begin{array}{l}\tfrac12(1+z_2)\,;\\ 1+\alpha_2\ \ ;\end{array} -t\right]F^{1:0;2}_{1:0;2}\left[\begin{array}{l}1-p_2\qquad :-;1+\alpha_1+\beta_1,\tfrac12(1+z_1)\,;\\ -\alpha_2-\beta_2:-;1+\alpha_1,\ p_1\qquad\qquad ;\end{array} t,-t\right].$$

Also, we have the following generating functions:

(5.18)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!(\mu)_n}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{(3)}\left[\begin{array}{l}\lambda :: -\ ;-\ ;-\ :-\ ;1+\alpha_1+\beta_1,\tfrac12(1+z_1)\ ;1+\alpha_2+\beta_2,\tfrac12(1+z_2)\ ;\\ \mu :: -\ ;-\ ;-\ :-\ ;1+\alpha_1,\ p_1\qquad\qquad ;1+\alpha_2,\ p_2\qquad\qquad ;\end{array} t,-t,-t\right];$$

(5.19)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n(-\alpha_2)_n}{n!}F_n^{(\alpha_1-n,\,\beta_1;\,\alpha_2-n,\,\beta_2)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{(3)}\left[\begin{array}{l}- :: -\alpha_2\ ;-\ ;-\alpha_1\ :-\ ;1+\alpha_1+\beta_1,\tfrac12(1+z_1)\ ;1+\alpha_2+\beta_2,\tfrac12(1+z_2)\ ;\\ - :: -\quad ;-\ ;-\quad :-\ ;p_1\qquad\qquad ;p_2\qquad\qquad ;\end{array} t,t,t\right];$$

(5.20)
$$\sum_{n=0}^{\infty}\frac{(1-p_1)_n(1-p_2)_n}{n!}F_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(p_1-n,z_1;p_2-n,z_2)t^n$$
$$= F^{(3)}\left[\begin{array}{l}- :: 1-p_2\,;-;1-p_1:-;1+\alpha_1+\beta_1,\tfrac12(1+z_1)\,;1+\alpha_2+\beta_2,\tfrac12(1+z_2)\,;\\ - :: -\qquad ;-;-\qquad :-;1+\alpha_1\qquad\qquad ;1+\alpha_2\qquad\qquad ;\end{array} t,t,t\right];$$

(5.21)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n(-\alpha_2)_n}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n}F_n^{(\alpha_1-n,\,\beta_1-n;\,\alpha_2-n,\,\beta_2-n)}(p_1,z_1;p_2,z_2)t^n$$
$$= F^{(3)}\left[\begin{array}{l}- :: -\alpha_2\qquad ;-;-\alpha_1\qquad :-;\tfrac12(1+z_1)\,;\tfrac12(1+z_2)\,;\\ - :: -\alpha_2-\beta_2\,;-;-\alpha_1-\beta_1:-;p_1\qquad ;p_2\qquad ;\end{array} t,-t,-t\right];$$

(5.22)
$$\sum_{n=0}^{\infty}\frac{(1-p_1)_n(1-p_2)_n}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n}F_n^{(\alpha_1,\,\beta_1-2n;\,\alpha_2,\,\beta_2-2n)}(p_1-n,z_1;p_2-n,z_2)t^n$$
$$= F^{(3)}\left[\begin{array}{l}- :: 1-p_2\qquad ;-;1-p_1\qquad :-;\tfrac12(1+z_1)\,;\tfrac12(1+z_2)\,;\\ - :: -\alpha_2-\beta_2\,;-;-\alpha_1-\beta_1:-;1+\alpha_1\quad ;1+\alpha_2\quad ;\end{array} t,-t,-t\right].$$

6. Generalized Pasternak's polynomials of two variables

The generalized Pasternak's polynomials $F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)$ are defined by

(6.1)

$$F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2) = \sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(\frac{1}{2}(1+z_1+m_1))_r(1+\alpha_2+\beta_2+n)_s(\frac{1}{2}(1+z_2+m_2))_s}{r!s!(1+\alpha_1)_r(1+m_1)_r(1+\alpha_2)_s(1+m_2)_s}.$$

The relation (6.1) can also be expressed in terms of double hypergeometric function as follows:

(6.2)

$$F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2) = F_{0:2;2}^{1:2;2}\left[\begin{matrix} -n & : 1+\alpha_1+\beta_1+n, \frac{1}{2}(1+z_1+m_1) & ; 1+\alpha_2+\beta_2+n, \frac{1}{2}(1+z_2+m_2) & ; \\ - & : 1+\alpha_1, 1+m_1 & ; 1+\alpha_2, 1+m_2 & ; \end{matrix} 1,1\right],$$

where we have used a special case of the double hypergeometric function defined by (1.4).

The definition (6.1) can also be represented as follows:

$$\begin{aligned}&F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)\\ &= \sum_{r=0}^{n}\frac{(-n)_r(1+\alpha_1+\beta_1+n)_r(\frac{1}{2}(1+z_1+m_1))_r}{r!(1+\alpha_1)_r(1+m_1)_r}F_{n-r,m_2}^{(\alpha_2,\,\beta_2+r)}(z_2),\end{aligned} \tag{6.3}$$

where $F_{n,m_2}^{(\alpha,\,\beta)}(z_2)$ is the well known generalized Pasternak's polynomial of one variable defined by (1.3).

The relationships between generalized Pasternak's polynomials of two variables and generalized Pasternak's polynomials of single variable are as follows:

$$F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,-1-m_2) = F_{n,m_1}^{(\alpha_1,\,\beta_1)}(z_1) \tag{6.4}$$

and

$$F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(-1-m_1,z_2) = F_{n,m_2}^{(\alpha_2,\,\beta_2)}(z_2). \tag{6.5}$$

7. Generating functions for generalized Pasternak's polynomials of two variables

The polynomials $F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)$ admit the following generating functions:

(7.1)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$
$$=F_{2:0;2;2}^{3:0;1;1}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\;:-\;;\\(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\;:-\;;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1+m_1):1)\;;(\tfrac{1}{2}(1+z_2+m_2):1)\;;\\(1+\alpha_1:1),(1+m_1:1)\;;(1+\alpha_2:1),(1+m_2:1)\;;\end{array}t,-t,-t\right],$$

where we have used a special case of generalized Lauricella function (see, for example, [5, p. 37]).

(7.2)
$$\sum_{n=0}^{\infty}\frac{1}{n!(\mu)_n}F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$
$$=F_{3:0;2;2}^{2:0;1;1}\left[\begin{array}{l}(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\;:-\;;\\(\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\;:-\;;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1+m_1):1)\;;(\tfrac{1}{2}(1+z_2+m_2):1)\;;\\(1+\alpha_1:1),(1+m_1:1)\;;(1+\alpha_2:1),(1+m_2:1)\;;\end{array}t,-t,-t\right];$$

(7.3)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!(\mu)_n}F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$
$$=F_{3:0;2;2}^{3:0;1;1}\left[\begin{array}{l}(\lambda:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2)\;:-\;;\\(\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\;:-\;;\end{array}\right.$$
$$\left.\begin{array}{l}(\tfrac{1}{2}(1+z_1+m_1):1)\;;(\tfrac{1}{2}(1+z_2+m_2):1)\;;\\(1+\alpha_1:1),(1+m_1:1)\;;(1+\alpha_2:1),(1+m_2:1)\;;\end{array}t,-t,-t\right];$$

(7.4)
$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu)_n}F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$
$$=F_{3:0;2;2}^{4:0;1;1}\left[\begin{array}{l}(\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):\\(\mu:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1)\;:\end{array}\right.$$
$$\left.\begin{array}{l}-\;;(\tfrac{1}{2}(1+z_1+m_1):1)\;;(\tfrac{1}{2}(1+z_2+m_2):1)\;;\\-\;;(1+\alpha_1:1),(1+m_1:1);(1+\alpha_2:1),(1+m_2:1);\end{array}t,-t,-t\right];$$

(7.5)

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n(\lambda_2)_n}{n!(\mu_1)_n(\mu_2)_n} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$

$$= F_{4:0;2;2}^{4:0;1;1}\left[\begin{array}{l} (\lambda_1:1,1,1),(\lambda_2:1,1,1),(1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2): \\ (\mu_1:1,1,1),(\mu_2:1,1,1),(1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1): \end{array}\right.$$
$$\left.\begin{array}{l} -;(\tfrac{1}{2}(1+z_1+m_1):1)\quad ;(\tfrac{1}{2}(1+z_2+m_2):1)\quad ; \\ -;(1+\alpha_1:1),(1+m_1:1);(1+\alpha_2:1),(1+m_2:1); \end{array} t,-t,-t\right].$$

Special cases

If we put $\lambda = \mu$ in the generating function (7.3), we get

(7.6)

$$\sum_{n=0}^{\infty} \frac{1}{n!} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$

$$= F_{2:0;2;2}^{2:0;1;1}\left[\begin{array}{l} (1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):-\ ; \\ (1+\alpha_1+\beta_1:1,1,1),(1+\alpha_2+\beta_2:1,1,1):-\ ; \end{array}\right.$$
$$\left.\begin{array}{l} (\tfrac{1}{2}(1+z_1+m_1):1)\quad ;(\tfrac{1}{2}(1+z_2+m_2):1)\quad ; \\ (1+\alpha_1:1),(1+m_1:1)\ ;(1+\alpha_2:1),(1+m_2:1)\ ; \end{array} t,-t,-t\right].$$

When $\lambda_1 = 1+\alpha_1+\beta_1$ and $\lambda_2 = 1+\alpha_2+\beta_2$, then the generating function (7.4) reduces immediately to the form:

(7.7)

$$\sum_{n=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_n(1+\alpha_2+\beta_2)_n}{n!(\mu)_n} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(z_1,z_2)t^n$$

$$= F_{1:0;2;2}^{2:0;1;1}\left[\begin{array}{l} (1+\alpha_1+\beta_1:1,2,1),(1+\alpha_2+\beta_2:1,1,2):-\ ; \\ (\mu:1,1,1)\qquad\qquad :-\ ; \end{array}\right.$$
$$\left.\begin{array}{l} (\tfrac{1}{2}(1+z_1+m_1):1)\quad ;(\tfrac{1}{2}(1+z_2+m_2):1)\quad ; \\ (1+\alpha_1:1),(1+m_1:1)\ ;(1+\alpha_2:1),(1+m_2:1)\ ; \end{array} t,-t,-t\right].$$

Further, we have the following generating functions:

$$\sum_{n=0}^{\infty} \frac{1}{n!} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(z_1,z_2)t^n$$
$$= e^t\ {}_2F_2\left[\begin{array}{c} 1+\alpha_1+\beta_1, \tfrac{1}{2}(1+z_1+m_1)\ ; \\ 1+\alpha_1,\ \ 1+m_1\ ; \end{array} -t\right] \tag{7.8}$$
$$\times {}_2F_2\left[\begin{array}{c} 1+\alpha_2+\beta_2, \tfrac{1}{2}(1+z_2+m_2)\ ; \\ 1+\alpha_2,\ \ 1+m_2\ ; \end{array} -t\right];$$

(7.9)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!}F^{(\alpha_1-n,\ \beta_1;\ \alpha_2,\ \beta_2-n)}_{n,m_1,m_2}(z_1,z_2)t^n$$
$$=(1-t)^{\alpha_1}\ {}_2F_1\left[\begin{array}{c}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;\\ 1+m_1\ ;\end{array}\ t\right]$$
$$\times {}_3F_2\left[\begin{array}{c}-\alpha_1,1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ 1+\alpha_2,\ 1+m_2\ ;\end{array}\ \frac{-t}{1-t}\right];$$

(7.10)
$$\sum_{n=0}^{\infty}\frac{(-\alpha_2)_n}{n!}F^{(\alpha_1,\ \beta_1-n;\ \alpha_2-n,\ \beta_2)}_{n,m_1,m_2}(z_1,z_2)t^n$$
$$=(1-t)^{\alpha_2}\ {}_2F_1\left[\begin{array}{c}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ 1+m_2\ ;\end{array}\ t\right]$$
$$\times {}_3F_2\left[\begin{array}{c}-\alpha_2,1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;\\ 1+\alpha_1,\ 1+m_1\ ;\end{array}\ \frac{-t}{1-t}\right];$$

(7.11)
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(\alpha_1,\ \beta_1-n;\ \alpha_2,\ \beta_2-n)}_{n,m_1,m_2}(z_1,z_2)t^n$$
$$=(1-t)^{-\lambda}$$
$$\times F^{1:2;2}_{0:2;2}\left[\begin{array}{l}\lambda\ :1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ -\ :1+\alpha_1,1+m_1\ ;1+\alpha_2,1+m_2\ ;\end{array}\ \frac{-t}{1-t},\frac{-t}{1-t}\right];$$

(7.12)
$$\sum_{n=0}^{\infty}\frac{1}{n!(\frac{1}{2}(1-z_1-m_1))_n}F^{(\alpha_1,\ \beta_1-n;\ \alpha_2,\ \beta_2-n)}_{n,m_1,m_2}(z_1-n,z_2)t^n$$
$$={}_1F_2\left[\begin{array}{c}1+\alpha_1+\beta_1\ ;\\ 1+\alpha_1,1+m_1\ ;\end{array}\ t\right]$$
$$\times F^{0:0;2}_{1:0;2}\left[\begin{array}{l}-\ :-\ ;1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ \frac{1}{2}(1-z_1-m_1)\ :-\ ;1+\alpha_2,1+m_2\ ;\end{array}\ t,-t\right];$$

(7.13)
$$\sum_{n=0}^{\infty}\frac{1}{n!(\frac{1}{2}(1-z_2-m_2))_n}F^{(\alpha_1,\ \beta_1-n;\ \alpha_2,\ \beta_2-n)}_{n,m_1,m_2}(z_1,z_2-n)t^n$$
$$={}_1F_2\left[\begin{array}{c}1+\alpha_2+\beta_2\ ;\\ 1+\alpha_2,1+m_2\ ;\end{array}\ t\right]$$
$$\times F^{0:0;2}_{1:0;2}\left[\begin{array}{l}-\ :-\ ;1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;\\ \frac{1}{2}(1-z_2-m_2)\ :-\ ;1+\alpha_1,1+m_1\ ;\end{array}\ t,\ -t\right];$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_1)_n}{n!(-\alpha_1-\beta_1)_n}F_{n,m_1,m_2}^{(\alpha_1-n,\ \beta_1-n;\ \alpha_2,\ \beta_2-n)}(z_1,z_2)t^n$$

$$= {}_1F_1\left[\begin{array}{ll}\frac{1}{2}(1+z_1+m_1) & ;\\ 1+m_1 & ;\end{array} -t\right] \tag{7.14}$$

$$\times F_{1:0;2}^{1:0;2}\left[\begin{array}{ll}-\alpha_1 & :-\ ;1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ -\alpha_1-\beta_1 & :-\ ;1+\alpha_2,\ 1+m_2\ ;\end{array} t,-t\right];$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha_2)_n}{n!(-\alpha_2-\beta_2)_n}F_{n,m_1,m_2}^{(\alpha_1,\ \beta_1-n;\ \alpha_2-n,\ \beta_2-n)}(z_1,z_2)t^n$$

$$= {}_1F_1\left[\begin{array}{ll}\frac{1}{2}(1+z_2+m_2) & ;\\ 1+m_2 & ;\end{array} -t\right] \tag{7.15}$$

$$\times F_{1:0;2}^{1:0;2}\left[\begin{array}{ll}-\alpha_2 & :-\ ;1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;\\ -\alpha_2-\beta_2 & :-\ ;1+\alpha_1,\ 1+m_1\ ;\end{array} t,-t\right];$$

$$\sum_{n=0}^{\infty}\frac{1}{n!(-\alpha_1-\beta_1)_n}F_{n,m_1,m_2}^{(\alpha_1,\ \beta_1-2n;\ \alpha_2,\ \beta_2-n)}(z_1,z_2)t^n$$

$$= {}_1F_2\left[\begin{array}{ll}\frac{1}{2}(1+z_1+m_1) & ;\\ 1+\alpha_1,\ 1+m_1 & ;\end{array} t\right] \tag{7.16}$$

$$\times F_{1:0;2}^{0:0;2}\left[\begin{array}{ll}- & :-\ ;1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2)\ ;\\ -\alpha_1-\beta_1 & :-\ ;1+\alpha_2,\ 1+m_2\ ;\end{array} t,\ -t\right];$$

$$\sum_{n=0}^{\infty}\frac{1}{n!(-\alpha_2-\beta_2)_n}F_{n,m_1,m_2}^{(\alpha_1,\ \beta_1-n;\ \alpha_2,\ \beta_2-2n)}(z_1,z_2)t^n$$

$$= {}_1F_2\left[\begin{array}{ll}\frac{1}{2}(1+z_2+m_2) & ;\\ 1+\alpha_2,\ 1+m_2 & ;\end{array} t\right] \tag{7.17}$$

$$\times F_{1:0;2}^{0:0;2}\left[\begin{array}{ll}- & :-\ ;1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1)\ ;\\ -\alpha_2-\beta_2 & :-\ ;1+\alpha_1,\ 1+m_1\ ;\end{array} t,\ -t\right].$$

Proof of (7.8).

$$\sum_{n=0}^{\infty}\frac{1}{n!}F_{n,m_1,m_2}^{(\alpha_1,\ \beta_1-n;\ \alpha_2,\ \beta_2-n)}(z_1,z_2)t^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s(\frac{1}{2}(1+z_1+m_1))_r(\frac{1}{2}(1+z_2+m_2))_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s(1+m_1)_r(1+m_2)_s}t^n$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s(\frac{1}{2}(1+z_1+m_1))_r(\frac{1}{2}(1+z_2+m_2))_s}{n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s(1+m_1)_r(1+m_2)_s}t^n(-t)^r(-t)^s$$

$$= e^t\ {}_2F_2\left[\begin{array}{ll}1+\alpha_1+\beta_1,\frac{1}{2}(1+z_1+m_1); & \\ 1+\alpha_1,\ \ 1+m_1 & ;\end{array} -t\right]{}_2F_2\left[\begin{array}{ll}1+\alpha_2+\beta_2,\frac{1}{2}(1+z_2+m_2); & \\ 1+\alpha_2,\ \ 1+m_2 & ;\end{array} -t\right].$$

Where for ${}_2F_2$ one is referred to [4]. This completes the proof of (7.8). □

The other results (7.9) to (7.17) will be established by using the similar argument as in (7.8).

Further,we have the following generating functions:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!(\mu)_n} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(z_1,z_2)t^n$$

(7.18)
$$= F^{(3)}\left[\begin{array}{l} \lambda \;::-\;;-\;;-\;:-\;;1+\alpha_1+\beta_1,\tfrac{1}{2}(1+z_1+m_1)\;; \\ \mu \;::-\;;-\;;-\;:-\;;1+\alpha_1,\;1+m_1\;; \end{array}\right.$$
$$\left.\begin{array}{l} 1+\alpha_2+\beta_2,\tfrac{1}{2}(1+z_2+m_2)\;; \\ 1+\alpha_2,\;1+m_2\;; \end{array} t,-t,-t\right];$$

$$\sum_{n=0}^{\infty} \frac{(-\alpha_1)_n(-\alpha_2)_n}{n!} F_{n,m_1,m_2}^{(\alpha_1-n,\,\beta_1;\,\alpha_2-n,\,\beta_2)}(z_1,z_2)t^n$$

(7.19)
$$= F^{(3)}\left[\begin{array}{l} -\;::-\alpha_2\;;-\;;-\alpha_1\;:-\;;1+\alpha_1+\beta_1,\tfrac{1}{2}(1+z_1+m_1)\;; \\ -\;::-\;;-\;;-\;:-\;;1+m_1\;; \end{array}\right.$$
$$\left.\begin{array}{l} 1+\alpha_2+\beta_2,\tfrac{1}{2}(1+z_2+m_2)\;; \\ 1+m_2\;; \end{array} t,t,t\right];$$

(7.20)
$$\sum_{n=0}^{\infty} \frac{1}{n!(\frac{1}{2}(1-z_1-m_1))_n(\frac{1}{2}(1-z_2-m_2))_n}$$
$$F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n)}(z_1-n,z_2-n)t^n$$
$$= F^{(3)}\left[\begin{array}{l} -\;::-\;;-\;;-\;:-\;;1+\alpha_1+\beta_1\;; \\ -\;::\tfrac{1}{2}(1-z_2-m_2)\;;-\;;\tfrac{1}{2}(1-z_1-m_1)\;:-\;;1+\alpha_1,\;1+m_1\;; \end{array}\right.$$
$$\left.\begin{array}{l} 1+\alpha_2+\beta_2\;; \\ 1+\alpha_2,\;1+m_2\;; \end{array} t,t,t\right];$$

$$\sum_{n=0}^{\infty} \frac{(-\alpha_1)_n(-\alpha_2)_n}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n} F_{n,m_1,m_2}^{(\alpha_1-n,\,\beta_1-n;\,\alpha_2-n,\,\beta_2-n)}(z_1,z_2)t^n$$

(7.21)
$$= F^{(3)}\left[\begin{array}{l} -\;::-\alpha_2\;;-\;;-\alpha_1\;:-\;;\tfrac{1}{2}(1+z_1+m_1)\;; \\ -\;::-\alpha_2-\beta_2\;;-\;;-\alpha_1-\beta_1\;:-\;;1+m_1\;; \end{array}\right.$$
$$\left.\begin{array}{l} \tfrac{1}{2}(1+z_2+m_2)\;; \\ 1+m_2\;; \end{array} t,-t,-t\right];$$

$$\sum_{n=0}^{\infty} \frac{1}{n!(-\alpha_1-\beta_1)_n(-\alpha_2-\beta_2)_n} F_{n,m_1,m_2}^{(\alpha_1,\,\beta_1-2n;\,\alpha_2,\,\beta_2-2n)}(z_1,z_2)t^n$$

(7.22)
$$= F^{(3)}\left[\begin{array}{l} -\;::-\;;-\;;-\;:-\;;\tfrac{1}{2}(1+z_1+m_1)\;; \\ -\;::-\alpha_2-\beta_2\;;-\;;-\alpha_1-\beta_1\;:-\;;1+\alpha_1,\;1+m_1\;; \end{array}\right.$$
$$\left.\begin{array}{l} \tfrac{1}{2}(1+z_2+m_2)\;; \\ 1+\alpha_2,\;1+m_2\;; \end{array} t,t,t\right].$$

Proof of (7.18).

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!(\mu)_n} F_{n,m_1,m_2}^{(\alpha_1,\, \beta_1-n;\, \alpha_2,\, \beta_2-n)}(z_1, z_2)t^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!(\mu)_n} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s(\frac{1}{2}(1+z_1+m_1))_r(\frac{1}{2}(1+z_2+m_2))_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s(1+m_1)_r(1+m_2)_s} t^n \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-1)^{r+s}(\lambda)_n(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s(\frac{1}{2}(1+z_1+m_1))_r(\frac{1}{2}(1+z_2+m_2))_s}{r!s!(n-r-s)!(\mu)_n(1+\alpha_1)_r(1+\alpha_2)_s(1+m_1)_r(1+m_2)_s} t^n \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\lambda)_{n+r+s}(1+\alpha_1+\beta_1)_r(\frac{1}{2}(1+z_1+m_1))_r(1+\alpha_2+\beta_2)_s(\frac{1}{2}(1+z_2+m_2))_s}{n!r!s!(\mu)_{n+r+s}(1+\alpha_1)_r(1+m_1)_r(1+\alpha_2)_s(1+m_2)_s} \\
&\quad \times t^n(-t)^r(-t)^s \\
&= F^{(3)}\left[\begin{array}{l} \lambda :: - \;;- \;;- \;: - \;;1+\alpha_1+\beta_1, \frac{1}{2}(1+z_1+m_1) \;; \\ \mu :: - \;;- \;;- \;: - \;;1+\alpha_1,\ 1+m_1 \;; \end{array}\right. \\
&\qquad\qquad \left.\begin{array}{l} 1+\alpha_2+\beta_2, \frac{1}{2}(1+z_2+m_2) \;; \\ 1+\alpha_2,\ 1+m_2 \;; \end{array} t, -t, -t\right],
\end{aligned}$$

where $F^{(3)}[x, y, z]$ denotes a general triple hypergeometric series [6, p. 69, Eq. (39)]. This completes the proof of (7.18). □

A similar argument as in getting (7.18) will establish the other results (7.19) to (7.22).

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