Commun. Korean Math. Soc. **28** (2013), No. 1, pp. 79–85
http://dx.doi.org/10.4134/CKMS.2013.28.1.079

AN IDEAL - BASED ZERO-DIVISOR GRAPH OF POSETS

Balasubramanian Elavarasan and Kasi Porselvi

Abstract. The structure of a poset P with smallest element 0 is looked at from two view points. Firstly, with respect to the Zariski topology, it is shown that $Spec(P)$, the set of all prime semi-ideals of P, is a compact space and $Max(P)$, the set of all maximal semi-ideals of P, is a compact T_1 subspace. Various other topological properties are derived. Secondly, we study the semi-ideal-based zero-divisor graph structure of poset P, denoted by $G_I(P)$, and characterize its diameter.

1. Preliminaries

Throughout this paper, $(P, \leq)$ denotes a poset with a least element 0, and all prime and maximal semi-ideals of P are assumed to be proper. For $M \subseteq P$, let $(M)^l := \{x \in P \ : \ x \leq m \text{ for all } m \in M\}$ denote the lower cone of M in P, and dually let $(M)^u := \{x \in P \ : \ m \leq x \text{ for all } m \in M\}$ be the upper cone of M in P. For $A, B \subseteq P$, we write $(A, B)^l$ instead of $(A \cup B)^l$ and dually for the upper cones. If $M = \{x_1, x_2, \ldots, x_n\}$ is finite, then we use the notation $(x_1, x_2, \ldots, x_n)^l$ instead of $(\{x_1, x_2, \ldots, x_n\})^l$ (and dually). We use $Spec(P)$ and $Max(P)$ for the spectrum of prime semi-ideals and the maximal semi-ideals of P, respectively.

Following [10], a nonempty subset I of P, I is called a semi-ideal of P if $b \in I$ and $a \leq b$, then $a \in I$. A proper semi-ideal I of P is called prime if for any $a, b \in P$, $(a, b)^l \subseteq I$ implies $a \in I$ or $b \in I$. In [5], Radomir Halaš, in which he has used the term ideals for the semi-ideals of a poset, defined a class of n-prime semi-ideals in posets, a semi-ideal I is called n-prime if for pairwise distinct elements $x_1, x_2, \ldots, x_n \in P$, if $(x_1, x_2, \ldots, x_n)^l \subseteq I$, then at least $(n-1)$ of n-subsets $(x_2, x_3, \ldots, x_n)^l$, $(x_1, x_3, \ldots, x_n)^l, \ldots, (x_1, x_2, \ldots, x_{n-1})^l$ is a subset of I. From Theorem 3 of [5], we can observe that every prime semi-ideal of P is n-prime. For any semi-ideal J of P and $a \in P$, we define $V(a) = \{I \in Spec(P) \ : \ a \in I\}$ and $D(I) = Spec(P) \backslash V(I)$. Let $V(J) = \cap_{a \in J} V(a)$. Then $F = \{V(J) \ : \ J \text{ is an semi-ideal of } P\}$ is closed under finite unions and arbitrary intersections, so that there is a topology on $Spec(P)$ for which F is

Received May 16, 2011; Revised November 23, 2011.
2010 *Mathematics Subject Classification.* 05C99, 06B35.
Key words and phrases. posets, semi-ideals, prime semi-ideals, zero-divisor graph.

the family of closed sets. This is called the Zariski topology. It is easy to see that, for any subset A of P, $(A)^l$ is a semi-ideal of P. If $A = \{a\}$, for any $a \in P$, then $(a)^l$ is the smallest semi-ideal containing a, and also $V(a) = V((a)^l)$. Also $B = \{D(a) : a \in P\}$ form a basis for a topology on $Spec(P)$. It is also clear that $Max(P) \subseteq Spec(P)$.

In [2], I. Beck introduced the idea of a zero-divisor graph of a commutative ring. Let the zero-divisors of R be the vertices and connect two vertices a and b by an edge in case $ab = 0$. Later in [1], D. F. Anderson and P. S. Livingston have considered only non-zero zero-divisors as vertices of the zero-divisor graph of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R)\backslash\{0\}$, the set of non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. In [9], S. P. Redmond generalized this notion by replacing elements whose product is zero with elements whose product lies in some ideal I of R.

In [6], R. Halaš and M. Jukl have introduced the concept of a graph structure of posets, let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of P, denoted by $G(P)$, is an undirected graph whose vertices are just the elements of P with two distinct vertices x and y are joined by an edge if and only if $(x, y)^l = \{0\}$, and proved some interesting results related with clique and chromatic number of this graph structure.

In [7], V. Joshi introduced the zero divisor graph $G_I(P)$ of a poset P (with 0) with respect to an ideal I, and proved $G_I(P)$ is connected with its diameter 3, also and if $G_I(P)$ contains a cycle, then the core K of $G_I(P)$ is a union of 3-cycles and 4-cycles.

In this paper, we study the zero divisor graph $G_I(P)$ of a poset P with respect to a semi-ideal I as semi-ideal need not be an ideal in poset. Let P be a poset and J be a semi-ideal of P. Then the graph of P with respect to the semi-ideal J, denoted by $G_J(P)$, is the graph whose vertices are the set $\{x \in P\backslash J : (x, y)^l \subseteq J$ for some $y \in P\backslash J\}$ with distinct vertices x and y are adjacent if and only if $(x, y)^l \subseteq J$. If $J = \{0\}$, then $G_J(P) = G(P)$, and J is a prime semi-ideal of P if and only if $G_J(P) = \phi$. For distinct vertices x and y of a graph G, let $d(x, y)$ be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices.

Following [5], let I be a semi-ideal of P. Then the extension of I by $x \in P$ is meant the set $(I : x) = \{a \in P : (a, x)^l \subseteq I\}$. For any subset S of P, we define $I_S = \{a \in P : (a, s)^l \subseteq I$ for all $s \in S\}$. Note that $I_S = \cap_{s \in S}(I : s)$, if $S = \{a\}$, then $I_S = (I : s)$. Let $\mathbb{P}$ be the intersection of all prime semi-ideals of P. Then we set $Supp(a) = \cap_{x \in (\mathbb{P}:a)} V(x)$. In this paper the notations of graph theory are from [3], the notations of posets are from [5] and [7], and the notations of topology are from [4] and [8].

2. Topological space of $Spec(P)$

In this section, we associate the poset properties of P and the topological properties of $Spec(P)$. We start this section with the following useful lemma.

Lemma 2.1. *Let P be a poset and A a subset of P. Then*

(i) *If $x \in A$, then $V(A) \subseteq V(x)$ and $D((\mathbb{P} : x)) \subseteq V(x)$.*
(ii) *If $V(A) = \phi$, then $A = P$.*
(iii) *$D(A) = \phi$ if and only if $A \subseteq \mathbb{P}$.*
(iv) *$V(\{0\}) = Spec(P)$ and $V(P) = \phi$.*
(v) *$V(I) \cup V(J) = V(I \cap J)$ for any semi-ideals I, J of P.*
(vi) *$\cap_{i \in A} V(I_i) = V(\cup_{i \in A} I_i)$, I_i is a semi-ideal of P for each $i \in A$.*

Lemma 2.2. *Let P be a poset. If A is a subset of $Spec(P)$, then there exists a semi-ideal $J = \cap A$ of P with $cl(A) = V(J)$. In particular, if A is a closed subset of $Spec(P)$, then $A = V(J)$ for some semi-ideal J of P.*

Proof. Let A be a subset of $Spec(P)$ and $J = \cap A$. Then it is easy to verify that $cl(A) \subseteq V(J)$ as $A \subseteq V(J)$. Let $P_1 \in V(J)$ and let $D(x)$ be any arbitrary element in B such that $P_1 \in D(x)$. Suppose that $D(x) \cap A = \phi$. Then $x \in J$, and so $P_1 \in V(x)$, a contradiction. Thus $D(x) \cap A \neq \phi$, and hence, the result follows from Theorem 17.5 of [8]. □

With the help of Lemma 2.2, we have the following remark and some important characterizations of $Spec(P)$.

Remark 2.3. Let P be a poset. Then

(i) The closure of $I \in Spec(P)$ is $V(I)$.
(ii) A point $I \in Spec(P)$ is closed if and only if $I \in Max(P)$.
(iii) If $I, J \in Spec(P)$ with $cl(I) = cl(J)$, then $I = J$.

Theorem 2.4. *Let S be a subset of P. Then $\mathbb{P}_S = \cap V(\mathbb{P}_S)$.*

Proof. Clearly, $\mathbb{P}_S \subseteq \cap V(\mathbb{P}_S)$. Let $a \in \cap V(\mathbb{P}_S)$. Suppose on the contrary that $a \in P \backslash \mathbb{P}$. Then $(a, s)^l \nsubseteq I$ for some $I \in Spec(P)$ and some $s \in S$ which implies $a \notin I$ and $s \notin I$. So we can get $\mathbb{P}_S \subseteq I$. Thus $a \notin I \in V(\mathbb{P}_S)$, a contradiction. □

Theorem 2.5. *Let P be a poset and $a, b \in P$. Then $int\ V(a) \subseteq int\ V(b)$ if and only if $(\mathbb{P} : a) \subseteq (\mathbb{P} : b)$.*

Proof. Let $int\ V(a) \subseteq int\ V(b)$ for any $a, b \in P$ and $x \in (\mathbb{P} : a)$. Then $Spec(P) \backslash V(x) \subseteq int\ V(a) \subseteq int\ V(b) \subseteq V(b)$, which gives $(b, x)^l \subseteq \mathbb{P}$, so $x \in (\mathbb{P} : b)$. Conversely, let $(\mathbb{P} : a) \subseteq (\mathbb{P} : b)$ and let $I \in int\ V(a)$. Suppose $I \notin V(b)$. By Lemma 2.2, since $I \notin Spec(P) \backslash int\ V(a)$, then there is $0 \neq c \in P$ with $Spec(P) \backslash int\ V(a) \subseteq V(c)$ and $c \notin I$ which imply $(a, c)^l \subseteq \mathbb{P}$. Clearly $(b, c)^l \nsubseteq I$. Then $c \in (\mathbb{P} : a)$ and $c \notin (\mathbb{P} : b)$, a contradiction. Thus $I \in V(b)$ and hence $int\ V(a) \subseteq V(b)$ which implies $int\ V(a) \subseteq int\ V(b)$. □

Theorem 2.6. *Let P be a poset. Then $cl(D(a)) = V((\mathbb{P} : a)) = Supp(a) = Spec(P)\backslash int\ V(a)$ for every $a \in P$.*

Proof. It is easy to verify that $D(a) \subseteq V((\mathbb{P} : a))$ which implies $cl(D(a)) \subseteq V((\mathbb{P} : a))$. Let $I \in V((\mathbb{P} : a))$ and $D(x)$ be any arbitrary element in B such that $I \in D(x)$. We now claim that $D(x) \cap D(a) \neq \phi$. If $I \notin D(a)$ and suppose $D(x) \cap D(a) = \phi$, then $D((x, a)^l) \subseteq D(x) \cap D(a) = \phi$ which implies $(x, a)^l \subseteq \mathbb{P}$. Then $x \in I$, a contradiction to $I \in D(x)$. Thus $D(x) \cap D(a) \neq \phi$ and hence $V((\mathbb{P} : a)) \subseteq cl(D(a))$. By the definition, we have $V((\mathbb{P} : a)) = Supp(a)$. It remains to prove that $cl(D(a)) = Spec(P)\backslash int\ V(a)$.

Let $I_1 \in cl(D(a))$ and suppose that $I_1 \in int\ V(a)$. Then there exists an open set U of $Spec(P)$ with $I_1 \in U \subseteq V(a)$, and so $I_1 \notin Spec(P)\backslash U$, a contradiction as $Spec(P)\backslash U$ is a closed set containing $D(a)$. So $cl(D(a)) \subseteq Spec(P)\backslash int\ V(a)$. Let $I_1 \in Spec(P)\backslash int\ V(a)$ and let $D(x)$ be any arbitrary element in B with $I_1 \in D(x)$. Suppose that $D(x) \cap D(a) = \phi$. Then $I_1 \in D((\mathbb{P} : a)) \subseteq V(a)$, a contradiction. □

Lemma 2.7. *Let P be a poset with greatest element e. Then $Spec(P)$ does not contains any clopen subset.*

Proof. Suppose that A is a clopen subset of $Spec(P)$ and let $J = \cap A$ and $J_1 = \cap A^c$. Then by Lemma 2.2 $A = cl(A) = V(J)$ and $A^c = V(J_1)$, and so $V(J) \cap V(J_1) = \phi$ which gives $e \in P = J \cup J_1$, a contradiction. □

Lemma 2.8. *Let P be a poset with greatest element e. If $F \subseteq Spec(P)$ is a closed set and $D(K)$ is an open set in $Spec(P)$ satisfying $F \cap Max(P) \subseteq D(K)$, then $F \subseteq D(K)$.*

Proof. Suppose that there is $I \in F$ with $I \notin D(K)$. Then $K \cup L \subseteq I$, since $F = V(L)$ for some semi-ideal L of P. Hence, each maximal semi-ideal M containing I is also in F. Then $M \in F \cap Max(P)$, and so $M \in D(K)$, a contradiction. □

Theorem 2.9. *Let P be a poset with greatest element e. Then*

(i) *$Max(P)$ is a compact T_1 subspace.*
(ii) *If $Spec(P)$ is normal, then $Max(P)$ is a Hausdorff space.*

Proof. (i) Let $B = \{D(s_i) : s_i \in J\}$ be the basis of P for any subset J of P, and suppose that $Max(P) = (\cup_{s_j \in J} D(s_i)) \cap Max(P)$. Then $\phi = \cap_{s_i \in J}(Max(P)\backslash D(s_i)) = (\cap_{s_i \in J} V(s_i)) \cap Max(P) = V(\cup_{s_i \in J}(s_i)^l) \cap Max(P)$ which implies $e \in (s_i)^l$ and $e = s_i$ for some $s_i \in J$. So $Max(P) = D(s_i)$. Let M_1 and M_2 be two distinct elements in $Max(P)$. Then $M_1 \in D(M_2)$ and $M_2 \in D(M_1)$, and so $Max(P)$ is a T_1 space.

(ii) Let M_1 and M_2 be distinct elements in $Max(P)$. Then $\{M_1\}$ and $\{M_2\}$ are closed subsets in both $Spec(P)$ and $Max(P)$. If $Spec(P)$ is normal, then there exist disjoint open sets $D(I)$ and $D(J)$ such that $\{M_1\} \subseteq D(I)$ and $\{M_2\} \subseteq D(J)$ for some semi-ideals I and J of P, respectively. So, $M_1 \in$

$D(I) \cap Max(P)$, and $M_2 \in D(J) \cap Max(P)$, which imply $Max(P)$ is a Hausdorff space. □

3. Properties of semi-ideal-based zero-divisor graphs

In this section, we associate the poset properties of P and the graph properties of semi-ideal-based zero-divisor graphs of poset. Although the proof of the following three theorems are just smilar of that for Theorem 2.4, Lemma 2.12 and Theorem 2.13 given in [7] to semi-ideal I of P.

Theorem 3.1 ([7])**.** *Let I be a semi-ideal of P. Then $G_I(P)$ is connected and $diam(G_I(P)) \leq 3$.*

Theorem 3.2 ([7])**.** *Let I be a semi-ideal of P and if $a - x - b$ is a path in $G_I(P)$, then either $I \cup \{x\}$ is a semi-ideal of P or $a - x - b$ is contained in a cycle of length ≤ 4.*

In view of above theorem, we have the following corollary.

Corollary 3.3. *Let $|G_I(P)| \geq 3$ and $I \cup \{x\}$ be not a semi-ideal of P for any $x \notin I$. Then any edge in $G_I(P)$ is contained in a cycle of length ≤ 4, and therefore $G_I(P)$ is a union of triangles and squares.*

Theorem 3.4 ([7])**.** *Let I be a semi-ideal of P. If $G_I(P)$ contains a cycle, then the core K of $G_I(P)$ is a union of triangles and rectangles. Moreover, any vertex in $G_I(P)$ is either a vertex of the core K of $G_I(P)$ or else is an end vertex of $G_I(P)$.*

Lemma 3.5. *Let I be a semi-ideal of P. Then a pentagon or hexagon can not be a $G_I(P)$.*

Proof. Suppose that $G_I(P)$ is $a - b - c - d - e - a$, a pentagon. Then by Theorem 3.2, $I \cup \{a\}$ is a semi-ideal of P. Then in the pentagon, $(a, b)^l \subseteq I$ and $(a, e)^l \subseteq I$. Since $I \cup \{a\}$ is a semi-ideal, and $(a, c)^l \nsubseteq I$, we have $a \leq c$. Similarly, we can show that $a \leq d$. Thus $a \in (c, d)^l \subseteq I$, a contradiction to $a \notin I$. The proof for the hexagon is the same. □

Theorem 3.6. *If $I \cup \{x\}$ is not a semi-ideal of P for any $x \in P \backslash I$ and $|G_I(P)| \geq 3$, then every pair of vertices in $G_I(P)$ is contained in a cycle of length ≤ 6.*

Proof. Let $a, b \in G_I(P)$. If $(a, b)^l \subseteq I$, then $a - b$ is an edge of triangles or rectangles by Corollary 3.3. If $a - x - b$ is a path in $G_I(P)$, then it is contained in a cycle of length ≤ 4. If $a - x - y - b$ is a path in $G_I(P)$, then we find cycles $a - x - y - c - a$ and $b - y - x - d - b$ where $c \neq x$ and $d \neq y$. This gives cycle $a - x - d - b - y - c - a$ of length 6. □

Lemma 3.7. *Let P be a poset and let $a, b \in G_{\mathbb{P}}(P)$. Then*

(i) *$Supp(a) \cup Supp(b) \neq Spec(P)$ if and only if $Supp(a) \cup Supp(b) \subseteq V(c)$ for some $c \in G_{\mathbb{P}}(P)$.*

(ii) *$D(a) \cap D(b) \neq \phi$ if and only if there exists $c \in G_{\mathbb{P}}(P)$ such that $\phi \neq D(a) \cap D(b) \subseteq V(c)$.*

Proof. (i) Suppose $Supp(a) \cup Supp(b) \neq Spec(P)$. Then there exists an element $P \in Spec(P)$ with $x, y \notin P$ for some $x \in (\mathbb{P} : a)$ and $y \in (\mathbb{P} : b)$. So $(x, y)^l \nsubseteq \mathbb{P}$. So there exists $t \in (x, y)^l$ with $t \notin \mathbb{P}$. It is easy to verify that $t \in G_{\mathbb{P}}(P)$ and $Supp(a) \cup Supp(b) \subseteq V(t)$. Conversely, let $Supp(a) \cup Supp(b) \subseteq V(c)$ for some $c \in G_{\mathbb{P}}(P)$ and suppose that $Supp(a) \cup Supp(b) = Spec(P)$. Then $c \in \mathbb{P}$, a contradiction. Hence, $Supp(a) \cup Supp(b) \neq Spec(P)$.

(ii) Obvious. □

Now by Theorem 3.1, and Lemma 3.7, we have the following characterizations of the diameter of $G_{\mathbb{P}}(P)$.

Theorem 3.8. *Let P be a poset and let $a, b \in G_{\mathbb{P}}(P)$ be distinct elements. Then*

(i) *For any $c \in G_{\mathbb{P}}(P)$, we have c is adjacent to both a and b if and only if $Supp(a) \cup Supp(b) \subseteq V(c)$.*
(ii) *$d(a, b) = 1$ if and only if $D(a) \cap D(b) = \phi$.*
(iii) *$d(a, b) = 2$ if and only if $D(a) \cap D(b) \neq \phi$ and $Supp(a) \cup Supp(b) \neq Spec(P)$.*
(iv) *$d(a, b) = 3$ if and only if $D(a) \cap D(b) \neq \phi$ and $Supp(a) \cup Supp(b) = Spec(P)$.*

Proof. (i) and (ii) are trivial.

(iii) Let $a, b \in G_{\mathbb{P}}(P)$. Then $d(a, b) = 2$ if and only if $(a, b)^l \nsubseteq \mathbb{P}$ and there exists $c \in G_{\mathbb{P}}(P)$ such that c is adjacent to both a and b if and only if $D(a) \cap D(b) \neq \phi$ and $Supp(a) \cup Supp(b) \subseteq V(c)$ if and only if $D(a) \cap D(b) \neq \phi$ and $Supp(a) \cup Supp(b) \neq Spec(P)$ by Lemma 3.7.

(iv) By Theorem 3.1, $d(a, b) = 3$ if and only if $d(a, b) \neq 1, 2$ if and only if $D(a) \cap D(b) \neq \phi$ and $Supp(a) \cup Supp(b) = Spec(P)$ by (i) and (ii). □

Theorem 3.9. *Let I be a semi-ideal of P and let $a \in G_I(P)$. If a is adjacent to every other vertex in $G_I(P)$, then $(I : a)$ is a prime semi-ideal of P.*

Proof. Let $(x, y)^l \subseteq (I : a)$ for $x \in P$. Then $(a, x, y)^l \subseteq I$ and so $x \in (I : t)$ for all $t \in (y, a)^l$. Suppose that $y \notin (I : a)$. Then there exists $t_1 \in (y, a)^l$ such that $t_1 \notin I$. We now claim that $I_{t_1} = I_a$. Clearly $(I : a) \subseteq (I : t_1)$. Now let $p \in (I : t_1)$. If $p \in I$, then $p \in (I : a)$. Otherwise $p \notin I$. It is clear that $p \in G_{\mathbb{P}}(P)$. Since a is adjacent to every vertex, therefore $(p, a)^l \subseteq I$. So $(I : a) = (I : t_1)$. Since $x \in (I : t_1)$, we have $x \in (I : a)$. □

Lemma 3.10. *Let P be a poset. If $x \in P$ and $(I : x)$ is maximal among $(I : a) = \{y \in P \ : \ (a, y)^l \subseteq I\}$, then $(I : x)$ is a prime semi-ideal of P.*

Proof. Suppose that $(a, b)^l \subseteq (I : x)$ and $a \notin (I : x)$. Then $(a, b, x)^l \subseteq I$. Let $z \in (a, x)^l \backslash I$. Then $(b, z)^l \subseteq (a, b, x)^l \subseteq I$, thus $b \in (I : z)$. Since $(I : x) \subseteq (I : z)$

and $z \notin I$, we have $(I : z) \neq P$. By the maximality of $(I : x)$, we have $(I : x) = (I : z)$, hence $b \in (I : z) = (I : x)$. □

Acknowledgments. The authors express their sincere thanks to the referee for his/her valuable comments and suggestions which improve the paper a lot.

References

[1] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.
[2] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.
[3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, Amsterdam, 1976.
[4] R. Engelking, *General Topology*, Heldermann-Verlag, 2000.
[5] R. Halaš, *On extensions of ideals in posets*, Discrete Math. **308** (2008), no. 21, 4972–4977.
[6] R. Halaš and M. Jukl, *On Beck's coloring of posets*, Discrete Math. **309** (2009), no. 13, 4584–4589.
[7] V. Joshi, *Zero divisor graph of a poset with respect to an ideal*, Order: DOI 10.1007/s11083-011-9216-2.
[8] J. R. Munkres, *Topology*, Prentice-Hall of Indian, New Delhi, 2005.
[9] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), no. 9, 4425–4443.
[10] P. V. Venkatanarasimhan, *Semi-ideals in posets*, Math. Ann. **185** (1970), 338–348.

Balasubramanian Elavarasan
Department of Mathematics
School of Science and Humanities
Karunya University
Coimbatore-641 114, Tamilnadu, India
E-mail address: belavarasan@gmail.com

Kasi Porselvi
Department of Mathematics
School of Science and Humanities
Karunya University
Coimbatore-641 114, Tamilnadu, India
E-mail address: porselvi94@yahoo.co.in