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SOME RESULTS ON THE LOCALLY EQUIVALENCE ON A NON-REGULAR SEMIGROUP

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Abstract. On any semigroup S, there is an equivalence relation ϕ^S, called the *locally equivalence relation*, given by $a\ \phi^S b \Leftrightarrow aSa = bSb$ for all $a, b \in S$. In Theorem 4 [4], Tiefenbach has shown that if ϕ^S is a band congruence, then $G_a := [a]_{\phi^S} \cap (aSa)$ is a group. We show in this study that $G_a := [a]_{\phi^S} \cap (aSa)$ is also a group whenever a is any idempotent element of S. Another main result of this study is to investigate the relationships between $[a]_{\phi^S}$ and aSa in terms of semigroup theory, where ϕ^S may not be a band congruence.

1. Introduction

On any semigroup S, there is an equivalence relation ϕ^S, called the *locally equivalence relation*, given by

$$a\phi^S b \Leftrightarrow aSa = bSb \text{ for all } a, b \in S.$$

If S is a band, then ϕ^S is a congruence, because ϕ^S separates idempotents of S by [3]. Also recall that ϕ^S is not a congruence in general. In Theorem 4 [4], Tiefenbach has shown that if ϕ^S is a band congruence, that is, $a\ \phi^S a^2$ for all $a \in S$, then $G_a := [a]_{\phi^S} \cap (aSa)$ is a group and G_a equals $a[a]_{\phi^S}a$, where $[a]_{\phi^S}$ is the ϕ^S-class determined by $a \in S$.

Firstly, we want to identify the relationships between $[a]_{\phi^S}$ and aSa, and then we show that $G_a := [a]_{\phi^S} \cap (aSa)$ is a group whenever a is any idempotent element of S.

To achieve this, let us consider S as the union of two disjoint subsets U_0^S and U_1^S, where

$$U_0^S := \{a \in S \mid [a]_{\phi^S} \setminus aSa = \emptyset\} \text{ and } U_1^S := \{a \in S \mid [a]_{\phi^S} \setminus aSa \neq \emptyset\}.$$

It is well known fact that $S = U_0^S$ when S is a regular semigroup and moreover, F. Pastijn has declared in [2, p. 161], that the two-sided implication "$a\phi^S b \Leftrightarrow a\mathcal{H}b$" is true, where $\mathcal{H}$ is one of the Green's relation on S. Thus by Corollary 2.2.6 [1], G_e is a group where e is an idempotent element of S. But

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ϕ^S doesn't have to be a band congruence. For example, let $S = \{a, b, e, f, 0\}$ be a regular semigroup whose multiplication table is given by:

.	a	b	e	f	0
a	0	e	0	a	0
b	f	0	b	0	0
e	a	0	e	0	0
f	0	b	0	f	0
0	0	0	0	0	0

It is easy to check that $[a]_{\phi^S} = \{a\}$, but $[a^2]_{\phi^S} = [0]_{\phi^S} \cap [a]_{\phi^S} = \emptyset$ and therefore ϕ^S is not a band congruence.

In this study, we also try to identify the possible relationships between a non-regular semigroup S and U_i^S $(i = 0, 1)$ in terms of the semigroup theory.

For undefined terms in semigroup theory, see [1].

2. Basic results

In this section, we establish some basic properties concerning a semigroup S and the set U_i^S $(i = 0, 1)$, and also we give some concrete examples related with obtained results for a non-regular semigroup.

Let us start by stating the following lemmas.

Lemma 2.1. *Let S be a semigroup and $a \in S$. If $a \in U_i^S$, then $[a]_{\phi^S} \subseteq U_i^S$.*

The proof of this lemma is straightforward.

Lemma 2.2. *If S is a semigroup, then either $\langle a\rangle = U_0^{\langle a\rangle}$ or $\langle a\rangle = U_1^{\langle a\rangle}$ for all $a \in S$.*

Proof. Let $a \in S$. According the Theorem 1.2.2 [1], either $\langle a\rangle \cong (N, +)$ or there exist positive integers m (the index of a) and r (the period of a).

If $\langle a\rangle \cong (N, +)$, then $\langle a\rangle = U_1^{\langle a\rangle}$, since $a^s \notin a^s\langle a\rangle a^s$ for all $s \in N$.

If m is the index and r is the period of a, then by Theorem 1.2.2 [1], $\langle a\rangle = \{a, a^2, \ldots, a^{m+r-1}\}$. If $m = 1$, then unambiguously $\langle a\rangle$ is a group and it follows that $\langle a\rangle = U_0^{\langle a\rangle}$. If $m \geq 2$, then by the definition of m $a^s \notin a^s\langle a\rangle a^s$ for all $1 \leq s < m$, whence $a^s \in U_1^{\langle a\rangle}$. The definition of r implies that $a^{m+2r} = a^m$. Then since

$$a^{m-1}a^t a^{m-1} = a^m a^{t+m-2} = a^{m+2r}a^{t+m-2} = a^m a^{2r-2+t}a^m$$

for all $1 \leq t < m + r - 1$ we obtain $a^{m-1}\langle a\rangle a^{m-1} \subseteq a^m\langle a\rangle a^m$. Also, since $a^m\langle a\rangle a^m \subseteq a^{m-1}\langle a\rangle a^{m-1}$, we have $a^m \in [a^{m-1}]_{\phi^{\langle a\rangle}}$. Since $a^{m-1} \in U_1^{\langle a\rangle}$, we have $a^m \in U_1^{\langle a\rangle}$ by Lemma 2.1.

Finally to complete the proof of this lemma, we have to show that $a^{m+k} \in U_1^{\langle a\rangle}$ for all $1 \leq k < r - 1$ and $r > 1$. Let $1 \leq k < r - 1$ and $t \in N$. Then, $a^{m+k} \in [a^m]_{\phi^{\langle a\rangle}}$ for all $1 \leq k < r - 1$, since

$$a^{m+k}a^t a^{m+k} = a^{m+r+k}a^t a^{m+r+k} = a^{m+k+1}a^{2r-2+t}a^{m+k+1}$$

and

$$a^{m+k+1}\langle a\rangle a^{m+k+1} \subseteq a^{m+k}\langle a\rangle a^{m+k}.$$

Hence, since $a^m \in U_1^{\langle a\rangle}$, $a^{m+k} \in U_1^{\langle a\rangle}$ for all $1 \leq k < r-1$ by Lemma 2.1. Consequently, $\langle a\rangle = U_1^{\langle a\rangle}$. □

Let S be a semigroup and T be a proper subsemigroup of S. We try to get an answer of the question: "if $a \in U_i^S$, is $a \in U_i^T$ $(i = 0, 1)$?". The next lemma shows that the answer is yes in the case $i = 0$ and T is an ideal.

Lemma 2.3. *Let S be a semigroup, T be a proper ideal of S and $a \in U_0^S \cap T$. Then $[a]_{\phi^S} = [a]_{\phi^T}$ if and only if $a \in U_0^T$.*

Proof. Take $a \in U_0^S \cap T$. Since $a \in U_0^S$, we have $[a]_{\phi^S} \subseteq aSa$. Let $b \in [a]_{\phi^S}$. Then since $a, b \in aSa = bSb$, it follows that there exist $x, y \in S$ such that $a = bxb, b = aya$. Hence $a, b \in T$ and also $aTa = bTb$ since T is an ideal and it follows that $b \in [a]_{\phi^T}$, that is, $[a]_{\phi^S} \subseteq [a]_{\phi^T}$.

Now by the hypothesis, $[a]_{\phi^T} \subseteq aTa \subseteq aSa$. Let $b \in [a]_{\phi^T}$. Then there exist $z, t \in T$ such that $b = aza$, $a = btb$ and it follows that $aSa = bSb$ which implies that $b \in [a]_{\phi^S}$ as desired.

Conversely, let $b \in [a]_{\phi^T}$. By $[a]_{\phi^S} \subseteq aSa$ and the hypothesis, one can get the set inclusion $[a]_{\phi^T} \subseteq aSa$. Then there exist $p, q \in S$ such that $b = bpb$, $a = aqa$. Therefore, since qa, pb are idempotent elements and T is ideal, $a, b \in aTa$ as desired.

The next lemma shows that the equality $U_i^{S\times T} = U_i^S \times U_i^T$ $(i = 0, 1)$ holds for $i = 0$ when S and T are semigroups. □

Lemma 2.4. *Let S and T be semigroups. Then $U_0^{S\times T} = U_0^S \times U_0^T$ and $U_1^{S\times T} = (U_0^S \times U_1^T) \cup (U_1^S \times U_1^T) \cup (U_1^S \times U_0^T)$.*

Proof. Let $(a, b), (c, d) \in S \times T$. It is easily seen that

$$(a, b)\phi^{S\times T}(c, d) \text{ if and only if } a\phi^S c \text{ and } b\phi^T d$$

by the definition of multiplication on $S \times T$. This implies that $[a]_{\phi^S} \times [b]_{\phi^T} = [(a, b)]_{\phi^{S\times T}}$.

Firstly, if (a, b) is an element of $U_0^{S\times T}$, then

$$[a]_{\phi^S} \times [b]_{\phi^T} = [(a, b)]_{\phi^{S\times T}} \subseteq (aSa) \times (bTb),$$

and so that $U_0^{S\times T} \subseteq U_0^S \times U_0^T$. Similarly, one can see that $U_0^S \times U_0^T \subseteq U_0^{S\times T}$, as desired.

Secondly, if (a, b) is an element of $U_1^S \times U_1^T$, then there exist $c \in [a]_{\phi^S}$ and $d \in [b]_{\phi^T}$ such that $c \notin aSa$ and $d \notin bSb$, that is, $(c, d) \notin (a, b)(S \times T)(a, b)$. Also, since $[a]_{\phi^S} \times [b]_{\phi^T} = [(a, b)]_{\phi^{S\times T}}$, we get $(a, b) \in U_1^{S\times T}$. Similarly, $U_1^S \times U_0^T$ and $U_0^S \times U_1^T$ are subsets of $U_1^{S\times T}$. On the other hand if $(a, b) \in U_1^{S\times T}$, then there exists $(c, d) \in [(a, b)]_{\phi S\times T}$ such that either $c \notin aSa$ or $d \notin bSb$. Also, the equality $[a]_{\phi^S} \times [b]_{\phi^T} = [(a, b)]_{\phi^{S\times T}}$ holds. Thus, the proof is completed. □

In the following Lemma 2.5, we get a positive answer of the question "Is the structure of the set U_i^S $(i = 0, 1)$ preserved under isomorphism?".

Lemma 2.5. *Let S and T be semigroups and $\theta : S \to T$ be an isomorphism. Then the equality $U_i^S\theta = U_i^T$ $(i = 0, 1)$ holds.*

The proof of this lemma is obvious since $U_0^S \cap U_1^S = \emptyset$ and θ is an isomorphism.

3. The ϕ^S-class with an idempotent element

In this section, first we show that for any semigroup S, G_e is a group where e is an idempotent element of S. To proof of this, we need the following next lemma.

Lemma 3.1. *Let S be a semigroup and e be an idempotent element of S. Then $[e]_{\phi^S}$ has only one idempotent element and $[e]_{\phi^S}$ is a subsemigroup of S.*

Proof. Firstly, we show that $[e]_{\phi^S}$ is a semigroup. Let $a, b \in [e]_{\phi^S}$. By the definition of ϕ^S, $eSe = aSa = bSb$. Since e is idempotent, then e is the identity element of aSa. Thus $a^3e = a^3 = ea^3$. It follows that $a^6 \in a^3Sa^3 = a^6Sa^6 = a^{12}Sa^{12}$. Hence there exists $y \in S$ such that $a^{12}ya^6 (= a^6ya^{12})$ is the identity element of a^3Sa^3. Further, since $e \in aSa$, there exists $x \in S$ such that $e = axa$. It follows that $(ax)e = ax(axa) = (axa)xa = e(xa)$. Therefore, since $a^2Sa^2 \subseteq aSa$ and

$$\begin{aligned} aSa &= e(aSa)e \\ &= (axa)(aSa)(axa) \\ &= (ax)e(a^2Sa^2)e(xa) \\ &= e(xa)(a^2Sa^2)(ax)e \\ &= ex(a^3Sa^3)xe \\ &= ex(a^{12}ya^6)(a^3Sa^3)(a^{12}ya^6)xe \\ &= (e(xa))(a^{11}ya^6)(a^3Sa^3)(a^{12}ya^5)((ax)e) \\ &= (ax)e(a^{11}ya^6)(a^3Sa^3)(a^{12}ya^5)e(xa) \\ &= (axa)(a^{10}ya^6)(a^3Sa^3)(a^{12}ya^4)(axa) \\ &= e(a^{10}ya^6)(a^3Sa^3)(a^{12}ya^4)e \\ &= a^2(a^8ya^6)(a^3Sa^3)(a^{12}ya^2)a^2, \end{aligned}$$

we have $e \in aSa = a^2Sa^2$. Thus there exists $t \in S$ such that $e = a^2ta^2$. Hence we get

$$ae = a(a^2ta^2) = a^3ta^2 = (ea^3)(ta^2) = (ea)(a^2ta^2) = eae.$$

Similarly, one can show that $ea = eae$ which implies that $ae = ea$. Using the same arguments we have $eb = be$. Since $ea = ae$, $aSa = eSe = bSb$ and e is idempotent, we obtain

$$\begin{aligned} eSe &= e(eSe)e \\ &= e(aSa)e \\ &= (ea)S(ae) \\ &= (ae)S(ea) \\ &\subseteq a(eSe) \\ &\subseteq abSb. \end{aligned}$$

Also, since $eb = be$, then we get

$$\begin{aligned} eSe &= e(eSe)e \\ &\subseteq e(abSb)e \\ &= (ea)(bS)(be) \\ &= (ae)(bS)(eb) \\ &= a(be)Seb \\ &= ab(aS)ab \\ &\subseteq abSab. \end{aligned}$$

On the other hand since $aSa = eSe = bSb$, then we have

$$abSab \subseteq (eSe)b \subseteq (bSb)b \subseteq bSb \subseteq eSe$$

which implies that $ab \in [e]_{\phi^S}$, as desired.

The rest of the proof of this lemma follows from the fact that ϕ^S separates idempotents. □

Now we are ready to proof the following theorem.

Theorem 3.2. *Let S be a non-regular semigroup and e be an idempotent element of S. Then $G_e = [e]_{\phi^S} \cap eSe$ is a group.*

Proof. By Theorem 5.1.1 [1], one can show that any regular semigroup with only one idempotent element is a group. To prove this theorem it is enough to show that $e \in U_1^S$ and $G_e \neq \{e\}$.

For all $a \in G_e$, since $a \in eSe$ and e is an idempotent element, it is easy to see that $ea = ae = a$. Let $a, b \in G_e \setminus \{e\}$. Assume that there exists an element c of $[e]_{\phi^S} \setminus G_e$ such that $ab = c$, then $ec = e(ab) = (ea)b = ab = a(be) = (ab)e = ce$, which is a contradiction by $c \in [e]_{\phi^S} \setminus G_e$. Consequently, for all $a, b \in G_e \setminus \{e\}$ we have $ab \in G_e$. Since $[e]_{\phi^S}$ is a semigroup by Lemma 3.1, then G_e is a semigroup. On the other hand since G_e is a regular semigroup with only one idempotent element, then G_e is a group. This completes the proof of this theorem. □

Finally, for any semigroup S we try to identify the possible relationships between the index of $e \neq k \in [e]_{\phi^S}$ and $e \in U_i^S (i = 0, 1)$.

Notice that G_e in the statement of Theorem 3.2 is the $\mathcal{H}$-class containing e that is denoted by H_e.

Theorem 3.3. *Let S be a semigroup and e be an idempotent element of S. If $e \neq k \in [e]_{\phi^S}$ has finite order, then there exists a positive integer t such that $k^t = e$. In particular, the index of k should be greater than or equal 3.*

Proof. Let $e \neq k \in [e]_{\phi^S}$. Suppose that k has finite order. Then there exist positive integers m (the index of k) and r (the period of k) such that $k^{m+r} = k^m$ and the order of k is $m+r-1$. By the proof of Lemma 3.1, $kSk = k^2Sk^2$. Then we obtain $k^3 \in k^6Sk^6$. Thus there exists $x \in S$ such that $k^3 = k^6xk^6$. Simple calculations show that $k^6xk^3 = k^3xk^6$ is the identity element of $kSk = eSe$. Then, since e is the identity element of eSe, we get $e = k^6xk^3 = k^3xk^6$. Further, since $k^n \in kSk$ for every integer $n \geq 3$,

$$k^ne = k^n = ek^n. \tag{1}$$

We will complete the proof by investigating the following cases:

Case 1: Let $m = 1$ and assume that $r = 1$. Then, since $k = k^2$, we get $k = k^6 = k^3 = k^6xk^6 = k^6xk^3 = e$, which contradicts $k \neq e$. Hence $r \geq 2$.

In the case $r = 2$ we have $k = k^3$. Therefore, one can obtain that $k^2 = k^4 = k^3k = (k^6xk^6)k = (k^6xk^2)k = e$ from $k^2 = k^4 = k^6$.

In the case $r = 3$ we have $k = k^4$. Since $k^3 = k^6$, then $k^3 = e$.

In the case $r \geq 4$, we have $k^r = k^3k^{r-3} = (k^6xk^6)k^{r-3} = k^6xk^{r+1}k^2 = k^6xkk^2 = e$. Therefore $\langle k \rangle$ is a group by (1).

Case 2: Let $m = 2$. Simple calculations show that for $r = 1, 2$, $k^2 = e$ and for $r = 3$, $k^6 = e$. For $r \geq 4$, it follows that $k^r = k^3k^{r-3} = (k^6xk^6)k^{r-3} = k^6xk^{r+2}k = k^6xk^2k = e$.

Case 3: Let $m = 3$. For $r = 1, 2, 3$, simple calculations show us that $k^6 = e$. For $r \geq 4$, we obtain that $k^r = k^3k^{r-3} = (k^6xk^6)k^{r-3} = k^6xk^{r+3} = k^6xk^3 = e$.

Case 4: If $4 \leq m \leq 6$, then there exists $0 \leq t \leq 2$ integer such that $m+t = 6$. Hence $k^6 = k^{m+t} = k^{m+r+t}$. It follows that $k^3 = k^6xk^6 = k^6xk^{m+r+t} = (k^6xk^6)k^r = k^{r+3}$. This is a contradiction by the definition of m and r.

Case 5: If $m \geq 7$, then $k^{m-3} = k^{m-6}k^3 = k^{m-6}(k^6xk^6) = k^mxk^6 = k^{m+r}xk^6 = k^{m+r-6}(k^6xk^6) = k^{m+r-6}k^3 = k^{m+r-3}$, which is a contradiction by the definition of m and r. $\square$

We have the following corollary.

Corollary 3.4. *Let S be a semigroup and e be an idempotent element of S. If $e \neq k \in [e]_{\phi^S}$ has finite order and the index of k is 2 or 3, then $e \in U_1^S$.*

Proof. Take m and r as the index and period of k, respectively, we provide a proof for the case $m = 2$ only, by the proof of Theorem 3.3 and the hypothesis. Assume that $e \in U_0^S$. Then $k \in U_0^S$ by Lemma 2.1 and the hypothesis. Hence, if $k \in eSe$, then we have

$$k^ne = k^n = ek^n \tag{2}$$

for every integer $n \geq 1$. For $r = 1$, since $k^2 = k^3 = e$, by the proof of *Case* 2 in Theorem 3.3, then $k = ke = kk^2 = e$ by (2), which is a contradiction with $k \neq e$. For $r = 2$, since $k^2 = e$, by the proof of *Case* 2 in Theorem 3.3, then $k^3 = kk^2 = ke = k$ by (2), this is a contradiction by the definition of m and r. A similar argument gives a contradiction for the case $r \geq 3$ which completes the proof of this corollary. □

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