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A NOTE ON PRIMITIVE SUBGROUPS OF FINITE SOLVABLE GROUPS

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Abstract. In [5], Johnson introduced the primitivity of subgroups and proved that a finite group G is supersolvable if every primitive subgroup of G has a prime power index in G. In that paper, he also posed an interesting problem: what a group looks like if all of its primitive subgroups are maximal. In this note, we give the detail structure of such groups in solvable case. Finally, we use the primitivity of some subgroups to characterize T-group and the solvable PST_0-groups.

1. Introduction

All groups considered in this note are finite. We use the conventional notions and notation, as in [2]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ stands for all prime divisors of $|G|$. We use $M < \cdot\ G$ to denote M is maximal in G. We say a group G is elementary abelian if G is abelian of square-free exponent.

In [5], Johnson introduced the primitivity of subgroups. A subgroup H of a group G is called primitive if it is a proper subgroup of the intersection of all subgroups properly containing H. Johnson proved that a finite group G is supersolvable if every primitive subgroup of G has a prime power index in G. Clearly, any maximal subgroup M of a group G is primitive because G is the only subgroup which properly containing M. However, a primitive subgroup may be not a maximal subgroup. A_4, the alternating group of degree 4, is a counterexample. Let H be a subgroup of order 2. H is primitive in A_4 because the unique Sylow 2-subgroup B_4 and G are the subgroups which containing H. But H is not maximal in A_4. So primitivity can be regarded as a generalization of maximality. Therefore it is worthy of characterizing the groups all of whose primitive subgroups are maximal.

For convenience, we use $\mathcal{Z}$ to denote the class of finite groups with property that all primitive subgroups are maximal.

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A natural question is to determine the structure of group G in $\mathcal{Z}$ and it was posed as an open question in [5].

In this paper we try to answer the question and we also try to discuss the condition that the direct product in $\mathcal{Z}$ is still in $\mathcal{Z}$ and some other properties.

In [3], Guo gave the detailed structure of a group G satisfying that every primitive subgroups of G has a prime power index which is very helpful in our proofs. He proved:

Theorem 1.1 ([3]). *Let G be a finite group. Then the following statements are equivalent*:

(1) *Every primitive subgroup of G has a prime power index in G.*

(2) *$G = [D]M$ is a supersolvable group, where D and M are nilpotent Hall subgroups of G, D is the nilpotent residual of G and $G = DN_G(D \cap X)$ for every primitive subgroup X of G. In particular, every maximal subgroup of D is normal in G.*

In this paper, we try to characterize the groups with property that primitivity always means maximality. We get the the detailed structure in solvable case. The following theorem is obtained.

Theorem 1.2. *Let G be a finite group. Then the following statements are equivalent.*

(1) *G is solvable and $X <\cdot\, G$ for any primitive subgroup X of G.*

(2) *Every primitive subgroup of G has a prime index in G.*

(3) *$G = [D]M$ is supersolvable with $\Phi(G) = 1$. Where M is an elementary abelian Hall subgroup of G, D is an abelian Hall subgroup of G of odd order, D is a direct product of some elementary abelian Sylow subgroups, D is the nilpotent residual of G, every subgroup of D is normal in G and every element of G induces a power automorphism in D, and $D \cap X <\cdot\, D$ for every primitive subgroup X of G not containing D.*

(4) *$G = [D]M$ is supersolvable. Where D and M are nilpotent Hall subgroups of G, D is the nilpotent residual of G, D is a direct product of some elementary abelian Sylow subgroups. $D \cap X <\cdot\, D$ for every primitive subgroup X of G not containing D. Every maximal subgroup of D is normal in G.*

A group G is said to be a T-group (or PST-group) if normality (S-quasinormality, respectively) is a transitive relation in G, that is, for the groups $H \leq K \leq G$, H is normal (S-permutable, respectively) in K and K is normal (S-permutable, respectively) in G always imply that H is normal (S-permutable, respectively) in G.

A group G is called a T_0-group (or PST_0-group) if $G/\Phi(G)$ is a T-group (PST-group, respectively). For more information about the transitivity of subgroup, please see ([1], [6]). In this paper, we show that $G \in T_0$ if $G \in \mathcal{Z}$. We also obtain a theorem concerning the solvable PST_0-groups.

Theorem 1.3. *Let G be a finite group. Then the following statements are equivalent.*

(1) *every primitive subgroup of* G *containing* $\Phi(G)$ *has a prime power index in* G.

(2) $G/\Phi(G) = [D/\Phi(G)]M/\Phi(G)$, $D/\Phi(G)$ *and* $M/\Phi(G)$ *are Hall subgroups with co-prime orders,* $D/\Phi(G)$ *is abelian and is the nilpotent residual of* $G/\Phi(G)$*, the elements of* $G/\Phi(G)$ *induces a power automorphism in* $D/\Phi(G)$.

(3) G *is a solvable* PST_0*-group.*

2. Preliminaries

Lemma 2.1 ([3])**.** *Let* G *be a non-identity group. Then the following statements hold*:

(1) *For every proper subgroup* H *of* G*, there is a set of primitive subgroups* $\{X_i|\ i \in I\}$ *in* G *such that* $H = \bigcap_{i\in I} X_i$;

(2) *If* $H \leq G$ *and* T *is a primitive subgroup of* H*, then* $T = H \cap X$ *for some primitive subgroup of* G;

(3) *If* $K \trianglelefteq G$ *and* $K \leq H \leq G$*, then* H *is a primitive subgroup of* G *if and only if* H/K *is a primitive subgroup of* G/K;

(4) *Let* P *and* Q *be subgroups of* G *with* $(|P|, |Q|) = 1$*. Suppose that* H *is a subgroup of* G *such that* $HP \leq G$ *and* $HQ \leq G$*, then* $HP \cap HQ = H$*. In addition, if we suppose that* H *is a primitive subgroup of* G*, then* $P \leq H$ *or* $Q \leq H$.

Lemma 2.2. *Suppose that* G *is nilpotent. Then the following statements are equivalent*:

(1) *Every primitive subgroup of* G *is maximal in* G.

(2) G *is elementary abelian, that is,* $\Phi(G) = 1$.

Proof. To prove (1) $\Rightarrow$ (2), we just need to prove $\Phi(G) = 1$. Suppose that $\Phi(G) \neq 1$, then we take a minimal subgroup L of $\Phi(G)$. By Lemma 2.1, there is a primitive subgroup X of G such that $1 = X \cap L$, since the identity subgroup is a primitive subgroup of L. By the hypothesis of (1), X is maximal in G, it follows that $L \leq X$, a contradiction. Thus $\Phi(G) = 1$.

(2) $\Rightarrow$ (1). For any primitive subgroup X of G, by Lemma 2.1(4), $|G : X|$ is a power of a prime, say p^n, p is a prime and n is a positive integer. So we may suppose G is a p-group. Then G is an elementary abelian p-group. It follows that G/X is an elementary abelian p-group, $\Phi(G/X) = 1$. Consequently

$$X = \bigcap_{X \leq K \text{ and } K <\cdot\, G} K.$$

Since X is a primitive subgroup of G, $X = K$ for some $K <\cdot\, G$. Thus X is maximal in G. This completes the proof. □

Lemma 2.3. *Let* G *be an abelian* p*-group,* p *is a prime. Suppose that the action of a group* H *on* G *is co-prime and that* H *acts trivially on* $\Omega_1(G) = \langle x \in G \mid x^p = 1\rangle$*. Then* H *acts trivially on* G.

3. Main results

Theorem 3.1. *Let G be a finite group. Then the following statements are equivalent.*

(1) *G is solvable and every primitive subgroup X of G is a maximal subgroup of G.*

(2) *Every primitive subgroup of G has a prime index in G.*

(3) *$G = [D]M$ is supersolvable with $\Phi(G) = 1$. Where M is an elementary abelian Hall subgroup of G, D is an abelian Hall subgroup of G of odd order, D is a direct product of some elementary abelian Sylow subgroups, D is the nilpotent residual of G, every subgroup of D is normal in G and every element of G induces a power automorphism in D, and $D \cap X < \cdot\, D$ for every primitive subgroup X of G not containing D.*

(4) *$G = [D]M$ is supersolvable. Where D and M are nilpotent Hall subgroups of G, D is the nilpotent residual of G, D is a direct product of some elementary abelian Sylow subgroups. $D \cap X < \cdot\, D$ for every primitive subgroup X of G not containing D. Every maximal subgroup of D is normal in G.*

Proof. (1) $\Rightarrow$ (2). Suppose (1) is true. Since G is solvable, and so every maximal subgroup of G has a prime power index in G. By (1) every primitive subgroup X of G is a maximal subgroup of G, so X is of prime power index in G. By Theorem 1.1, G is supersolvable. Consequently every primitive subgroup of G has a prime index in G.

(2) $\Rightarrow$ (1) is obvious. This completes (1) $\Leftrightarrow$ (2).

Next, we prove (2) $\Rightarrow$ (3). By Theorem 1.1, if (2) is true, then $G = [D]M$ is a supersolvable group, where D and M are nilpotent Hall subgroups of G, D is the nilpotent residual of G.

We will try to prove (2) $\Rightarrow$ (3) through the following 5 steps and give a more detailed structure of G.

Step 1. M is elementary abelian.

Note that G/D is nilpotent and G/D satisfies the hypothesis of (2) by Lemma 2.1(3). By Lemma 2.2(2), $M \cong G/D$ is an elementary abelian group.

Step 2. $\Phi(G) = 1$.

Suppose that $\Phi(G) \neq 1$, then can choose a minimal subgroup L of prime order in $\Phi(G)$. Since the identity subgroup is a primitive of L, by Lemma 2.1 (2), there is a primitive subgroup X of G such that $1 = X \cap L$. Since (2) implies (1), X is maximal in G, it follows that $L \leq X$ and so $1 = X \cap L = L$, a contradiction. Thus $\Phi(G) = 1$.

Step 3. D is abelian. Every subgroup of D is normal in G and every element of G induces a power automorphism in D.

By Theorem 1.1, every maximal subgroup of D is normal in G. By Step 2, $\Phi(D) \leq \Phi(G) = 1$ and so D is a direct product of some elementary abelian Sylow subgroups. So every subgroup H of D is the intersection of the maximal subgroup of D which containing H. Thus every subgroup of D is normal in G.

In particular, every element of G induces a power automorphism in D. This proves Step 3.

Step 4. D is of odd order.

Let $D^2 = \langle d^2 \mid d \in D\rangle$. D^2 is characteristic in D and so $D \trianglelefteq G$. Since every element of it is of order no more than 2, D/D^2 is an elementary abelian 2-group (it maybe the identity group). By Step 3, every subgroup of D/D^2 is normal in G/D^2. Since every normal subgroup of order 2 is contained in the center. Thus $D/D^2 \le Z(G/D^2)$. Therefore the nilpotent residual D has the property $D \le [D, G] \le D^2$. Hence $D = D^2$. By (3), $D = D_2 \times D_{p_1} \times \cdots \times D_{p_k}$ is a direct product of the elementary abelian Sylow subgroups of D, where p_i are odd primes. $D^2 = D$ implies that $D_2 = 1$ and so D is of odd order. Step 4 holds.

Step 5. $D \cap X < \cdot\, D$ for every primitive subgroup X of G not containing D.

For any primitive subgroup X of G, by the hypothesis, $|G : X|$ is a prime. If D is not contained in X, then $G = DX$ since D is a Hall subgroup. This yields that $|D : D \cap X| = |DX : X|$ is a prime. This implies that $D \cap X < \cdot D$ for every primitive subgroup X of G not containing D. Step 5 holds.

(3) $\Rightarrow$ (4) is trivial.

Finally we prove (4) $\Rightarrow$ (2). Suppose that (4) is true. Firstly, we will show the hypotheses are inherited by quotient groups of G.

For any normal subgroup N of G, $G/N = [DN/N]MN/N$ is supersolvable. DN/N is an abelian group of odd order and is the nilpotent residual of G/N. DN/N and MN/N are Hall subgroups. DN/N is a direct product of some elementary abelian Sylow subgroups. And it is easy to see that every maximal subgroup of DN/N is normal in G/N. MN/N is an elementary abelian Hall subgroup of G/N. For any primitive subgroup X/N of G/N not containing DN/N, then X is a primitive subgroup of G and D is not contained in X. By the hypotheses, $|D : D \cap X|$ is a prime. Then

$$|DN/N : DN/N \cap X/N| = \frac{|D|}{|D \cap N|} : \frac{|X \cap D|}{|X \cap D \cap N|} = \frac{|D|}{|X \cap D|} = |D : X \cap D|,$$

and so $|DN/N : DN/N \cap X/N|$ is a prime, that is, $DN/N \cap X/N < \cdot\, DN/N$. Thus the hypotheses are inherited by quotient groups of G.

Secondly, if G satisfies (4), then the hypotheses of Theorem 1.1(2) satisfied. For any primitive subgroup X of G, $|G : X| = p^n$ with p a prime, we will prove that $n = 1$. Suppose it is false and let G be a counterexample with minimal order. Then $|G : X| = p^n$ and $n > 1$ is an integer. Suppose that $X_G \neq 1$. Since G/X_G satisfies the hypotheses of (3) and X/X_G is a primitive subgroup of G/X_G, by the minimal choice of G, $|G : X| = |G/X_G : X/X_G|$ is a prime, a contradiction. So we may suppose $X_G = 1$. By Lemma 2.1(4), $F(G) = O_q(G)$, where q is the maximal prime divisor of $|G|$. If $D = 1$, then $G = M$ is an elementary abelian group. According to Lemma 2.2, X has a prime index in G, a contradiction. So D is a non-identity Hall subgroup of G.

Since $X_G = 1$, D is not contained in X. Consequently, $|G : X| = p^n$ implies that $G = DX$, and thus $|G : X| = |DX : X| = |D : D \cap X|$. Since $D \cap X < \cdot\, D$, $|G : X| = |D : D \cap X|$ is a prime, a final contradiction. This completes the proof of the theorem. □

It is also interesting to determine the structure of a group G that every proper subgroup of G is a primitive subgroup. The following observation is that $\Phi(G)$ plays the essential role.

Theorem 3.2. *Let G be a finite group. Then the following statements are equivalent.*

(1) *Every proper subgroup X of G is a primitive subgroup.*
(2) *$\Phi(G)$ is a primitive subgroup of G.*
(3) *G is a cyclic p-group for some prime p.*

Proof. (1) ⇒ (2) and (3) ⇒ (1) is trivial. We only need to prove (2) ⇒ (3). Suppose that $\Phi(G)$ is a primitive subgroup of G. Since $\Phi(G)$ is the intersection of the maximal subgroups of G, $\Phi(G)$ is the only maximal subgroup of G. Thus G is a cyclic p-group for some prime p, and (2) ⇒ (3) is proved. □

For many class of groups $\mathcal{X}$, such as abelian groups, nilpotent groups, solvable groups, the direct product of the groups in $\mathcal{X}$ is in $\mathcal{X}$. We also have interest to know if the groups with that every primitive subgroup is a maximal subgroup have such a property or not.

The following example show that $\mathcal{Z}$ is not closed under direct products.

Example 3.3. Clearly, the symmetric group of degree 3, $S_3 \in \mathcal{Z}$. Put $G = S_3 \times S_3 = ([\langle a_1\rangle]\langle b_1\rangle) \times ([\langle a_2\rangle]\langle b_2\rangle)$. $D = \langle a_1\rangle \times \langle a_2\rangle$ is the nilpotent residual of G. If $G \in \mathcal{Z}$, then every subgroup of D is normal in G by Theorem 3.1. Consequently, $\langle (a_1,\ a_2)\rangle \trianglelefteq G$. By the defining relation of S_3, ${a_1}^{b_1} = {a_1}^2$. This implies that $(a_1,\ a_2)^{(b_1,\ 1)} = (a_1,\ a_2)^i$, where i is an integer with $1 \le i \le 3$. Consequently, $({a_1}^2,\ a_2) = ({a_1}^i,\ {a_1}^i)$, and thus $2 = i = 1$, a contradiction. So $\mathcal{Z}$ is not closed under direct products.

Inspired by the above example, we see that direct product of the groups in $\mathcal{Z}$ may be not in $\mathcal{Z}$. We try to find the condition such that the property is true for the direct product. we have the following theorem.

Theorem 3.4. *Suppose that the groups A and B are in $\mathcal{Z}$. Then $G := A \times B \in \mathcal{Z}$ if and only if $(|D(A)|, |D(B)|) = 1$ and $X \cap D < \cdot\, D$ for any primitive subgroup X of G not containing D. Here, $D(A)$ and $D(B)$ denote the nilpotent residuals of A and B, respectively; D denotes the direct product of $D(A)$ and $D(B)$.*

Proof. We first prove the "if" part. Assume that $G = A \times B$ with the properties that $(|D(A)|, |D(B)|) = 1$ and $X \cap D < \cdot\, D$ for any primitive subgroup X of G not containing D. Since A and B lies in $\mathcal{Z}$, we have that $A = [D(A)]M(A)$ and $B = [D(B)]M(B)$ have the properties in Theorem 3.1(4). We have that

$G = [D]M$ where $D = D(A) \times D(B)$ and $M = M(A) \times M(B)$. We can directly check that G satisfies the properties in Theorem 3.1(4). Therefore $G = A \times B \in \mathcal{Z}$.

Now we prove the "only if " part. Since the nilpotent residual of $A \times B$ is $D(A) \times D(B)$, if $G := A \times B \in \mathcal{Z}$, then $X \cap D < \cdot\, D$ for any primitive subgroup X of G not containing D.

Now we only need to prove that $(|D(A)|,\ |D(B)|) = 1$. Suppose that $(|D(A)|, |D(B)|) \neq 1$ and pick a prime p dividing $(|D(A)|, |D(B)|)$. Put $P = P_1 \times P_2$, P_1 and P_2 are Sylow p-subgroups of $D(A)$ and $D(B)$, respectively. Then P is an abelian Sylow p-subgroup of G. Since $G \in \mathcal{Z}$, by Theorem 3.1 (3), every subgroup of P is normal in G. Take any two elements $a \in P_1$ and $b \in P_2$ such that $|a| = |b| = p$. By the hypothesis, $\langle (a,\ b) \rangle \trianglelefteq G$. Take any p'-element $t \in A$, then

$$(a,\ b)^{(t,\ 1)} = (a^t,\ b) = (a,\ b)^i,\ 1 \leq i < p.$$

Since $a \in D(A)$ and the elements of G induces a power automorphism in $D(A)$, then $a^t = a^l$, $1 \leq l < p$. Consequently, we have $(a^l,\ b) = (a^i, b^i)$, and so $l = i = 1$. Then $a^t = a$, this implies that every p'-element of A centralizes every minimal subgroup of P_1. Since P_1 is an abelian p-subgroup, by Lemma 2.3, $O^p(A) \leq C_A(P_1)$. From the fact that P_1 is an abelian Sylow p-subgroup of A, $A = P_1 O^p(A) \leq C_A(P_1)$, and so $P_1 \leq Z(A)$. This is contrary to the fact that $D(A)$ is the nilpotent residual of A. Thus $(|D(A)|, |D(B)|) = 1$.

This proves the theorem. □

Immediately, we have the following corollary.

Corollary 3.5. *A group $A \in \mathcal{Z}$, then $A \times A \in \mathcal{Z}$ if and only if $D(A) = 1$, that is, A is elementary abelian.*

Proof. Suppose that $A \times A \in \mathcal{Z}$. By Theorem 3.4, $(|D(A)|, |D(A)|) = 1$. This implies that $D(A) = 1$. By Theorem 3.1, A is elementary abelian. Conversely, if A is elementary abelian, then so is G. By Lemma 2.2, $G \in \mathcal{Z}$. This completes the proof of the corollary. □

As applications of Theorem 3.1, we give some interesting characterizations of T-group PST_0-group by using the primitivity of subgroups.

According to Statement (3) in Theorem 3.1, the following is obvious.

Corollary 3.6. *Let G be a finite group. Then G is a T-group(or T_0-group) if every primitive subgroup of G has a prime index in G.*

Proof. See [7, Theorem 13.4.4]. □

Theorem 3.7. *Let G be a finite group. Then the following statements are equivalent.*

(1) *Every primitive subgroup of G containing $\Phi(G)$ has a prime power index in G.*

(2) *$G/\Phi(G) = [D/\Phi(G)]M/\Phi(G)$, $D/\Phi(G)$ and $M/\Phi(G)$ are Hall subgroups with co-prime orders, $D/\Phi(G)$ is abelian and is the nilpotent residual of $G/\Phi(G)$, the elements of $G/\Phi(G)$ induce the power automorphisms in $D/\Phi(G)$.*

(3) *G is a solvable PST_0-group.*

Proof. Without loss of generality we may suppose $\Phi(G) = 1$.

(1) $\Rightarrow$ (2). By Theorem 1.1, it suffices to prove D is abelian and every subgroup of D is normal in G .

By Theorem 1.1, every maximal subgroup of D is normal in G. Since $\Phi(G) = 1$, D is elementary abelian. Then every subgroup of D is the intersection of some maximal subgroups of D. Thus every subgroup of D is normal in G.

Put $D^2 = \langle d^2 \mid d \in D\rangle$, D^2 is characteristic in D. Then D/D^2 is an elementary abelian 2-group. By the previous paragraph, every subgroup of D/D^2 is normal in G/D^2. Since every normal subgroup of order 2 is contained in the center. Thus $D/D^2 \leq Z(G/D^2)$, $D = [D, G] \leq D^2$. It follows that $D = D^2$, and so D is of odd order. By Step 3, G is a Dedekind group of odd order, thus D is abelian. This proves (1) $\Rightarrow$ (2).

By Theorem 1.1, (2) $\Rightarrow$ (1) is obvious.

The equivalence between (2) and (3) is the Main Theorem in [1]. □

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