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INTERSECTIONAL SOFT SETS AND APPLICATIONS TO BCK/BCI-ALGEBRAS

Young Bae Jun, Min Su Kang, and Kyoung Ja Lee

Abstract. The notion of intersectional soft sets is introduced, and several examples are given. The application of intersectional soft sets to BCK/BCI-algebras is discussed.

1. Introduction

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [20]. In response to this situation Zadeh [21] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [22]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [18]. Maji et al. [16] and Molodtsov [18] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [18] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [16] described the application of soft set theory

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to a decision making problem. Maji et al. [15] also studied several operations on the theory of soft sets. Chen et al. [6] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. After than, many algebraic properties of soft sets are studied (see [1, 3, 7, 9, 10, 11, 12, 13, 14, 19, 23]).

In this paper, we introduce the notion of intersectional soft sets and provide several examples. We apply this notion to BCK/BCI-algebras, and obtain many useful results.

2. Basic results on BCK/BCI-algebras

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,

where $x \leq y$ if and only if $x*y = 0$. A BCK-algebra X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$. A commutative BCK-algebra will be written by cBCK-algebra for short. A BCI-algebra X is said to be p-semisimple if $0 * (0 * x) = x$ for all $x \in X$. A BCI-algebra X is said to be associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$. A nonempty subset S of a BCK/BCI-algebra X is called a *BCK/BCI-subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. We refer the reader to the books [8, 17] for further information regarding BCK/BCI-algebras.

3. Basic results on soft sets

Molodtsov [18] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 3.1 ([18])**.** A pair $(\mathscr{F}, A)$ is called a *soft set* over U, where F is a mapping given by

$$\mathscr{F} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\mathscr{F}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $(\mathscr{F}, A)$. A soft set over U can be represented by the set of ordered pairs:

$$(\mathscr{F}, A) = \{(x, \mathscr{F}(x)) \mid x \in A,\ \mathscr{F}(x) \in \mathscr{P}(U)\}.$$

Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [18].

Definition 3.2 ([15])**.** For two soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over U, we say that $(\mathscr{F}, A)$ is a *soft subset* of $(\mathscr{G}, B)$, denoted by $(\mathscr{F}, A) \widetilde{\subseteq} (\mathscr{G}, B)$, if

(i) $A \subseteq B$,
(ii) $(\forall e \in A)\,(\mathscr{F}(e)$ and $\mathscr{G}(e)$ are identical approximations$)$.

Definition 3.3 ([15])**.** Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cap B$,
(ii) $(\forall e \in C)$ $(\mathscr{H}(e) = \mathscr{F}(e)$ or $\mathscr{G}(e)$, (as both are same set)).

In this case, we write $(\mathscr{F}, A)\widetilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 3.4 ([15])**.** Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cup B$,
(ii) for all $e \in C$,

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cup \mathscr{G}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\mathscr{F}, A)\widetilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$.

4. Intersectional soft sets

In this section, let U denote an initial universe set and assume that E, a set of parameters, has a binary operation $\hookrightarrow$.

Definition 4.1. For any non-empty subset A of E, a soft set $(\mathscr{F}, A)$ over U is said to be *intersectional* over U if it satisfies:

$$(\forall x, y \in A)\,(x \hookrightarrow y \in A \;\Rightarrow\; \mathscr{F}(x) \cap \mathscr{F}(y) \subseteq \mathscr{F}(x \hookrightarrow y))\,. \tag{4.1}$$

Example 4.2. Suppose that there are five houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5\}\,.$$

Let a set of parameters $E = \{e_1, e_2, e_3, e_4\}$ be a set of status of houses which stand for the parameters "beautiful", "cheap", "in good location" and "in green surroundings, respectively, with the following binary operation:

$\hookrightarrow$	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_1	e_4	e_3
e_3	e_3	e_4	e_3	e_4
e_4	e_4	e_3	e_4	e_3

(1) For a subset $A = \{e_1, e_3, e_4\}$ of E, consider a soft set $(\mathscr{F}, A)$ over U as follows:

$$(\mathscr{F}, A) = \{(e_1, \{h_1, h_2, h_3, h_4, h_5\}), (e_3, \{h_2, h_3, h_5\}), (e_4, \{h_1, h_2, h_3, h_4\})\}\,.$$

Then $(\mathscr{F}, A)$ is an intersectional soft set over U.

(2) Let $B = \{e_1, e_2, e_3\}$. Then the soft set $(\mathscr{G}, B)$ over U which is given by

$$(\mathscr{G}, B) = \{(e_1, \{h_1, h_2, h_3, h_4, h_5\}), (e_2, \{h_2, h_3, h_5\}), (e_3, \{h_1, h_3, h_4\})\}$$

is an intersectional soft set over U.

(3) The soft set $(\mathscr{H}, E)$ over U given by

$$\begin{aligned}(\mathscr{H}, E) = \big\{&(e_1, \{h_1, h_2, h_3, h_4, h_5\}), (e_2, \{h_2, h_3, h_4, h_5\}),\\ &(e_3, \{h_1, h_2, h_3, h_4\}), (e_4, \{h_1, h_2, h_4\})\big\}\end{aligned}$$

is not an intersectional soft set over U since

$$\mathscr{H}(e_2) \cap \mathscr{H}(e_3) = \{h_2, h_3, h_4\} \nsubseteq \{h_1, h_2, h_4\} = \mathscr{H}(e_2 \hookrightarrow e_3).$$

Theorem 4.3. *Let $(\mathscr{F}, A)$ be a soft subset of $(\mathscr{G}, B)$. If $(\mathscr{G}, B)$ is intersectional, then so is $(\mathscr{F}, A)$.*

Proof. Let $x, y \in A$ with $x \hookrightarrow y \in A$. Then $x \hookrightarrow y \in B$ since $A \subseteq B$. Hence

$$\mathscr{F}(x) \cap \mathscr{F}(y) = \mathscr{G}(x) \cap \mathscr{G}(y) \subseteq \mathscr{G}(x \hookrightarrow y) = \mathscr{F}(x \hookrightarrow y).$$

Therefore $(\mathscr{F}, A)$ is an intersectional soft set over U. □

The converse of Theorem 4.3 may not be true as seen in the following example.

Example 4.4. Let U and E be the initial universe set and a set of parameters, respectively, which are provided in Example 4.2. Consider the intersectional soft set $(\mathscr{G}, B)$ over U which is described in Example 4.2(2). Let $(\mathscr{H}, E)$ be a soft set over U given by

$$(\mathscr{H}, E) = \big\{(e_1, \{h_1, h_2, h_3, h_4, h_5\}), (e_2, \{h_2, h_3, h_5\}), (e_3, \{h_1, h_3, h_4\}), (e_4, \{h_1, h_2, h_4\})\big\}.$$

Then $(\mathscr{G}, B)$ is a soft subset of $(\mathscr{H}, E)$. Since

$$\mathscr{H}(e_2) \cap \mathscr{H}(e_3) = \{h_3\} \nsubseteq \{h_1, h_2, h_4\} = \mathscr{H}(e_4) = \mathscr{H}(e_2 \hookrightarrow e_3),$$

$(\mathscr{H}, E)$ is not intersectional.

5. Applications to BCK/BCI-algebras

In what follows let X and A be a BCK/BCI-algebra and a nonempty subset of X, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ unless otherwise specified. A set-valued function $\mathscr{F} : A \to \mathscr{P}(X)$ can be defined as

$$\mathscr{F}(x) = \{y \in X \mid x\, R\, y\}$$

for all $x \in A$. The pair $(\mathscr{F}, A)$ is then a soft set over X. For any element x of a BCI-algebra X, we define the order of x, denoted by $o(x)$, as

$$o(x) = \min\{n \in \mathbb{N} \mid 0 * x^n = 0\}.$$

Definition 5.1 ([9])**.** Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called a *soft BCK/BCI-algebra* over X if $\mathscr{F}(x)$ is a BCK/BCI-subalgebra of X for all $x \in A$.

Definition 5.2. A soft set $(\mathscr{F}, A)$ over X is called an *intersectional soft BCK/BCI-algebra* over X if it satisfies:

$$(\forall x, y \in A)\,(x * y \in A \;\Rightarrow\; \mathscr{F}(x) \cap \mathscr{F}(y) \subseteq \mathscr{F}(x * y))\,. \tag{5.1}$$

Let us illustrate this definition using the following examples.

Example 5.3. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let $(\mathscr{F}, A)$ be a soft set over X, where $A = X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{F}(x) = \{y \in X \mid x\, R\, y \;\Leftrightarrow\; y \in x^{-1} I\}$$

for all $x \in A$ where $I = \{0, a\}$ and $x^{-1}I = \{y \in X \mid x \wedge y \in I\}$. Then

$$(\mathscr{F}, A) = \big\{(0, \{0, a, b, c, d\}), (a, \{0, a, b, c, d\}), (b, \{0, a, c, d\}), (c, \{0, a, b, d\}), (d, \{0, a, b, c\})\big\}$$

which is an intersectional soft BCK-algebra over X.

Example 5.4. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = X$ and let $\mathscr{F} : A \to \mathscr{P}(X)$ be a set-valued function defined as follows:

$$\mathscr{F}(x) = \{y \in X \mid x\, R\, y \iff y = x^n,\ n \in \mathbb{N}\}$$

for all $x \in A$ where $x^n = x * x * \cdots * x$ in which x appears n-times. Then

$$(\mathscr{F}, A) = \big\{(0, \{0\}), (a, \{0, a\}), (b, \{0, b\}), (c, \{0, c\})\big\}$$

which is not an intersectional soft BCI-algebra over X since

$$\mathscr{F}(b) \cap \mathscr{F}(b) = \{0, b\} \nsubseteq \{0\} = \mathscr{F}(0) = \mathscr{F}(b * b).$$

Example 5.5. Let $X = \{0, a, b, c, d, e, f, g\}$ and consider the following Cayley table:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then $(X; *, 0)$ is a BCI-algebra (see [5]).

(1) Let $(\mathscr{F}, A)$ be a soft set over X, where $A = X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined as follows:

$$\mathscr{F}(x) = \{0\} \cup \{y \in X \mid x\, R\, y \iff o(x) = o(y)\}$$

for all $x \in A$. Then

$$(\mathscr{F}, A) = \Big\{(0, \{0, a, b, c\}), (a, \{0, a, b, c\}), (b, \{0, a, b, c\}), (c, \{0, a, b, c\}), (d, \{0, d, e, f, g\}), (e, \{0, d, e, f, g\}), (f, \{0, d, e, f, g\}), (g, \{0, d, e, f, g\})\Big\}$$

which is not an intersectional soft BCI-algebra over X since

$$\mathscr{F}(d) \cap \mathscr{F}(e) = \{0, d, e, f, g\} \nsubseteq \{0, a, b, c\} = \mathscr{F}(0) = \mathscr{F}(d * e).$$

(2) If we take $B = \{0, a, b, c\} \subset X$ and define a set-valued function $\mathscr{G} : B \to \mathscr{P}(X)$ by

$$\mathscr{G}(x) = \{y \in X \mid x\,R\,y \Leftrightarrow o(x) = o(y)\}$$

for all $x \in B$, then $(\mathscr{G}, B)$ is an intersectional soft BCI-algebra over X.

(3) If we take $C = \{0, d, e, f, g\} \subset X$ and

$$\mathscr{H} : C \to \mathscr{P}(X),\ x \mapsto \{y \in X \mid o(x) = o(y)\},$$

then

$$\begin{aligned}(\mathscr{H}, C) = \Big\{&(0, \{0, a, b, c\}), (d, \{d, e, f, g\}), (e, \{d, e, f, g\}),\\ &(f, \{d, e, f, g\}), (g, \{d, e, f, g\})\Big\}\end{aligned}$$

which is not an intersectional soft BCI-algebra over X since

$$\mathscr{H}(d) \cap \mathscr{H}(f) = \{d, e, f, g\} \nsubseteq \{0, a, b, c\} = \mathscr{H}(0) = \mathscr{H}(d * f).$$

Note that the soft set $(\mathscr{F}, A)$ in Example 5.4 is a soft BCI-algebra (see [9]). Hence we know that a soft BCK/BCI-algebra may not be an intersectional soft BCK/BCI-algebra. We also know that an intersectional soft BCK/BCI-algebra may not be a soft BCK/BCI-algebra in the following example.

Example 5.6. Consider a BCI-algebra X which is given in Example 5.4. Let $(\mathscr{H}, A)$ be a soft set over X where $A = X$ and

$$\mathscr{H} : A \to \mathscr{P}(X),\ x \mapsto \begin{cases} X & \text{if } x = 0,\\ X \setminus \{0\} & \text{if } x \in \{a, b, c\}.\end{cases}$$

Then $(\mathscr{H}, A)$ is an intersectional soft BCI-algebra over X, but it is not a soft BCI-algebra over X.

Theorem 5.7. *For a subset A of X containing 0, if $(\mathscr{F}, A)$ is an intersectional soft BCK/BCI-algebra over X, then*

$$(\forall x \in A)\,(\mathscr{F}(x) \subseteq \mathscr{F}(0))\,. \tag{5.2}$$

Proof. For any $x \in A$, we have $\mathscr{F}(0) = \mathscr{F}(x * x) \supseteq \mathscr{F}(x) \cap \mathscr{F}(x) = \mathscr{F}(x)$. □

Theorem 5.8. *Let $(\mathscr{F}, A)$ be an intersectional soft BCK/BCI-algebra over X, where A is a subset of X containing 0. For a given point $x \in A$, the following are equivalent:*

(1) $\mathscr{F}(x) = \mathscr{F}(0)$.
(2) $\mathscr{F}(y) \subseteq \mathscr{F}(x * y)$ *for all $y \in A$ with $x * y \in A$.*

Proof. Assume that (2) is valid. Taking $y = 0$ and using (a1), we have $\mathscr{F}(0) \subseteq \mathscr{F}(x * 0) = \mathscr{F}(x)$. It follows from Theorem 5.7 that $\mathscr{F}(x) = \mathscr{F}(0)$.

Conversely, suppose $\mathscr{F}(x) = \mathscr{F}(0)$. Then

$$\mathscr{F}(y) = \mathscr{F}(0) \cap \mathscr{F}(y) = \mathscr{F}(x) \cap \mathscr{F}(y) \subseteq \mathscr{F}(x * y)$$

by (5.2). □

Definition 5.9. For any subset A of a BCI-algebra X, an intersectional soft BCI-algebra $(\mathscr{F}, A)$ over X is said to be *strong* if it satisfies:

$$(\forall x \in A)\,(0 * x \in A \Rightarrow \mathscr{F}(0 * x) = \mathscr{F}(x))\,. \tag{5.3}$$

Example 5.10. The intersectional soft BCI-algebra $(\mathscr{G}, B)$ over X which is given in Example 5.5(2) is strong.

Theorem 5.11. *For a subset A of a BCI-algebra X, every strong intersectional soft BCI-algebra $(\mathscr{F}, A)$ over X satisfies the following condition*:

$$(\forall x, y \in A)\,(0 * y \in A \Rightarrow \mathscr{F}(x) \cap \mathscr{F}(y) \subseteq \mathscr{F}(x * (0 * y)))\,. \tag{5.4}$$

Proof. Suppose that an intersectional soft BCI-algebra $(\mathscr{F}, A)$ over X is strong. Let $x, y \in A$ be such that $0 * y \in A$. Using (5.3), we have

$$\mathscr{F}(x) \cap \mathscr{F}(y) = \mathscr{F}(x) \cap \mathscr{F}(0 * y) \subseteq \mathscr{F}(x * (0 * y)),$$

which is the desired result. □

Theorem 5.12. *For a subset A of a p-semisimple BCI-algebra X, if an intersectional soft BCI-algebra $(\mathscr{F}, A)$ over X satisfies the condition* (5.4)*, then it is strong.*

Proof. Let $x \in A$ be such that $0 * x \in A$. Then

$$\mathscr{F}(x) = \mathscr{F}(0) \cap \mathscr{F}(x) = \mathscr{F}(0) \cap \mathscr{F}(0 * (0 * x)) \subseteq \mathscr{F}(0 * (0 * (0 * x))) = \mathscr{F}(0 * x).$$

Since $0 * (0 * x) \in A$, it follows that $\mathscr{F}(0 * x) \subseteq \mathscr{F}(0 * (0 * x)) = \mathscr{F}(x)$. Hence $(\mathscr{F}, A)$ satisfies the condition (5.3), and so it is strong. □

Definition 5.13. Let $(\mathscr{F}, A)$ be a soft set over X and γ be a subset of X. The *γ-support* of $(\mathscr{F}, A)$, denoted by $\gamma(\mathscr{F}, A)$, is defined to be the set:

$$\gamma\,(\mathscr{F}, A) := \{x \in A \mid \gamma \subseteq \mathscr{F}(x)\}\,.$$

Proposition 5.14. *For any two soft sets $(\mathscr{F}, A)$ and $(\mathscr{F}, B)$ over X and a subset γ of X where A and B are subsets of X, we have*

$$A \subseteq B \Rightarrow \gamma\,(\mathscr{F}, A) \subseteq \gamma\,(\mathscr{F}, B)\,.$$

Proof. Straightforward. □

Proposition 5.15. *For any two soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over X and a subset γ of X where A and B are subsets of X, we have*

(1) $\gamma\,(\mathscr{F}, A) \cup \gamma\,(\mathscr{G}, B) \subseteq \gamma\left((\mathscr{F}, A)\,\widetilde{\cup}\,(\mathscr{G}, B)\right)$.

(2) $\gamma(\mathscr{F},A)\cap\gamma(\mathscr{G},B)\subseteq\gamma\left((\mathscr{F},A)\,\widetilde{\cap}\,(\mathscr{G},B)\right).$

Proof. (1) Let $x\in\gamma(\mathscr{F},A)\cup\gamma(\mathscr{G},B)$. Then $x\in\gamma(\mathscr{F},A)$ or $x\in\gamma(\mathscr{G},B)$, that is, $[x\in A$ and $\gamma\subseteq\mathscr{F}(x)]$ or $[x\in B$ and $\gamma\subseteq\mathscr{G}(x)]$. Denote by $(\mathscr{H},C):=(\mathscr{F},A)\,\widetilde{\cup}\,(\mathscr{G},B)$ where $C=A\cup B$. If $x\in A\setminus B$, then $x\in C$ and $\gamma\subseteq\mathscr{F}(x)=\mathscr{H}(x)$. If $x\in B\setminus A$, then $x\in C$ and $\gamma\subseteq\mathscr{G}(x)=\mathscr{H}(x)$. If $x\in A\cap B$, then $x\in C$ and $\gamma\subseteq\mathscr{F}(x)\cup\mathscr{G}(x)=\mathscr{H}(x)$. Therefore $x\in\gamma\left((\mathscr{F},A)\,\widetilde{\cup}\,(\mathscr{G},B)\right)$, and hence (1) is valid. Similarly, we can prove the second result. □

Theorem 5.16. *Let $(\mathscr{F},A)$ be an intersectional soft BCK/BCI-algebra over X where A is a subset of X. If A is a subalgebra of X, then $\gamma(\mathscr{F},A)$ is a subalgebra of X for any $\gamma\subseteq X$.*

Proof. Let $x,y\in\gamma(\mathscr{F},A)$. Then $x,y\in A$, $\gamma\subseteq\mathscr{F}(x)$ and $\gamma\subseteq\mathscr{F}(y)$. Since A is a subalgebra, it follows that $x*y\in A$ and

$$\gamma\subseteq\mathscr{F}(x)\cap\mathscr{F}(y)\subseteq\mathscr{F}(x*y)$$

so that $x*y\in\gamma(\mathscr{F},A)$. Therefore $\gamma(\mathscr{F},A)$ is a subalgebra of X. □

Corollary 5.17. *Let $(\mathscr{F},A)$ be an intersectional soft BCK/BCI-algebra over X with $A=X$. Then $\gamma(\mathscr{F},A)$ is a subalgebra of X for any $\gamma\subseteq X$.*

For a soft set $(\mathscr{F},A)$ over X, we consider the set:

$$X_0:=\{x\in A\mid\mathscr{F}(x)=\mathscr{F}(0)\}.$$

Theorem 5.18. *For a subalgebra A of X, let $(\mathscr{F},A)$ be an intersectional soft BCK/BCI-algebra over X. Then the set X_0 is a subalgebra of X.*

Proof. Let $x,y\in X_0$. Then $x,y\in A$ and $\mathscr{F}(x)=\mathscr{F}(0)=\mathscr{F}(y)$. Since A is a subalgebra of X, we have $x*y\in A$. It follows from (5.1) that $\mathscr{F}(0)=\mathscr{F}(x)\cap\mathscr{F}(y)\subseteq\mathscr{F}(x*y)$ so from (5.2) that $\mathscr{F}(0)=\mathscr{F}(x*y)$. Hence $x*y\in X_0$; so X_0 is a subalgebra of X. □

Corollary 5.19. *Let $(\mathscr{F},A)$ be an intersectional soft BCK/BCI-algebra over X with $A=X$. Then the set X_0 is a subalgebra of X.*

In [4], Çağman et al. provided new definitions and various results on soft set theory.

Definition 5.20 ([4])**.** A soft set F_A on the universe U is defined to be the set of ordered pairs

$$F_A:=\{(x,f_A(x)):x\in E,\ f_A(x)\in\mathscr{P}(U)\},$$

where $f_A:E\to\mathscr{P}(U)$ such that $f_A(x)=\emptyset$ if $x\notin A$.

The function f_A is called approximate function of the soft set F_A. The subscript A in the notation f_A indicates that f_A is the approximate function of F_A.

In what follows, denote by $S(U)$ the set of all soft sets over U by Çağman et al. [4].

Definition 5.21 ([4]). For $F_A, F_B \in S(U)$, we say that F_A is a soft subset of F_B, denoted by $F_A \widetilde{\subseteq} F_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

Definition 5.22. Let X be a BCK/BCI-algebra and $F_X \in S(U)$. Then F_X is called a BCK/BCI-intersectional soft algebra if it satisfies:

$$(\forall x, y \in X)\,(f_X(x) \cap f_X(y) \subseteq f_X(x * y))\,. \tag{5.5}$$

Example 5.23. Consider the BCI-algebra $(\mathbb{Z}, *, 0)$ as the initial universe set U, where $a * b = a - b$ for all $a, b \in \mathbb{Z}$. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Define the approximate function f_X of F_X by

$$f_X(x) = \begin{cases} \mathbb{Z} & \text{if } \min\{n \in \mathbb{N} \mid 0 * x^n = 0\} = 1, \\ [-r, r] & \text{if } \min\{n \in \mathbb{N} \mid 0 * x^n = 0\} = r \neq 1, \\ [-\frac{r}{2}, \frac{r}{2}] & \text{if } \min\{n \in \mathbb{N} \mid 0 * x^n = 0\} > r. \end{cases}$$

Then $F_X = \{(0, \mathbb{Z}), (1, \mathbb{Z}), (a, [-1, 1]), (b, [-2, 2]), (c, [-1, 1])\}$ which is a BCI-intersectional soft algebra.

Lemma 5.24. *Let X be a BCK/BCI-algebra. If $F_X \in S(U)$ is a BCK/BCI-intersectional soft algebra, then*

$$(\forall x \in X)\,(f_X(x) \subseteq f_X(0))\,. \tag{5.6}$$

Proof. We have

$$f_X(x) = f_X(x) \cap f_X(x) \subseteq f_X(x * x) = f_X(0)$$

for all $x \in X$. □

Proposition 5.25. *Let X be a BCI-algebra. If $F_X \in S(U)$ is a BCI-intersectional soft algebra, then*

$$(\forall x, y \in X)\,(f_X(x) \cap f_X(y) \subseteq f_X(x * (0 * y)))\,. \tag{5.7}$$

Proof. Using Lemma 5.24, we have

$$\begin{aligned} f_X(x) \cap f_X(y) &= f_X(x) \cap f_X(0) \cap f_X(y) \\ &\subseteq f_X(x) \cap f_X(0 * y) \\ &\subseteq f_X(x * (0 * y)) \end{aligned}$$

for all $x, y \in X$. □

In considering the converse of Proposition 5.25, we need to strength the condition of a BCI-algebra X.

Lemma 5.26 ([8]). *A BCI-algebra X is associative if and only if* $0 * x = x$ *for all* $x \in X$.

Proposition 5.27. *Let X be a BCI-algebra and let $F_X \in S(U)$ satisfy the condition* (5.7). *If X is associative, then F_X is a BCI-intersectional soft algebra.*

Proof. Straightforward. □

Theorem 5.28. *Let X be a BCK/BCI-algebra and let $F_X \in S(U)$ be a BCK/BCI-intersectional soft algebra. Then*

$$(\forall x, y \in X)\,(f_X(y) \subseteq f_X(x * y) \Leftrightarrow f_X(x) = f_X(0)).$$

Proof. Assume that $f_X(y) \subseteq f_X(x * y)$ for all $x, y \in X$. Taking $y = 0$ induces $f_X(0) \subseteq f_X(x * 0) = f_X(x)$. It follows from (5.6) that $f_X(0) = f_X(x)$ for all $x \in X$.

Conversely suppose that $f_X(x) = f_X(0)$ for all $x \in X$. Then

$$f_X(y) = f_X(0) \cap f_X(y) = f_X(x) \cap f_X(y) \subseteq f_X(x * y)$$

for all $x, y \in X$. □

Definition 5.29. For any BCK/BCI-algebras X and Y, let $\mu : X \to Y$ be a function and $F_X, F_Y \in S(U)$.

(1) The soft set

$$\mu^{-1}(F_Y) = \left\{(x, \mu^{-1}(f_Y)(x) : x \in X,\ \mu^{-1}(f_Y)(x) \in \mathscr{P}(U)\right\},$$

where $\mu^{-1}(f_Y)(x) = f_Y(\mu(x))$, is called the soft pre-image of F_Y under μ.

(2) The soft set

$$\mu(F_X) = \{(y, \mu(f_X)(y) : y \in Y,\ \mu(f_X)(y) \in \mathscr{P}(U)\},$$

where

$$\mu(f_X)(y) = \begin{cases} \bigcup\limits_{x \in \mu^{-1}(y)} f_X(x) & \text{if } \mu^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

is called the soft image of F_X under μ.

Proposition 5.30. *For any BCK/BCI-algebras X and Y, let $\mu : X \to Y$ be a function. Then*

$$(\forall F_X \in S(U))\left(F_X \widetilde{\subseteq} \mu^{-1}(\mu(F_X))\right). \tag{5.8}$$

Proof. Note that $\mu^{-1}(\mu(x)) \neq \emptyset$ for all $x \in X$. Hence

$$f_X(x) \subseteq \bigcup_{a \in \mu^{-1}(\mu(x))} f_X(a) = \mu(f_X)(\mu(x)) = \mu^{-1}(\mu(f_X))(x)$$

for all $x \in X$, and therefore (5.8) is valid. □

Theorem 5.31. *Let $\mu : X \to Y$ be a homomorphism of BCK/BCI-algebras and $F_Y \in S(U)$. If F_Y is a BCK/BCI-intersectional soft algebra, then the soft pre-image $\mu^{-1}(F_Y)$ of F_Y under μ is also a BCK/BCI-intersectional soft algebra.*

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned}\mu^{-1}(f_Y)(x_1) \cap \mu^{-1}(f_Y)(x_2) &= f_Y(\mu(x_1)) \cap f_Y(\mu(x_2)) \\ &\subseteq f_Y\left(\mu(x_1) * \mu(x_2)\right) \\ &= f_Y\left(\mu(x_1 * x_2)\right) \\ &= \mu^{-1}(f_Y)(x_1 * x_2).\end{aligned}$$

Hence $\mu^{-1}(F_Y)$ is also a BCK/BCI-intersectional soft algebra. □

Theorem 5.32. *Let $\mu : X \to Y$ be a homomorphism of BCK/BCI-algebras and $F_X \in S(U)$. If F_X is a BCK/BCI-intersectional soft algebra and μ is injective, then the soft image $\mu(F_X)$ of F_X under μ is also a BCK/BCI-intersectional soft algebra.*

Proof. Let $y_1, y_2 \in Y$. If at least one of $\mu^{-1}(y_1)$ and $\mu^{-1}(y_1)$ is empty, then the inclusion

$$\mu(f_X)(y_1) \cap \mu(f_X)(y_1) \subseteq \mu(f_X)(y_1 * y_2)$$

is clear. Assume that $\mu^{-1}(y_1) \neq \emptyset$ and $\mu^{-1}(y_2) \neq \emptyset$. Then

$$\begin{aligned}\mu(f_X)(y_1) \cap \mu(f_X)(y_2) &= \left(\bigcup_{x_1 \in \mu^{-1}(y_1)} f_X(x_1)\right) \cap \left(\bigcup_{x_2 \in \mu^{-1}(y_2)} f_X(x_2)\right) \\ &= \bigcup_{\substack{x_1 \in \mu^{-1}(y_1) \\ x_2 \in \mu^{-1}(y_2)}} \left(f_X(x_1) \cap f_X(x_2)\right) \\ &\subseteq \bigcup_{\substack{x_1 \in \mu^{-1}(y_1) \\ x_2 \in \mu^{-1}(y_2)}} \left(f_X(x_1 * x_2)\right) \\ &= \bigcup_{x \in \mu^{-1}(y_1 * y_2)} f_X(x) \\ &= \mu(f_X)(y_1 * y_2).\end{aligned}$$

Therefore $\mu(F_X)$ is a BCK/BCI-intersectional soft algebra. □

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Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: skywine@gmail.com

Min Su Kang
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
E-mail address: sinchangmyun@hanmail.net

Kyoung Ja Lee
Department of Mathematics Education
Hannam University
Daejeon 306-791, Korea
E-mail address: lsj1109@hotmail.com