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EXTREME SETS OF RANK INEQUALITIES OVER BOOLEAN MATRICES AND THEIR PRESERVERS

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Abstract. We consider the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of matrices over nonbinary Boolean algebra. We characterize linear operators that preserve these sets of matrix ordered pairs as the form of $T(X) = PXP^T$ with some permutation matrix P.

1. Introduction

The linear preserver problem is one of the most active and fertile subjects in matrix theory during the past one hundred years, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Beasley and Guterman ([1]) investigated the rank inequalities of matrices over semirings. The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a pair of matrices which belongs to it and to act on this pair by various linear operators that preserve this variety. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [5]. For the details on linear operators preserving matrix invariants one can see [8] and [9]. Almost all researches on linear preserver problems over semirings have dealt with those semirings without zero-divisors to avoid the difficulties of multiplication arithmetic for the elements in those semirings ([2]-[6]). But nonbinary Boolean algebra is not the case. That is, all elements except 0 and 1 in the nonbinary Boolean algebra are zero-divisors. So there are few results on the linear preserver problems for the matrices over nonbinary Boolean algebra ([7], [10]). Kirkland and Pullman characterized the linear operators that preserve rank of matrices over nonbinary Boolean algebra in [7].

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In this paper, we construct the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of two matrix multiplication over nonbinary Boolean algebra and characterize the linear operators that preserve those extreme sets.

2. Preliminaries and basic results

A *semiring* S consists of a set S with two binary operations, addition and multiplication, such that:

- S is an abelian monoid under addition (the identity is denoted by 0);
- S is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);
- multiplication is distributive over addition on both sides;
- $s0 = os = 0$ for all $s \in S$.

A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

A semiring $\mathcal{S}$ is called a *Boolean algebra* if $\mathcal{S}$ is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M.

Let $S_k = \{a_1, a_2, \ldots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a Boolean algebra of subsets of $S_k = \{a_1, a_2, \ldots, a_k\}$, which is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. If $\mathbb{B}_k$ consists of only the empty subset and M, then it is called a *binary Boolean algebra*. If $\mathbb{B}_k$ is not a binary Boolean algebra, then it is called a *nonbinary Boolean algebra*. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes the nonbinary Boolean algebra, which contains at least 3 elements. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i,j)^{th}$ entry. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{B}_k$. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank* r if there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We denote $b(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

A *line* of a matrix A is a row or a column of the matrix A.

For $X, Y \in \mathbb{M}_{m,n}(\mathcal{S})$, the matrix $X \circ Y$ denotes the *Hadamard or Schur product*, i.e., the $(i,j)^{th}$entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called *linear* if it satisfies $T(X+Y) = T(X)+T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.

We say that an operator T *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs such that $(X, Y) \in \mathcal{P}$ implies $(T(X), T(Y)) \in \mathcal{P}$.

An operator T *strongly preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs such that $(X, Y) \in \mathcal{P}$ if and only if $(T(X), T(Y)) \in \mathcal{P}$.

An operator T is called a (P, Q, B)-*operator* if there exist permutation matrices P and Q and a matrix $B \in \mathbb{M}_{m,n}(\mathcal{S})$ with no zero entries such that $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$ or if for $m = n$, $T(X) = P(X \circ B)^T Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

If $\mathcal{S}$ is a field, then there is the usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathcal{S})$. It is well-known that the behavior of the function ρ with respect to matrix addition and multiplication is given by the following inequalities ([3]):

- the rank-sum inequalities:
$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq \rho(A) + \rho(B),$$
- Sylvester's laws:
$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}, \text{ and}$$
- the Frobenius inequality:
$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B are conformal matrices with entries from a field.

The arithmetic properties of Boolean rank for the matrix multiplications are restricted by the following list of inequalities ([1]): For arbitrary $A, B \in \mathbb{M}_n(\mathbb{B}_k)$,

(1) $\mathrm{b}(AB) \leq \min\{\mathrm{b}(A), \mathrm{b}(B)\}$,

(2) $\mathrm{b}(AB) \geq \begin{cases} 0 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) \leq n, \\ 1 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) > n. \end{cases}$

Now, we construct the following sets of matrix pairs that arise as either extremal cases in the inequalities (1) and (2):

$$\mathcal{R}_{MM}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = \min\{\mathrm{b}(X), \mathrm{b}(Y)\}\},$$
$$\mathcal{R}_{M0}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = 0\},$$
$$\mathcal{R}_{M1}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = 1\}.$$

In this paper, we characterize the linear operators that preserve these sets of matrix pairs.

Lemma 2.1. *Let P and Q be permutation matrices of m-square and n-square respectively. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is defined by $T(X) = PX$ or $T(X) = XQ$ for any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$. Then T preserves Boolean rank. That is $b(T(X)) = b(X)$.*

Proof. Let $A, B \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and P be an $m \times m$ permutation matrix. Since $\mathrm{b}(AB) \leq \min\{\mathrm{b}(A),\mathrm{b}(B)\}$, we have $\mathrm{b}(PX) \leq \min\{\mathrm{b}(P),\, \mathrm{b}(X)\} \leq \mathrm{b}(X)$. And $\mathrm{b}(X) = \mathrm{b}(IX) = \mathrm{b}((P^TP)X) = \mathrm{b}(P^T(PX)) \leq \mathrm{b}(PX)$. Hence $\mathrm{b}(PX) = \mathrm{b}(X)$. Similarly $\mathrm{b}(XQ) = \mathrm{b}(X)$ for all $n \times n$ permutation matrix Q. □

Theorem 2.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then the following conditions are equivalent*:

(a) *T is bijective*;

(b) *T is surjective*;

(c) *T is injective*;

(d) *there exists a permutation σ on $\{(i, j) \mid i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. (a), (b) and (c) are equivalent since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a finite set.

(d)⇒(b) For any $D \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, we may write

$$D = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j}.$$

Since σ is a permutation, there exist $\sigma^{-1}(i, j)$ and

$$D' = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}$$

such that

$$\begin{aligned} T(D') &= T\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}\Big) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma\sigma^{-1}(i,j)}E_{\sigma\sigma^{-1}(i,j)} = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j} = D. \end{aligned}$$

(a)⇒(d) We assume that T is bijective. Suppose that $T(E_{i,j}) \neq E_{\sigma(i,j)}$ where σ be a permutation on $\{(i, j) \mid i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. Then there exist some pairs (i, j) and (r, s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$ or some pairs $(i, j), (r, s)$ and (u, v) $((r, s) \neq (u, v))$ such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r, s)^{th}$ and $(u, v)^{th}$ entries of Z are zeros.

Case 1) Suppose that there exist some pairs (i, j) and (r, s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$. Since T is bijective, there exists $X_{r,s} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$

such that $T(X_{r,s}) = E_{r,s}$. Then $\alpha T(X_{r,s}) = \alpha E_{r,s} = T(E_{i,j})$, and hence $\alpha X_{r,s} = E_{i,j}$, which contradicts the fact that $\alpha \neq 1$.

Case 2) Suppose that there exist some pairs $(i,j),(r,s)$ and (u,v) such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros. Since T is bijective, there exist $X_{r,s}$, $X_{u,v}$ and $Z' \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = \alpha E_{r,s}$, $T(X_{u,v}) = \beta E_{u,v}$, and $T(Z') = Z$. Thus $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z = T(X_{r,s}) + T(X_{u,v}) + T(Z') = T(X_{r,s} + X_{u,v} + Z')$. So $E_{i,j} = X_{r,s} + X_{u,v} + Z'$, a contradiction. □

Remark 2.3. One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P,Q,B)-operators and if $m = n = 1$, then all operators under consideration are (P,P^T,B)-operators.

Henceforth we will always assume that $m, n \geq 2$.

Lemma 2.4. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps a line to a line and T be defined by the rule $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j) \mid i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ and $b_{i,j} \in \mathbb{B}_k$ are nonzero elements for $i = 1,2,\ldots,m; j = 1,2,\ldots,n$. Then T is a (P,Q,B)-operator.*

Proof. Since no combination of p rows and q columns can dominate J for any nonzero p and q with $p+q = m$, we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$, $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^TQ$, $T(C_j) \leq P(C_j)^TQ$. Since each nonzero entry of a cell lies in the intersection of a row and a column and T maps nonzero cells into nonzero (weighted) cells, it follows that $T(E_{i,j}) = Pb_{i,j}E_{i,j}Q = P(E_{i,j} \circ B)Q$, or, if $m = n$, $T(E_{i,j}) = P(b_{i,j}E_{i,j})^TQ = P(E_{i,j} \circ B)^TQ$ where $B = (b_{i,j})$ is defined by the action of T on the cells. □

3. Linear preservers of $\mathcal{R}_{Mm}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{Mm}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = \min\{\mathrm{b}(X), \mathrm{b}(Y)\}\}.$$

Theorem 3.1. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear operator. Then T is surjective and preserves $\mathcal{R}_{Mm}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be defined by $T(X) = PXP^T$ and $(X,Y) \in \mathcal{R}_{Mm}(\mathbb{B}_k)$. Then $\mathrm{b}(XY) = \min\{\mathrm{b}(X), \mathrm{b}(Y)\}$ and hence

$$\begin{aligned}\mathrm{b}(T(X)T(Y)) = \mathrm{b}(PXP^TPYP^T) &= \mathrm{b}(PXYP^T) = \mathrm{b}(XY)\\ &= \min\{\mathrm{b}(X), \mathrm{b}(Y)\} = \min\{\mathrm{b}(PXP^T), \mathrm{b}(PYP^T)\}\\ &= \min\{\mathrm{b}(T(X)), \mathrm{b}(T(Y))\}\end{aligned}$$

by Lemma 2.1. Thus $(T(X), T(Y)) \in \mathcal{R}_{Mm}(\mathbb{B}_k)$. That is T preserves $\mathcal{R}_{Mm}(\mathbb{B}_k)$.

Conversely, assume that T is surjective and preserves $\mathcal{R}_{Mm}(\mathbb{B}_k)$. By Theorem 2.2, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a permutation σ on $\{(i,j) \mid 1 \leq i,j \leq n\}$. Consider $(E_{i,j}, E_{j,h}) \in \mathcal{R}_{Mm}(\mathbb{B}_k)$ for all h. Then $\mathrm{b}(T(E_{i,j})T(E_{j,h}))$ $= \min\{\mathrm{b}(T(E_{i,j}), \mathrm{b}(T(_{j,h}))\} = 1$, but $T(E_{i,j})T(E_{j,h}) = E_{\sigma(i,j)}E_{\sigma(j,h)}$. It follows that $E_{\sigma(j,h)}$ is in the same row as $E_{\sigma(j,1)}$ for any $h = 1, 2, \ldots, n$. That is, T maps rows to rows; similarly T maps columns to columns. By Lemma 2.4 with $b_{i,j} = 1$, it follows that $T(X) = PXQ$ for some permutation matrices P and Q. Let us show that $Q = P^T$. Indeed $T(E_{i,j}) = E_{\pi(i),\tau(j)}$, where π is the permutation corresponding to P and τ is the permutation corresponding to Q^T. But $(E_{1,i}, E_{i,1}) \in \mathcal{R}_{Mm}(\mathbb{B}_k)$; thus $(E_{\pi(1),\tau(i)},\ E_{\pi(i),\tau(1)}) \in \mathcal{R}_{Mm}(\mathbb{B}_k)$ and hence $\pi \equiv \tau$. Therefore $Q = P^T$. □

4. Linear preservers of $\mathcal{R}_{M0}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{M0}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = 0\}.$$

In this section, we characterize the linear operator $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ that preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$.

A linear operator $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ is *singular* if $T(X) = O$ for some nonzero $X \in \mathbb{M}_n(\mathbb{B}_k)$; Otherwise T is *nonsingular*. Notice that if T is a (P, Q)-operator, then T is nonsingular.

Theorem 4.1. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a nonsingular linear operator. Assume that $T(J) \geq P_J$, a permutation matrix. Then T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be defined by $T(X) = PXP^T$ and $(X, Y) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. Then $\mathrm{b}(XY) = 0$ and hence $\mathrm{b}(T(X)T(Y)) = \mathrm{b}(PXP^TPYP^T) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY) = 0$ by Lemma 2.1. Thus $(T(X), T(Y)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. That is, T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$.

Conversely, assume that T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. Since $T(J) \geq P_J$, a permutation matrix, there are n different cells whose images have nonzero entries in every column. Assume that these cells can be chosen such that their nonzero entries are in fewer than n columns, say $X = E_1 + E_2 + \cdots + E_n$ is the sum of n such cells and X has no nonzero entry in column h. Then $(X, R_h) \in \mathcal{R}_{M0}(\mathbb{B}_k)$ and hence $(T(X), T(R_h)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$, since T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. But $T(X)$ has nonzero entry in every column, which implies $T(X)T(R_h) \neq O$, a contradiction. Thus, if T maps a column into two columns, then we have a contradiction from above. Furthermore, if T maps two columns into one column, there must be a column whose image is at least two column from $T(J) \geq P_J$ for some permutation matrix P_J. Thus in this case, we also have a contradiction as above. Consequently T maps a column into a column and all columns into all columns respectively. Hence T induces a permutation on the set of columns. Similarly T induces a permutation on the set of rows, and hence $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrices P and Q. Let us show that $Q = P^T$. Indeed we have that $T(E_{i,j}) = b_{i,j}E_{\pi(i),\tau(j)}$. If $Q \neq P^T$, then $\pi \neq \tau$. Thus, for some i, we have $\pi(i) \neq \tau(i)$ and hence for some $j \neq i$, we have $\pi(j) = \tau(i)$. Here $(E_{i,i}, E_{j,i}) \in \mathcal{R}_{M0}(\mathbb{B}_k)$ but $T(E_{i,i})T(E_{j,i}) = b_{i,i}b_{j,i}E_{\pi(i),\tau(i)}E_{\pi(j),\tau(i)} = b_{i,i}b_{j,i}E_{\pi(i),\tau(i)} \neq O$, and hence $(T(E_{i,i}), T(E_{j,i})) \notin \mathcal{R}_{M0}(\mathbb{B}_k)$; a contradiction. Thus $\pi = \tau$ and hence $T(X) = P(X \circ B)P^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$. Since T is nonsingular, all entries of B are nonzero and not zero divisors. But every elements α in $\mathbb{B}_k$ is a zero divisor if $\alpha \neq 1$. Thus $b_{i,j} = 1$. Hence $B = J$. Consequently $T(X) = PXP^T$. □

Corollary 4.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a surjective linear operator. Then T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. If T is a surjective linear operator, then T is bijective by Theorem 2.2. Thus T is nonsingular. Hence, T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if $T(X) = PXP^T$, by Theorem 4.1. □

Corollary 4.3. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear operator. Then T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. It is easy to see that operator of the form $T(X) = PXP^T$ strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$.

Conversely, suppose that T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. We claim that (1) $T(J) \geq P_J$, some permutation matrix, that is, $T(J)$ has a nonzero element in each row and each column and (2) T is a nonsingular operator. Then we apply Theorem 4.1.

Claim (1): $T(J) \geq P_J$. Assume, on the contrary, that $T(J)$ has a zero column (For the case of a zero row, the proof is quite similar). Up to a multiplication with permutation matrices, we may assume that there are nonzero elements in columns $1, 2, \ldots, t$ of $T(J)$ and all elements in the column $(t+1), \ldots, n$ are zero. Then there exist column matrices $C_{j_1}, C_{j_2}, \ldots, C_{j_s}$ whose images dominate all nonzero entries in columns 1 through t. Let $l \neq j_h$ for all h, $1 \leq h \leq s$. Thus $(C_{j_1}, C_{j_2}, \ldots, C_{j_s})R_l = O$. Since T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$, it follows that $T(C_{j_1}, C_{j_2}, \ldots, C_{j_s})T(R_l) = O$. Then all the entries in rows 1 through t of $T(R_l)$ are zero, since there is a nonzero element in each of the first t columns of $T(C_{j_1}, C_{j_2}, \ldots, C_{j_s})$. Therefore $T(E_{l,l})$ has nonzero entries only in rows $t+1, \ldots, n$ and only in columns $1, 2, \ldots, t$. Thus $T(E_{l,l})^2 = O$, equivalently, $(T(E_{l,l}), T(E_{l,l})) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. This is a contradiction since T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ and $(E_{l,l}, E_{l,l}) \notin \mathcal{R}_{M0}(\mathbb{B}_k)$. Thus $T(J)$ has neither a zero row nor a zero column, that is $T(J) \geq P_J$.

Claim (2): T is a nonsingular operator. Assume that there exists a nonzero matrix X such that $T(X) = O$. Then $(T(X), T(I)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. But $(X, I) \notin$

$\mathcal{R}_{M0}(\mathbb{B}_k)$. This contradicts the fact that T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. Thus T is a nonsingular operator.

Hence Theorem 4.1 is applicable, since claims (1) and (2) satisfy the conditions in Theorem 4.1. Consequently we obtain $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and for some permutation matrix P. $\square$

5. Linear preservers of $\mathcal{R}_{M1}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{M1}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 \mid \mathrm{b}(XY) = 1\}.$$

Lemma 5.1. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear operator defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ of $\{(i, j) \mid 1 \leq i, j \leq n\}$ and nonzero scalars $b_{i,j} \in \mathbb{B}_k$. Then T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. Clearly linear operators of the form $T(X) = PXP^T$ strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$.

Conversely, assume that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Consider $(E_{i,i}, E_{i,h}) \in \mathcal{R}_{M1}(\mathbb{B}_k)$ for all $h = 1, \ldots, n$. If $T(E_{i,i}) = b_{i,i}E_{r,s}$ for some r and s, then $T(E_{i,h}) = b_{i,h}E_{s,\tau(h)}$, where τ is some permutation, since $(T(E_{i,i}), T(E_{i,h})) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. That is, $T(R_i) \leq R_s$. Thus T induces a permutation on the rows. Similarly T induces a permutation on the columns. Thus, for some permutations π and τ, $T(E_{i,j}) = b_{i,j}E_{\pi(i),\tau(j)}$. Now $\mathrm{b}(T(E_{i,i})T(E_{i,j}))$ must be 1 and hence $\pi(i) = \tau(i)$. Therefore $\pi = \tau$ and we have that $T(X) = P(X \circ B)P^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, and P is the permutation corresponding to π. Now, if $B \neq J$, then $b_{p,q} \neq 1$ for some (p, q). But then, $(E_{i,i} + E_{i,q} + E_{p,i} + b_{p,q}E_{p,q}, I) \notin \mathcal{R}_{M1}(\mathbb{B}_k)$, while $(E_{i,i} + E_{i,q} + E_{p,i} + E_{p,q}, I) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. However $T(E_{i,i} + E_{i,q} + E_{p,i} + b_{p,q}E_{p,q}) = T(E_{i,i} + E_{i,q} + E_{p,i} + E_{p,q})$, which contradicts the fact that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Thus $B = J$ and hence $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$. $\square$

Theorem 5.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a surjective linear operator. Then T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. Assume that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Since T is surjective, we have $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i and j with $1 \leq i, j \leq n$ by Theorem 2.2. By Lemma 5.1 with $b_{i,j} = 1$, we obtain the result.

Conversely if $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrix P, then $T(XY) = P(XY)P^T = PXP^TPYP^T = T(X)T(Y)$. Thus $\mathrm{b}(T(X)T(Y)) = \mathrm{b}(T(XY)) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY)$. Hence $(X, Y) \in \mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if $(T(X), T(Y)) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. Therefore T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. $\square$

As a concluding remark, we have constructed the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of two matrix multiplication over nonbinary Boolean algebra and characterize the linear operators that preserve those extreme sets.

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