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시변지연과 임의 발생 외란을 고려한 불확실 선형 시스템에 대한 지연의존 강인 H_∞ 제어

Delay-dependent Robust H_∞ Control of Uncertain Linear Systems with Time-varying Delays and Randomly Occurring Disturbances

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Abstract - This paper proposes a new condition about delay-dependent robust H_∞ control of uncertain linear systems with time-varying delay and randomly occurring disturbances. The norm bounded uncertainties are subjected to the system matrices. Based on Lyapunov stability theory, a sufficient condition for designing a controller gain such that the closed-loop systems are asymptotically stable with H_∞ disturbance level γ is formulated in terms of linear matrix inequalities (LMIs). Finally, two numerical examples are included to show the effectiveness of the presented method.

Key Words : H_∞ control, Time-delays, Randomly occurring disturbances, Lyapunov method, LMIs

1. Introduction

Time-delays are encountered in various systems such as physical and chemical systems, process control systems, networked control systems, and so on. Stability for time-delay systems has been attracted by many researchers during the last decades because time-delays can lead to oscillation, poor performance and instability of the systems [1-8].

There are two cases for stability criteria of linear systems with time-delay; namely, delay-dependent ones and delay-independent ones. Delay-dependent stability conditions have been paid much attention than delay-independent ones in recent years. In case of delay-dependent criteria, the information on lengths of time-delays is utilized. So, delay-dependent criteria are less restrictive than delay-independent ones. In addition, much efforts have been made to enlarge upper bounds of time-delays as large as possible or enlarge the feasibility region of stability criteria for guaranteeing asymptotic stability of time-delay systems. The key lemma in deriving delay-dependent criteria is Jensen Inequality [5] which provides an upper bound of the negative-definite integral terms. Recently, in [2], upper bounds of double integral terms was obtained by proposing the idea of reciprocally convex optimization for time-delay systems.

It should be noted that many real systems have nonlinear dynamics. Also there exist unknown parameters in systems because the exact values of system parameters cannot be known. But sometimes, the nonlinear models can be approximated by linear systems with some uncertainties. In that case, robust stability means that system is stable despite model mismatch. That is, the systems having unknown parameters demand establishment of the robust conditions that can guarantee the stability. This topic issued dominantly in the control community during the last two decades [3, 6, 12-17]. To handle the topic, in this paper, unknown matrices in system parameters are assumed to be of Lebesgue measurable elements.

The theory of H_∞ control was presented by Zames [9] firstly. Since then, a great number of papers about this topic have been reported [3, 14-19]. The H_∞ control technique has been used to minimize the effects of the external disturbances. It is the objective of H_∞ control to design the controllers such that the closed-loop system is internally stable and its H_∞-norm of the transfer function between the controlled output and the disturbances will not exceed a given level γ.

On the other hand, complex networks model was handled by using the concept of randomly occurring nonlinearities in [11]. In addition, in [12], the concept of randomly occurring uncertainties was introduced. In many practical systems, external disturbances occur randomly. Therefore, it is worth to investigate the problem of H_∞

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control design for the time-delay system with randomly occurring disturbances by adding the probabilistic information for disturbances. However, unfortunately, this problem has not been tackled in any other literature.

Motivated by the above discussions, in this paper, the problem of an H_∞ control of uncertain linear systems with time-varying delay and randomly occurring disturbances will be investigated for the first time. Based on Lyapunov theory, a sufficient condition for designing a controller gain such that the closed-loop systems are asymptotically H_∞ stable with disturbance level is formulated in terms of LMIs. Finally, two numerical examples have been given to show the effectiveness of the presented criterion.

Notations: R^n denotes the n-dimensional Euclidean space, $R^{n\times m}$ is the set of $n\times m$ real matrices. $diag\{\cdots\}$ denotes the block diagonal matrix. L_2 is the space of square integrable functions on $[0,\infty)$. For two symmetric matrices A and B, $A>(\geq)B$ means that $A-B$ is (semi-) positive definite. A^T denotes the transpose of A. If the context allows it, the dimensions of these matrices are often omitted. $\mathrm{E}\{x\}$ and $\mathrm{E}\{x|y\}$ will, respectively, mean the expectation of x and the expectation of x condition on y. $X_{[f(t)]}\in R^{m\times n}$ means that the elements of the matrix $X_{[f(t)]}$ includes the value of $f(t)$; e.g., $X_{[f_0]}\equiv X_{[f(t)=f_0]}$. $\Pr\{\cdot\}$ means the occurrence probability of the event "$\cdot$".

2. Problem Statements

Consider the following linear system with time-varying delay:

$$\begin{cases}\dot{x}(t)=(A+\Delta A(t))x(t)+(A_d+\Delta A_d(t))x(t-h(t))\\ \qquad +B_w w(t)+Bu(t),\\ z(t)=Cx(t),\end{cases} \tag{1}$$

where $x(t)\in R^n$ is the state vector, $u(t)\in R^m$ is the vector of controlled input, $z(t)\in R^m$ is the vector of controlled output, $w(t)\in R^n$ is the disturbance input which belongs to $L_2[0,\infty)$. $A\in R^{n\times n}$, $A_d\in R^{n\times n}$, $B_w\in R^{n\times n}$, $B\in R^{m\times n}$ and $C\in R^{m\times n}$ are known real constant matrices which describe the nominal system of (1). $\Delta A(\cdot)$, $\Delta A_d(\cdot)$ are real matrix functions representing time-varying parameter uncertainties satisfying

$$[\Delta A(\cdot),\ \Delta A_d(\cdot)]=DF(t)[E_a,\ E_d], \tag{2}$$

where $F^T(t)F(t)\leq I$.

Also, $h(t)$ is a time-delay satisfying time-varying continuous function as follows:

$$0\leq h(t)\leq h_M,\ \ \dot{h}(t)\leq h_d\,, \tag{3}$$

where h_M is a positive scalar and h_d is any constant one.

It is assumed that disturbances occur randomly. Let us define $\rho(t)$ as a stochastic variable representing the occurring disturbances which satisfy a Bernoulli distribution as following:

$$\rho(t)=\begin{cases}1,\text{if disturbances are maximum},\\ 0,\text{if disturbances are decreased by } d\times 100\%,\end{cases} \tag{4}$$

where d is a decreasing rate of disturbances satisfying $0\leq d\leq 1$. Also, $\rho(t)$ satisfies the following condition

$$\Pr\{\rho(t)=1\}=\mathrm{E}\{\rho(t)\}=\rho_0 \quad , \tag{5}$$

where $0\leq \rho_0\leq 1$ is a known constant scalar reflecting the occurrence probability of maximum disturbances.

Let us consider the following time-delay system with randomly occurring disturbances given by

$$\begin{cases}\dot{x}(t)=(A+DF(t)E_a)x(t)+(A_d+DF(t)E_d)x(t-h(t))\\ \qquad +(\rho(t)+(1-\rho(t))d)B_w w(t)+Bu(t),\\ z(t)=Cx(t),\end{cases} \tag{6}$$

We are interested in designing a memoryless state feedback controller

$$u(t)=Kx(t), \tag{7}$$

where $K\in R^{m\times n}$ is a constant matrix.

The system (6), comparing with the system (1), has the term of $\rho(t)$. That is information of probabilistic which is occurrence of maximum disturbances. For considering the randomly occurring disturbances, throughout this paper, we handle the system (6).

The purpose of this paper is to develop a delay-dependent robust H_∞ condition satisfying following conditions:

(i) With zero disturbance, the closed loop system (6) with control input $u(t)$ is asymptotically stable.

(ii) With zero condition and a given constant $\gamma>0$, the following condition holds:

$$J=\mathrm{E}\left\{\int_0^\infty z^T(s)z(s)-\gamma^2 w^T(s)w(s)ds\right\}\leq 0 \tag{8}$$

$$\left(i.e.\ \sup_{w\neq 0,w\in L_2[0,\infty]}\frac{\|z(t)\|_2}{\|w(t)\|_2}\leq\gamma\right).$$

Then, the controller $u(t)$ is said to be the H_∞ stabilization controller with the disturbance attenuation γ. The parameter γ is called the H_∞-norm bound of the controller. In deriving a main result, we use the following definitions.

Lemma 1. (Jensen's inequality) [5] *For any constant matrix* $0 < M = M^T \in R^{n \times n}$, *scalar* $\gamma > 0$, *vector function* $x:[0,\gamma] \to R^n$ *such that the integrations in the following are well defined, then*

$$-\gamma \int_0^{\gamma} x^T(s) M x(s) ds \le -\left(\int_0^{\gamma} x^T(s) ds\right)^T M \left(\int_0^{\gamma} x^T(s) ds\right).$$

Lemma 2. (Lower bounds theorem) [2] *Let* $f_1, f_2, \ldots, f_N : R^m \to R^n$ *have positive values in an open subset* D *of* R^m. *Then, the reciprocally convex combination of* f_i *over* D *satisfies*

$$\min_{\left\{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1\right\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i + \max_{g_{i,j}(t)} \sum_{i \neq j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j} : R^m \mapsto R, g_{j,i}(t) = g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}.$$

Lemma 3. [19] *Given matrices* $Q = Q^T, H, E$ *and* $R = R^T > 0$ *of appropriate dimensions,*

$$Q + HFE + E^T F^T H^T < 0$$

for all F *satisfying* $F^T F \le R$ *if and only if there exists some* $\lambda > 0$ *such that*

$$Q + \lambda H H^T + \lambda^{-1} E^T R E < 0.$$

3. Main Results

In this section, we will present an H_∞ stability criteria for system (6). For the sake of simplicity on matrix representation, notations of several matrices are defined as:

$$\begin{aligned}
&\zeta^T(t) = \left[x^T(t)\ x^T(t-h(t))\ x^T(t-h_M)\ \dot{x}^T(t)\ w^T(t)\right],\\
&e_1 = [I\,0\,0\,0\,0]^T, e_2 = [0\,I\,0\,0\,0]^T,\\
&e_3 = [0\,0\,I\,0\,0]^T, e_4 = [0\,0\,0\,I\,0]^T,\\
&e_5 = [0\,0\,0\,0\,I]^T,\\
&X\overline{Q_1}X = U_1, X\overline{Q_2}X = U_2, X\overline{R}X = U_3, X\overline{M}X = U_4,\\
&\Xi_1 = e_1 U_1 e_1^T - e_3 U_1 e_3^T + e_1 X e_4^T + e_4 X e_1^T,\\
&\Xi_2 = e_1 U_2 e_1^T - (1-h_d) e_2 U_2 e_2^T,\\
&\Xi_3 = h_M^2 e_4 U_3 e_4^T - \begin{bmatrix} e_1^T - e_2^T \\ e_2^T - e_3^T \end{bmatrix}^T \begin{bmatrix} U_3 & U_4 \\ U_4^T & U_3 \end{bmatrix} \begin{bmatrix} e_1^T - e_2^T \\ e_2^T - e_3^T \end{bmatrix},\\
&\Xi_{4[\rho(t)]} = \alpha(e_1 + e_4)((AX+BY)e_1^T + A_d X e_2^T - X e_4^T\\
&\qquad + (\rho(t) + (1-\rho(t))d) B_w X e_5^T)\\
&\qquad + \alpha(e_1(AX+BY)^T + e_2 X A_d^T - e_4 X\\
&\qquad + (\rho(t) + (1-\rho(t))d) e_5 X B_w^T)(e_1^T + e_4^T),\\
&\Xi_5 = \alpha^2 (e_1 + e_4) D D^T (e_1^T + e_4^T),\\
&\Xi_6 = e_1 C^T C e_1^T - \gamma^2 e_5 e_5^T,\\
&\Xi_{[\rho(t)]} = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_{4[\rho(t)]} + \Xi_5 + \Xi_6,\\
&\Lambda = (e_1 X E_a^T + e_2 X E_d^T).
\end{aligned} \tag{9}$$

Then, the main result of this paper is as follows:

Theorem 1. *For given scalars* $\gamma > 0$, $0 \le \rho_0 \le 1$, $0 \le d \le 1$, $h_M > 0$, h_d *and* $\alpha > 0$, *there exists a state-feedback controller (7) such that the system (6) is asymptotically stable with* H_∞ *disturbance attenuation level* γ *for any time-varying delay* $h(t)$ *satisfying (3) if there exist positive definite matrices* X, U_1, U_2, U_3 *and any matrix* Y, U_4 *satisfying the following LMIs hold:*

$$\begin{bmatrix} U_3 & U_4 \\ U_4^T & U_3 \end{bmatrix} \ge 0, \tag{10}$$

$$\begin{bmatrix} \Xi_{[\rho_0]} & \Lambda \\ \Lambda^T & -I \end{bmatrix} < 0. \tag{11}$$

Moreover, if the above conditions are feasible, a desired controller gain matrix is given by $K = YX^{-1}$.

Proof. For positive definite matrices P, Q_1, Q_2 and R, let us consider the following Lyapunov-Krasovskii functional candidate:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{12}$$

where

$$\begin{aligned}
V_1(x_t) &= x^T(t) P x(t) + \int_{t-h_M}^{t} x^T(s) Q_1 x(s) ds,\\
V_2(x_t) &= \int_{t-h(t)}^{t} x^T(s) Q_2 x(s) ds,\\
V_3(x_t) &= h_M \int_{t-h_M}^{t} \int_s^t \dot{x}^T(u) R \dot{x}(u) du ds.
\end{aligned}$$

The infinitesimal operator L[13] of $V(x_t)$ defined as

$$LV(x_t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \{ \mathrm{E}\{ V(x_{t+\Delta}) | x_t \} - V(x_t) \}. \tag{13}$$

Then, from (12) and (13), we get

$$LV(x_t) = LV_1(x_t) + LV_2(x_t) + LV_3(x_t), \tag{14}$$

where

$$LV_1(x_t) = 2\dot{x}^T(t)Px(t) + x^T(t)Q_1x(t) - x^T(t-h_M)Q_1x(t-h_M),$$
$$LV_2(x_t) = x^T(t)Q_2x(t) - (1-\dot{h}(t))x^T(t-h(t))Q_2x(t-h(t)),$$
$$LV_3(x_t) = h_M^2\dot{x}^T(t)R\dot{x}(t) - h_M\int_{t-h_M}^{t}\dot{x}^T(s)R\dot{x}(s)ds.$$

An upper-bound of $LV_2(x_t)$ can be

$$LV_2(x_t) \le x^T(t)Q_2x(t) - (1-h_d)x^T(t-h(t))Q_2x(t-h(t)). \tag{15}$$

By Lemma 1, an upper-bound of $LV_3(x_t)$ can be estimated as

$$\begin{aligned}LV_3(x_t) &\le h_M^2\dot{x}^T(t)R\dot{x}(t) - \frac{1}{1-\phi}\left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right)^T R\left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right) + \frac{1}{\phi}\left(\int_{t-h_M}^{t-h(t)}\dot{x}(s)ds\right)^T R\left(\int_{t-h_M}^{t-h(t)}\dot{x}(s)ds\right)\\ &= h_M^2\dot{x}^T(t)R\dot{x}(t) - \zeta^T(t)\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}^T\begin{bmatrix}\frac{1}{1-\phi}R & 0\\ 0 & \frac{1}{\phi}R\end{bmatrix}\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}\zeta(t),\end{aligned} \tag{16}$$

where $\phi = \dfrac{h_M - h(t)}{h_M}$.

From Lemma 2, if the inequality for any matrix M holds

$$\begin{bmatrix}R & M\\ M^T & R\end{bmatrix} \ge 0, \tag{17}$$

then, we obtain

$$\begin{bmatrix}-\sqrt{\frac{\phi(t)}{1-\phi(t)}}I & 0\\ 0 & \sqrt{\frac{1-\phi(t)}{\phi(t)}}I\end{bmatrix}\begin{bmatrix}R & M\\ M^T & R\end{bmatrix} \times \begin{bmatrix}-\sqrt{\frac{\phi(t)}{1-\phi(t)}}I & 0\\ 0 & \sqrt{\frac{1-\phi(t)}{\phi(t)}}I\end{bmatrix} \ge 0, \tag{18}$$

which implies

$$\begin{bmatrix}\frac{1}{1-\phi(t)}R & 0\\ 0 & \frac{1}{\phi(t)}R\end{bmatrix} \ge \begin{bmatrix}R & M\\ M^T & R\end{bmatrix}. \tag{19}$$

Then, an upper bound of (16) can be written as

$$LV_3(x_t) \le h_M^2\dot{x}^T(t)R\dot{x}(t) - \zeta^T(t)\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}^T\begin{bmatrix}R & M\\ M^T & R\end{bmatrix}\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}\zeta(t). \tag{20}$$

Note that when $h(t)=0$ or $h(t)=h_M$, we have $\zeta^T(t)(e_1-e_2)=0$ or $\zeta^T(t)(e_2-e_3)=0$, respectively. So relation (20) still holds.

Let us define $B_{z[\rho(t)]}$ as follows:

$$B_{z[\rho(t)]} = B_{z1[\rho(t)]} + B_{z2}, \tag{21}$$

where

$$B_{z1[\rho(t)]} = (A+BK)e_1^T + A_de_2^T - Ie_4^T + (\rho(t)+(1-\rho(t))d)B_we_5^T,$$
$$B_{z2} = DF(t)(E_ae_1^T + E_de_2^T).$$

Since the equality $B_{z[\rho(t)]}\zeta(t)=0$ with the system (6) satisfies, the following equality holds for any matrix Z

$$2\zeta^T(t)(e_1+e_4)ZB_{z[\rho(t)]}\zeta(t) = 0, \tag{22}$$

From the equalities (14) - (22), we obtain a delay-dependent stability condition for the system (6) as follows:

$$LV(x_t) \le \zeta^T(t)\{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_{z[\rho(t)]}\}\zeta(t) < 0, \tag{23}$$

where

$$\Phi_1 = e_1Q_1e_1^T - e_3Q_1e_3^T + e_1Pe_4^T + e_4Pe_1^T,$$
$$\Phi_2 = e_1Q_2e_1^T - (1-h_d)e_2Q_2e_2^T,$$
$$\Phi_3 = h_M^2e_4Re_4^T - \begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}^T\begin{bmatrix}R & M\\ M^T & R\end{bmatrix}\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix},$$
$$\Phi_{z[\rho(t)]} = (e_1+e_4)ZB_{z[\rho(t)]} + ((e_1+e_4)ZB_{z[\rho(t)]})^T.$$

By utilizing Lemma 3 from (23), we get

$$\begin{aligned}LV(x_t) \le \zeta^T(t)\big[&\lambda\{\Phi_1+\Phi_2+\Phi_3\}\\ &+\lambda\{(B_{z1[\rho(t)]}^T Z^T(e_1+e_4)^T + (e_1+e_4)ZB_{z1[\rho(t)]})\}\\ &+\lambda^2(e_1+e_4)ZDD^TZ^T(e_1^T+e_4^T)\\ &+(e_1E_a^T+e_2E_d^T)(e_1E_a^T+e_2E_d^T)^T\big]\zeta(t) < 0.\end{aligned} \tag{24}$$

Also, $\lambda P, \lambda Q_1, \lambda Q_2, \lambda R, \lambda M$ and λZ replacing $\overline{P}, \overline{Q_1}, \overline{Q_2}, \overline{R}, \overline{M}$ and $\overline{Z}$ to (17) and (24), we obtain respectively

$$\begin{bmatrix}\overline{R} & \overline{M}\\ \overline{M}^T & \overline{R}\end{bmatrix} \ge 0, \tag{25}$$

$$
\begin{aligned}
LV(x_t) \le \zeta^T(t)\Big[&\left\{\overline{\Phi_1}+\overline{\Phi_2}+\overline{\Phi_3}\right\} \\
&+\left\{(B_{z1[\rho(t)]}^T\overline{Z}^T(e_1+e_4)^T+(e_1+e_4)\overline{Z}B_{z1[\rho(t)]})\right\} \\
&+(e_1+e_4)\overline{Z}DD^T\overline{Z}^T(e_1^T+e_4^T) \\
&+(e_1E_a^T+e_2E_d^T)(e_1E_a^T+e_2E_d^T)^T\Big]\zeta(t)<0, \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
&\overline{\Phi_1}=e_1\overline{Q_1}e_1^T-e_3\overline{Q_1}e_3^T+e_1\overline{P}e_4^T+e_4\overline{P}e_1^T, \\
&\overline{\Phi_2}=e_1\overline{Q_2}e_1^T-(1-h_d)e_2\overline{Q_2}e_2^T, \\
&\overline{\Phi_3}=h_M^2e_4\overline{R}e_4^T-\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}^T\begin{bmatrix}\overline{R} & \overline{M}\\ \overline{M}^T & \overline{R}\end{bmatrix}\begin{bmatrix}e_1^T-e_2^T\\ e_2^T-e_3^T\end{bmatrix}.
\end{aligned}
$$

Assume that $\overline{Z}=\alpha\overline{P}$, we have

$$
\begin{aligned}
LV(x_t) \le \zeta^T(t)\Big[&\left\{\overline{\Phi_1}+\overline{\Phi_2}+\overline{\Phi_3}\right\} \\
&+\alpha\left\{(B_{z1[\rho(t)]}^T\overline{P}(e_1+e_4)^T+(e_1+e_4)\overline{P}B_{z1[\rho(t)]})\right\} \\
&+\alpha^2(e_1+e_4)\overline{P}DD^T\overline{P}(e_1^T+e_4^T) \\
&+(e_1E_a^T+e_2E_d^T)(e_1E_a^T+e_2E_d^T)^T\Big]\zeta(t)<0, \qquad (27)
\end{aligned}
$$

where α is positive scalar.

To obtain the control gain, let us define $X=\overline{P}^{-1}$ and $KX=Y$. Then, pre and post multiplying to (25) and (27), respectively, by $diag\{X,X\}$ and $diag\{X,X,X,X,X\}$ lead to

$$
\begin{bmatrix}U_3 & U_4\\ U_4^T & U_3\end{bmatrix}\ge 0, \qquad (28)
$$

$$
\zeta^T(t)(\Xi_1+\Xi_2+\Xi_3+\Xi_{4[\rho(t)]}+\Xi_5+\Lambda\Lambda^T)\zeta(t)<0. \qquad (29)
$$

Since $V(x_t)|_{t=0}=0$ and $V(x_t)|_{t=\infty}\ge 0$ under the zero initial condition, we obtain from (8) as follows:

$$
\begin{aligned}
J=\ &\mathrm{E}\left\{\int_0^\infty z^T(s)z(s)-\gamma^2w^T(s)w(s)ds\right\} \\
=\ &\mathrm{E}\left\{\int_0^\infty z^T(s)z(s)-\gamma^2w^T(s)w(s)+LV(x_s)ds\right\} \\
&-\mathrm{E}\left\{\int_0^\infty LV(x_s)ds\right\}. \\
=\ &\mathrm{E}\left\{\int_0^\infty z^T(s)z(s)-\gamma^2w^T(s)w(s)+LV(x_s)ds\right\} \\
&-\mathrm{E}\{V(x_\infty)\}+\mathrm{E}\{V(x_0)\}, \\
\le\ &\mathrm{E}\left\{\int_0^\infty z^T(s)z(s)-\gamma^2w^T(s)w(s)+LV(x_s)ds\right\}, \\
\le\ &\mathrm{E}\left\{\int_0^\infty \zeta^T(t)(\Xi_{[\rho(t)]}+\Lambda\Lambda^T)\zeta(t)ds\right\}. \qquad (30)
\end{aligned}
$$

With the inequality (28), $\mathrm{E}\left\{\zeta^T(t)(\Xi_{[\rho(t)]}+\Lambda\Lambda^T)\zeta(t)\right\}<0$ implies that $J<0$ for all $t>0$. Futhermore, by condition (4), $\mathrm{E}\left\{\zeta^T(t)(\Xi_{[\rho(t)]}+\Lambda\Lambda^T)\zeta(t)\right\}<0$ is equivalent to

$$
\zeta^T(t)(\Xi_{[\rho_0]}+\Lambda\Lambda^T)\zeta(t)<0. \qquad (31)
$$

By Schur complement, the inequality (31) is equivalent to the following condition:

$$
\begin{bmatrix}\Xi_{[\rho_0]} & \Lambda\\ \Lambda^T & -I\end{bmatrix}<0. \qquad (32)
$$

Therefore, the inequality (32) is sufficient condition guaranteeing $\mathrm{E}\left\{\zeta^T(t)(\Xi_{[\rho(t)]}+\Lambda\Lambda^T)\zeta(t)\right\}<0$. As a result, if the LMIs (28) and (32) are satisfied, we have $\|z(t)\|_2\le\gamma\|w(t)\|_2$ for all nonzero $w(t)\in L_2[0,\infty)$. Since the (28) and (32) are equivalent to (10) and (11), this proof is completed. ■

Remark 1. The proposed disturbance model $(\rho(t)+(1-\rho(t))d)B_ww(t)$ presented in (6) is more practical than $B_ww(t)$ since it is assumed that $w(t)$ occurred with stochastic properties. This idea has not been proposed in any other literature, which motivates this research.

4. Numerical Examples

In this section, we provide two numerical examples to show the effectiveness of the proposed criterion.

Example 1. Consider the physical plant system which is the satellite system [20] shown in Figure 1. The satellite system consists of two ridged bodies joined by a flexible link which is modeled as a spring with torque constant k and viscous damping f.

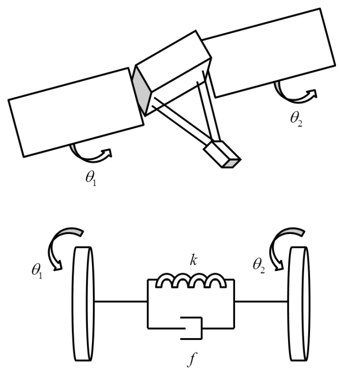


그림 1 위성시스템의 개략도

Fig. 1 Schematic diagram of satellite system

Denote that θ_1 and θ_2 are yaw angles for the two bodies (the main body and the instrumentation module). $u(t)$ is the control torque, and J_1 and J_2 are moments of inertia of the two bodies. The dynamic equations are given by

$$
\begin{aligned}
&J_1\ddot{\theta}_1(t)+f(\dot{\theta}_1(t)-\ddot{\theta}_2(t))+k(\theta_1(t)-\theta_2(t))=u(t), \\
&J_2\ddot{\theta}_2(t)+f(\dot{\theta}_1(t)-\ddot{\theta}_2(t))+k(\theta_1(t)-\theta_2(t))=0. \qquad (33)
\end{aligned}
$$

The above equations are represented a state-space as follows:

$$\Gamma\dot{x}(t)=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-k&k&-f&f\\k&-k&f&-f\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\\0\end{bmatrix}u(t), \tag{34}$$

where

$$\Gamma=diag\{1,1,J_1,J_2\},$$
$$x(t)=[x_1\ x_2\ x_3\ x_4]^T=[\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2]^T.$$

Considering the time-delay, uncertain and disturbances, we assumed that the satellite system is

$$\begin{cases}\dot{x}(t)=(A+DF(t)E_a)x(t)+(A_d+DF(t)E_d)x(t-h(t))\\ \qquad +(\rho(t)+(1-\rho(t))d)B_w w(t)+Bu(t),\\ z(t)=Cx(t),\end{cases} \tag{35}$$

where

$$A=\Gamma^{-1}\begin{bmatrix}0&0&1&0\\0&0&0&1\\-k&k&-f&f\\k&-k&f&-f\end{bmatrix},$$

$$A_d=\Gamma^{-1}\begin{bmatrix}0&0&0&0\\0&0&0&0\\-0.1k&0.1k&-0.1f&f0.1\\0.1k&-0.1k&0.1f&-0.1f\end{bmatrix},$$

$$B=\Gamma^{-1}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, B_w=0.5\Gamma^{-1}I, C=[1\,1\,0\,0],$$

$$D=\Gamma^{-1}, E_a=\begin{bmatrix}0\\0\\0.01\\0\end{bmatrix}, E_d=\begin{bmatrix}0\\0\\0.01\\0\end{bmatrix}, B=\Gamma^{-1}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}.$$

Also, the disturbances are defined as follows:

$$w(t)=[w_1^T(t),w_2^T(t),0,0]^T, \tag{36}$$

where

$$w_1(t)=\begin{cases}0.8(\sin(2\pi 10t)+1), & 5\le t\le 8,\\ 0, & otherwise,\end{cases}$$
$$w_2(t)=\begin{cases}\sin(2\pi 10t)+1, & 6\le t\le 9,\\ 0, & otherwise.\end{cases}$$

Also, $\rho(t)$ is Bernoulli distributed sequence defined by

$$\rho(t)=\begin{cases}1, \text{if disturbances are maximum},\\ 0, \text{if disturbances are decresed by } 0.4\times 100\%.\end{cases}$$

We choose $J_1=1, J_2=1, k=0.3$ and $f=0.0004$ (the values of k and f are chosen within their respective ranges).

By applying Theorem 1, the controller gain K that minimize value of γ when $h_M=1$, $h_d=0.5$ with various ρ_0 are listed in Table 1.

표 1 ρ_0에 따른 제어 이득 K, $\gamma_{\min}$, 그리고 α.

Table 1 Controller gain K, $\gamma_{\min}$ and α with different ρ_0.

Case	ρ_0	K	$\gamma_{\min}$	α
1	0.1	$\begin{bmatrix}-8.7398\\2.4163\\-5.3475\\-15.8547\end{bmatrix}^T$	0.75	0.68
2	0.5	$\begin{bmatrix}-8.4847\\2.4455\\-5.2721\\-15.1300\end{bmatrix}^T$	3.73	0.70
3	0.9	$\begin{bmatrix}-8.2409\\2.4644\\-5.1965\\-14.4460\end{bmatrix}^T$	6.70	0.72

Despite the same size of disturbances for each case, Table 1 shows that the H_∞ performance becomes worse as the occurrence probability of maximum disturbance increases.

For example, by comparing the results of Cases 1 and 2, when ρ_0 is increased from 0.1 to 0.9, $\gamma_{\min}$ is increased from 0.75 to 6.70. These results can be explained by decrease in the feasible region of asymptotically stability with $\|z(t)\|_2\le\gamma\|w(t)\|_2$ for all nonzero $w(t)\in L_2[0,\infty)$.

Moreover, Figure 2 shows that the closed-loop system is asymptotically stable with H_∞ disturbance attenuation level γ for any time-varying delay $h(t)$ satisfying (3).

Simulations in Figure 2 show the state responses in each case. As ρ_0 increase, disturbances have more influence on the state responses.

Example 2. Consider the system (6) with

$$A=\begin{bmatrix}-2&0\\0&-0.9\end{bmatrix}, A_d=\begin{bmatrix}-1&0\\-1&-1\end{bmatrix}, B=\begin{bmatrix}1\\1\end{bmatrix},$$
$$B_w=\begin{bmatrix}0.5&0\\0&0.5\end{bmatrix}, C=[1\,0],$$
$$D=I, E_a=\begin{bmatrix}0.1\\0\end{bmatrix}, E_d=\begin{bmatrix}0\\0.1\end{bmatrix} \tag{37}$$

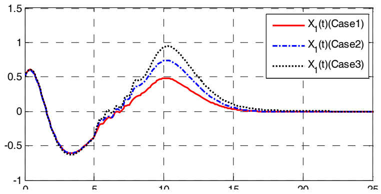
and $h_M=1.5$, $h_d=0.7$.

Moreover, the disturbances are defined as follows:

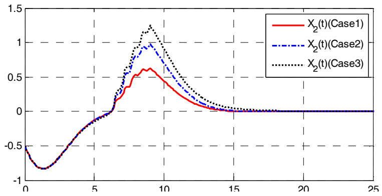
$$w(t)=[w_1^T(t),w_2^T(t)]^T, \tag{38}$$

where

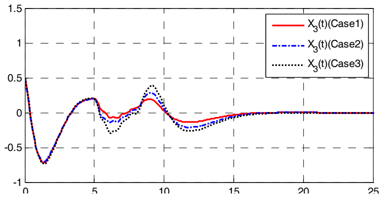
$$w_1(t)=\begin{cases}1, & 4\le t\le 7,\\ 0, & otherwise,\end{cases}$$
$$w_2(t)=\begin{cases}\sin(2\pi 10t)+1, & 5\le t\le 8,\\ 0, & otherwise.\end{cases}$$



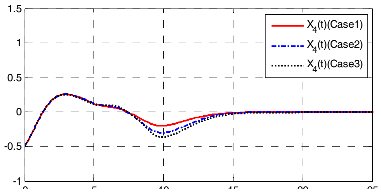
(a) Responses of $x_1(t)$ for each case



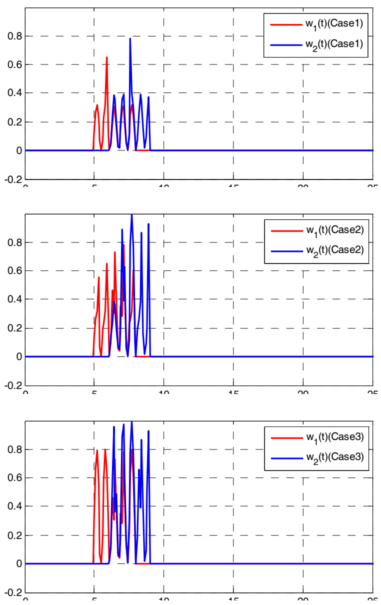
(b) Responses of $x_2(t)$ for each case



(c) Responses of $x_3(t)$ for each case



(d) Responses of $x_4(t)$ for each case



(e) Disturbances for each case

그림 2 표 1의 각각의 상황에 따른 위성 시스템의 시뮬레이션

Fig. 2 Simulations of satellite system for each case in Table 1

Also, $\rho(t)$ is considered with $d=0.2$.

By applying Theorem 1, minimum value of γ and controller gain K for system (6) when ρ_0 is 0.1, 0.5 and 0.9 are listed in Table 2.

표 2 ρ_0에 따른 제어 이득 K, $\gamma_{\min}$, 그리고 α.

Table 2 Controller gain K, $\gamma_{\min}$ and α with different ρ_0.

Case	ρ_0	K	$\gamma_{\min}$	α
1	0.1	$[-0.7800 \; -1.7459]$	0.283	0.34
2	0.5	$[-0.7846 \; -1.7531]$	1.414	0.34
3	0.9	$[-0.7842 \; -1.7525]$	2.545	0.34

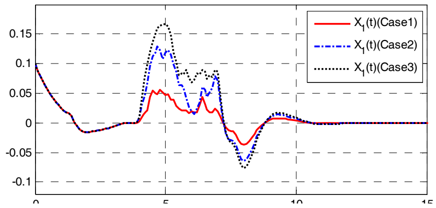
Table 2 shows results about effect of ρ_0 which increases minimum of H_∞ disturbance attenuation level γ. For example, we get $\gamma_{\min}=0.283$ for $\rho_0=0.1$ and $\gamma_{\min}=2.545$ for $\rho_0=0.9$.

Following figures show the linear system responses for each case in Table 2.

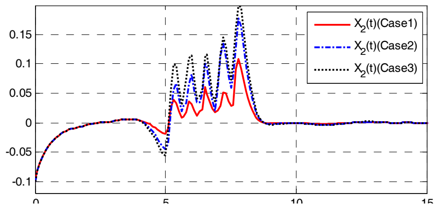
From Figure 3, we can see that the state responses for each case in Table 2. Furthermore, trajectories of disturbances show that maximum disturbances more frequently occur as ρ_0 increase. Thus, we can see that the state responses for Case 3 with $\rho_0=0.9$ are more worse then Case 1 with $\rho_0=0.1$. As a result, effects of disturbances on the state responses is affected by the occurrence probability of maximum disturbances, ρ_0.

5. Conclusions

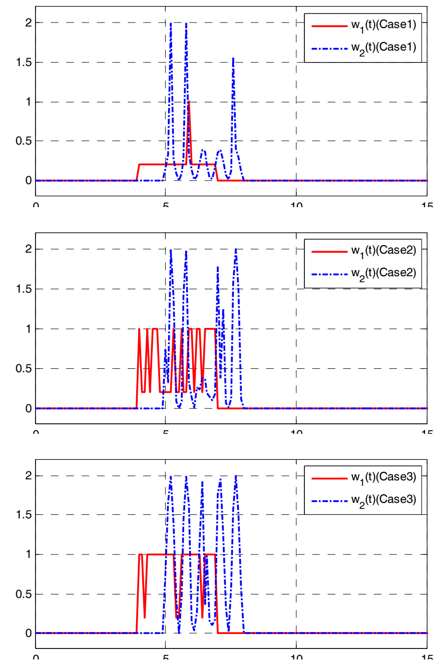
In this paper, the delay-dependent and probability dependent robust H∞ control for the uncertain linear systems with time-varying delay and randomly occurring disturbances has been addressed. Lyapunov-Krasovskii functional was structured to establish the stability criterion for the systems. We considered that the unknown matrices in uncertain systems are measured by Lebesgue elements which exist on current and delayed states. Also, randomly occurring disturbances have been considered that occurrence probability satisfies a Bernoulli distribution. The double integral terms of the Lyapunov-Krasovskii functional was handled using by reciprocally convex concept in [2]. Finally, based on LMI, designing linear memoryless state feedback robust controllers have been addressed and numerical examples have been given to show the effectiveness of the presented criterion.



(a) Responses of $x_1(t)$ for each case



(b) Responses of $x_2(t)$ for each case



(c) Disturbances for each case in Table 2

그림 3 표 2의 각각의 상황을 고려한 시뮬레이션
Fig. 3 Simulations for each case in Table 2

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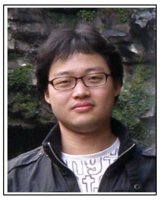

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