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ON MINIMAL NON-NSN-GROUPS

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Abstract. A finite group G is called an NSN-group if every proper subgroup of G is either normal in G or self-normalizing. In this paper, the non-NSN-groups whose proper subgroups are all NSN-groups are determined.

1. Introduction

The structure of the group whose subgroups are all normal (called a Dedekind group or a Hamiltonian group) has been completely classified by R. Dedekind, E. Wendt and R. Bare (see [9, Theorem 5.3.7]). Since then, many authors have dealt with generalizations of such kind of groups. We mention some of them here. Pic [8] considered finite groups in which every subgroup S is quasinormal, that is, S satisfies $SH = HS$ for all subgroups H of G, and Walls [11] studied groups with maximal subgroups of Sylow subgroups that are normal in G. Buckley et al. [2] dealt with groups in which all subgroups form at most two conjugate classes and Brandl [1] classified groups all of whose non-normal subgroups are conjugates.

If N is a normal subgroup of G, then N is normalized by all elements of G. For a normal subgroup, the number of elements of G normalizing N is up to maximum. On the other hand, if $N_G(N) = N$ for a proper subgroup N of G, then the number of elements of G normalizing N is up to minimum. Thus in some sense, the properties $N \trianglelefteq G$ and $N_G(N) = N$ can be viewed as two extreme cases in considering the number of elements normalizing N in G. Let $\mathscr{X}$ be a property of a group. A group G is called an $\mathscr{X}$-critical group or a minimal non-$\mathscr{X}$-group if G is not an $\mathscr{X}$-group but every proper subgroup of G is an $\mathscr{X}$-group. There are many remarkable examples of minimal non-$\mathscr{X}$-groups:

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minimal non-abelian groups (Miller and Moreno, [7]), minimal non-nilpotent groups (Schmidt), minimal non-supersoluble groups ([4]) and minimal non-p-nilpotent groups (Itô).

A group G whose every subgroup N has extreme numbers of elements normalizing N, that is, either $N_G(N) = G$ or $N_G(N) = N$, is called an NSN-group. The structure of NSN-groups has been investigated in [12]. In this paper, by applying the properties of an NSN-group, we classify all the minimal non-NSN-groups.

We first introduce the following definitions.

Definition 1.1. Let G be a finite group. Then G is called an NSN-group if every subgroup N of G is either normal in G or self-normalizing, that is, either $N_G(N) = G$ or $N_G(N) = N$.

Definition 1.2. A group G is called a minimal non-NSN-group if every proper subgroup of G is an NSN-group but G itself is not an NSN-group.

Our main results are as follows:

Main Theorem. *Suppose that G is a finite minimal non-NSN-group. Then G is solvable and G is isomorphic to one of the following groups*:

(1) $G = \langle x, y_1, y_2, \ldots, y_b \mid x^{p^a} = y_1^q = y_2^q = \cdots = y_b^q,\ y_iy_j = y_jy_i,\ i, j = 1, 2, \ldots, b,\ y_i^x = y_{i+1},\ i = 1, 2, \ldots, b-1,\ y_b^x = y_1^{d_1}y_2^{d_2}\cdots y_b^{d_b}\rangle$, *where* $f(z) = z^b - d_bz^{b-1} - \cdots - d_2z - d_1$ *is irreducible in* F_q *and* $f(z) | z^{p^a} - 1$.

(2) $G = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1,\ ba = abc,\ ca = ac,\ cb = bc\rangle$, *where* m *and* n *are natural numbers.*

(3) $G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ ba = a^{1+p^{m-1}}b\rangle$, *where* m *and* n *are natural numbers and* $m \geq 2$.

(4) $G = \langle a, b, c \mid a^{3m} = b^4 = c^4 = 1,\ b^2 = c^2,\ cb = b^{-1}c,\ a^{-1}ba = c,\ a^{-1}ca = cb\rangle$, *where* m *and* n *are natural numbers.*

(5) $G = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ b^{-1}ba = a^{-1}\rangle$.

(6) $G = PQ$, $P \trianglelefteq G$, *P is an elementary abelian p-group of rank* > 1, *Q is cyclic, and Q acts irreducibly on P.*

(7) $G = Q_8 \rtimes C_{3^m}$ *and* $Z(G) = \Omega_1(Q_8)\Phi(C_{3^m})$.

In the following (8)-(11), *p and q are distinct primes and* $p > q$.

(8) $G = \langle a, b, c \mid a^q = b^q = 1,\ c^p = 1,\ a^{-1}ca = c^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p\rangle$.

(9) $G = \langle a, b, c \mid a^{q^n} = b^q = 1,\ c^p = 1,\ a^{-1}ca = c^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p,\ n > 1\rangle$.

(10) $G = \langle a, b \mid a^{q^n} = 1,\ b^p = 1,\ a^{-1}ba = b^r\rangle$, *where* $n > 1$ *and the order of* r *modulo* p *is* q^2.

(11) $G = \langle a, b, c \mid a^p = b^p = 1,\ c^{q^n} = 1,\ c^{-1}ac = a^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p\rangle$.

(12) $G = C_p \times (C_r \rtimes C_{q^n})$, *where p, q and r are distinct primes. Moreover,* $Z(G) = C_p \times \Phi(C_{q^m})$.

(13) $G = C_r \rtimes (C_p \times C_q)$, *where* p, q *and* r *are distinct primes*, $r > q > p$, *and* $Z(G) = 1$.

Throughout this paper, only finite groups are considered and all our notations are standard. For example, we denote by $A \rtimes P$ the semidirect product of A and P; C_n denotes a cyclic group of order n and $\pi(G)$ denotes the set of all prime divisors of $|G|$. All unexplained notations can be found in [5] and [9].

2. Some preliminaries

In this section, we collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([5, 7.2.2]). *Suppose that the Sylow* p*-subgroups of* G *are cyclic, where* p *is the smallest prime divisor of* $|G|$. *Then* G *has a normal* p*-complement.*

Lemma 2.2 ([6]). *Suppose that* p'*-group* H *acts on a* p*-group* G. *Let*

$$\Omega(G) = \begin{cases} \Omega_1(G) & p > 2, \\ \Omega_2(G) & p = 2. \end{cases}$$

If H *acts trivially on* $\Omega(G)$, *then* H *acts trivially on* G *as well.*

Lemma 2.3 (Maschke's Theorem, [5, 8.4.6]). *Suppose that the action of* A *on an elementary abelian group* G *is coprime and* H *is an* A*-invariant direct factor of* G. *Then* H *has an* A*-invariant complement in* G.

Lemma 2.4 ([10]). *If* G *is a minimal nonabelian simple group, i.e. a nonabelian simple group all of whose proper subgroups are solvable, then* G *is isomorphic to one of the following simple groups*:

(1) $PSL(2, p)$, *where* p *is a prime with* $p > 3$ *and* $5 \nmid p^2 - 1$.

(2) $PSL(2, 2^q)$, *where* q *is a prime.*

(3) $PSL(2, 3^q)$, *where* q *is a prime.*

(4) $PSL(3, 3)$.

(5) *The Suzuki group* $Sz(2^q)$, *where* q *is an odd prime.*

In proving our main theorem, the following result will be frequently used.

Lemma 2.5 ([12, Main Theorem]). *Let* G *be a finite group. Then all subgroups of* G *are either normal or self-normalizing if and only if either*

(1) G *is a Dedekind group, or*

(2) $G = H \rtimes P$, *where* H *is an abelian normal Hall* p'*-subgroup and* $P = \langle x \rangle \in Syl_p(G)$, $\langle x^p \rangle = O_p(G) = Z(G)$, *where* p *is the minimal prime dividing the order of* G. *Furthermore,* x *induces a fixed-point-free power automorphism of order* p *on* H.

3. Proof of Main Theorem

We first note that the classification of minimal non-Dedekind groups was given in [3] and [7]. We list them in the following lemma.

Lemma 3.1. *Let G be a minimal non-Dedekind group. Then G is solvable and G is isomorphic to one of the following groups*:

(1) $G = \langle x, y_1, y_2, \ldots, y_b \mid x^{p^a} = y_1^q = y_2^q = \cdots = y_b^q,\ y_i y_j = y_j y_i,\ i, j = 1, 2, \ldots, b,\ y_i^x = y_{i+1},\ i = 1, 2, \ldots, b-1,\ y_b^x = y_1^{d_1} y_2^{d_2} \cdots y_b^{d_b} \rangle$, *where* $f(z) = z^b - d_b z^{b-1} - \cdots - d_2 z - d_1$ *is irreducible in* F_q *and* $f(z) \mid z^{p^a} - 1$.

(2) $G = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1,\ ba = abc,\ ca = ac,\ cb = bc \rangle$, *where m and n are natural numbers.*

(3) $G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ ba = a^{1+p^{m-1}} b \rangle$, *where m and n are natural numbers and $m \geq 2$.*

(4) $G = \langle a, b, c \mid a^{3m} = b^4 = c^4 = 1,\ b^2 = c^2,\ cb = b^{-1}c,\ a^{-1}ba = c,\ a^{-1}ca = cb \rangle$, *where m and n are natural numbers (minimal non-3-closed groups).*

(5) $G = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ b^{-1}ba = a^{-1} \rangle$.

By Lemma 3.1, it is enough to discuss minimal non-NSN-groups which are not minimal non-Dedekind groups. By Lemma 2.5, in what follows, every Sylow subgroup of G is a Dedekind 2-group or an abelian group of odd order. We will use this fact frequently in our following proof.

Lemma 3.2. *Let G be a minimal non-NSN-group. Then G is solvable.*

Proof. Suppose that G is not solvable. By Lemma 2.5, every proper subgroup of G is solvable and hence $G/\Phi(G)$ is a minimal simple group, where $\Phi(G)$ is the Frattini subgroup of G. Let H be the 2-complement of $\Phi(G)$. Then $H \trianglelefteq G$ and H is abelian since H is an NSN-group of odd order. We have following claims.

(1) $H = 1$.

Consider $H \neq 1$. Let $P \in Syl_p(H)$, where p is any prime in $\pi(H)$. Then $P \trianglelefteq G$. Let $S_2 \in Syl_2(G)$ and $K = S_2 P$. Then K is a proper subgroup of G, and hence K is an NSN-group by hypothesis. If K is an NSN-group as in (2) of Lemma 2.5, then S_2 is cyclic, which concludes that G has normal 2-complement, a contradiction. Hence we may assume that K is nilpotent. But it follows in this case that $S_2 \leq C_G(P) \trianglelefteq G$. Using the simplicity of $G/\Phi(G)$, we conclude that $S_2 \leq C_G(P)\Phi(G)$, which concludes that G is solvable, a contradiction.

(2) Every subgroup of order $2^m p$ (p an odd prime) of $\overline{G} = G/\Phi(G)$ is 2-nilpotent.

Assume that G possesses a subgroup L containing $S_0 = \Phi(G)$ such that L/S_0 is not a 2-nilpotent group of order $2^m p$. Then L contains a minimal non-2-nilpotent subgroup D with order $2^n p$ for some natural number n. Hence $D = S^* P$ is a minimal non-nilpotent group with a normal Sylow 2-subgroup S^* and $|P| = p$. Since given that G is non-solvable, D is a proper subgroup of G, D is an NSN-group by the hypothesis. Hence D is nilpotent by Lemma 2.5, a contradiction.

(3) Conclusion.

Now, we assert that there is no simple group listed in Lemma 2.4 isomorphic to $\overline{G}$. Then we get that G is solvable. In fact, if $\overline{G}$ is isomorphic to one of $PSL(2,p)$, $PSL(2,3^q)$ or $PSL(3,3)$, then $\overline{G}$ has a subgroup isomorphic to A_4, the alternating group of degree 4, a contradiction to (2). If $\overline{G} \cong PSL(2,2^q)$ or $Sz(2^q)$, then $\overline{G}$ is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group, again a contradiction to (2). Hence $\overline{G}$ cannot be any one of $PSL(2,2^q)$ nor $Sz(2^q)$. Thus the proof is completed. □

By Lemma 3.2, we always assume in the following that G is a solvable minimal non-NSN-group.

Lemma 3.3. *Let G be a minimal non-NSN-group. Then there exist at most two distinct primes $p,q \in \pi(G)$ such that the Sylow p-subgroup and the Sylow q-subgroup of G are not normal in G.*

Proof. Since G is solvable by Lemma 3.2, there is a normal maximal subgroup M of G such that $|G : M| = r$ for some prime r. By our assumption, M is an NSN-group. If M is a Dedekind group, then the unique possibility is that r-Sylow subgroups are not normal in G, which then the lemma follows. If M is not a Dedekind group, then M is a group described in (2) of Lemma 2.5. In this case, the possible non-normal Sylow subgroups are r-Sylow subgroups and one Sylow subgroup in M, from which the lemma follows. □

Lemma 3.4. *Let G be a minimal non-NSN-group. Suppose that there exist exactly two distinct primes $p,q \in \pi(G)$ such that the Sylow p-subgroup and the Sylow q-subgroup of G are not normal in G. Then $|\pi(G)| > 2$.*

Proof. Assume that $|\pi(G)| = 2$. Then there exists a normal maximal subgroup M of G such that $|G : M| = r$ for some prime r. By hypothesis, M is an NSN-group. If M is nilpotent, then G has a normal Sylow subgroup, a contradiction. Hence M is not nilpotent. Let Q be the non-normal Sylow subgroup of M and set $M = P^* \rtimes Q$, where $P^* \in Syl_p(M)$. Surely, it follows that $p = r$ since otherwise, P^* is a normal Sylow subgroup of G. Hence $G = PQ$, where $P \in Syl_p(G)$. By Lemma 2.5(2), we know that $p > q$ and that Q is cyclic. Moveover, the r-Syolw subgroup is normal by Lemma 2.1, a contradiction. Therefore, we get $|\pi(G)| > 2$. □

Lemma 3.5. *Let G be a minimal non-NSN-group. Suppose that there exist exactly two distinct primes $p,q \in \pi(G)$ such that the Sylow p-subgroup and the Sylow q-subgroup of G are not normal in G. Then $|\pi(G)| = 3$.*

Proof. If $|\pi(G)| \neq 3$, then $|\pi(G)| \geq 4$ by Lemma 3.4. Since G is solvable, we may assume that $\{P_1, P_2, \ldots, P_k, \ldots, P_s\}$ is a Sylow system of G such that P_1, P_2 are not normal in G. Consider the subgroup $K = P_1P_2$. One has that $K < G$ is an NSN-group and that K is a group described in (2) of Lemma 2.5.

Without loss of generality, assume that $P_2 \trianglelefteq K$ and P_1 is cyclic. Let P_k be a Sylow subgroup not normalizing P_2. Consider the subgroup $G_1 = P_1P_2P_k < G$. One has that G_1 is also an NSN-group, but $P_2 \trianglelefteq P_1P_2 \leq N_{G_1}(P_2)$. It follows that $N_{G_1}(P_2) = G_1$, a contradiction to the choice of P_k. This concludes the Lemma. □

Lemma 3.6. *Let G be a minimal non-NSN-group. Suppose there exists a unique prime $q \in \pi(G)$ such that the Sylow q-subgroup of G is not normal in G. Then G must be one of the following groups*:

(1) $G = C_p \times (C_r \rtimes C_{q^n})$, *where p, q, r are distinct primes and $Z(G) = C_p \times \Phi(C_{q^n})$.*

(2) $G = PQ$, $P \trianglelefteq G$, *P is an elementary abelian p-group of rank > 1, Q is cyclic, and Q acts irreducibly on P.*

(3) $G = Q_8 \rtimes C_{3^m}$ *and* $Z(G) = \Omega_1(Q_8)\Phi(C_{3^m})$.

In the following (4)-(7), *p and q are distinct primes such that $p > q$.*

(4) $G = \langle a, b, c \mid a^q = b^q = 1,\ c^p = 1,\ a^{-1}ca = c^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p\rangle$.

(5) $G = \langle a, b, c \mid a^{q^n} = b^q = 1,\ c^p = 1,\ a^{-1}ca = c^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p,\ n > 1\rangle$.

(6) $G = \langle a, b \mid a^{q^n} = 1,\ b^p = 1,\ a^{-1}ba = b^r\rangle$, *where $n > 1$ and the order of r modulo p is q^2.*

(7) $G = \langle a, b, c \mid a^p = b^p = 1,\ c^{q^n} = 1,\ c^{-1}ac = a^r,\ [b,\ a] = [b,\ c] = 1,\ r \not\equiv 1 \pmod p,\ r^q \equiv 1 \pmod p\rangle$.

Proof. Let $Q \in Syl_q(G)$ be a non-normal Sylow subgroup of G. Then $G = A \rtimes Q$ by hypothesis. Here, we have that A is nilpotent and all Sylow subgroups of G are Dedekind groups by our hypothesis. Our proof is divided into two cases:

Case 1. $|\pi(G)| \geq 3$.

If $PQ = P \rtimes Q$ for every Sylow p-subgroup P satisfying $p \neq q$, then we have $\Phi(Q) \leq Z(G)$ and every subgroup of P is normal in PQ by Lemma 2.5(2). Hence they are all normal in G as A is nilpotent. We first claim that there exists a Sylow subgroup P of G satisfying $N_P(Q) \neq 1$. In fact, there exists a proper subgroup H such that H is neither normal nor self-normalizing by hypothesis, that is, $H < N_G(H) < G$. Since all subgroups of Sylow p-subgroups are normal in G for every $p \neq q$, H itself is an NSN-group. By Frattini argument, we have $N_G(H) = N_{N_G(H)}(Q)H = N_{N_G(H)}(Q)N > NQ$, which implies that $N_{N_G(H)}(Q) > Q$ and our claim holds. Now, RQ is a proper subgroup of G and so RQ is an NSN-group. Hence $RQ = R \times Q$, that is, $G = P \times (B \rtimes Q)$ for some subgroup B of G. We now claim that P and B are groups of prime order. Suppose that P has a nontrivial subgroup P^*. Then $G_2 = P^* \times (B \rtimes Q)$ is an NSN-group and $Q \ntrianglelefteq G_2$. But $P^*Q \leq N_{G_2}(Q)$, which implies that B normalizes Q, and so does G, a contradiction. This implies that P is a group of prime order. We claim that B is a group of prime power order. Otherwise, let R be any Sylow subgroup of B. Then $G_3 = P \times (RQ)$ is an NSN-group. Since Q is not normal in G, we may assume that $R \nleq N_G(Q)$ without loss of generality. In this case, $Q \ntrianglelefteq G_3$, but $PQ \leq N_{G_3}(Q)$, a contradiction. Hence $R = B$ is a Sylow subgroup of G and $G = P \times (R \rtimes Q)$. Assume that $\Omega_1(R) \neq R$. Then $P \times (\Omega_1(R)Q)$ is an NSN-group by hypothesis and so $\Omega_1(R) \leq C_G(Q)$ by Lemma 2.5. Now by applying Lemma 2.2 we have that $R \leq C_G(Q)$, a contradiction. This contradiction concludes that $R = \Omega_1(R)$ is an elementary abelian r-group. Moreover, every subgroups of R is normal in RQ as RQ is an NSN-group. Let $R = \langle v_1\rangle \times \langle v_2\rangle \times \cdots \times \langle v_n\rangle$. If $n > 1$, then $P \times (\langle v_i\rangle Q)$ is an NSN-group for each $i \in \{1, 2, \ldots, n\}$. It follows that $\langle v_i\rangle \leq C_G(Q)$ for each $i \in \{1, 2, \ldots, n\}$. Therefore, $R \leq C_G(Q)$, a contradiction. Thus $n = 1$ and so $B = R$ is of prime order. Moreover, since RQ is a non-nilpotent NSN-group, we have that Q is cyclic by Lemma 2.5(2). Hence $G = C_p \times (C_r \rtimes C_{q^n})$, where p, q, r are distinct primes and $Z(G) = C_p \times \Phi(C_{q^n})$, that is, G is of type (1).

Case 2. $|\pi(G)| = 2$.

In this case, $G = P \rtimes Q$, where P is a Sylow p-subgroups of G. We have the following two claims.

Claim 1. If $N_P(Q) = 1$, then G is isomorphic to one of the groups of types (2) or (4)-(7).

Suppose that $|P| = p$. If Q has two maximal subgroups Q_1 and Q_2, then Q_1P and Q_2P are NSN-groups. In this case, not both Q_1 and Q_2 can centralize P. By Lemma 2.5(2), we have $p > q$ and $\Phi(Q_i) \leq Z(G)$, where $i = 1, 2$. If all maximal subgroups of Q are cyclic, then Q is isomorphic to the quaternion group Q_8 or an elementary abelian q-subgroup of order q^2 by the structure of Dedekind groups.

Assume that $Q = Q_8$. Then $G = P \rtimes Q_8$ and $Z(G) = \Omega_1(Q_8)$. However, $Q_8/Z(G) = Q_8/C_{Q_8}(P) \cong N_G(P)/C_G(P)$ is cyclic, which is impossible since $Q_8/\Omega_1(Q_8)$ is a Klein 4-group. Hence $Q \neq Q_8$.

Assume that Q is an elementary abelian q-subgroup of order q^2. Then $G = P \rtimes Q$. Since $Q \cong Q/Z(G) = Q/C_Q(P) \cong N_G(P)/C_G(P) \leq \mathrm{Aut}(P)$ is cyclic, we have that $|Z(G)| = q$. Hence $G = P \rtimes Q = (P \rtimes \langle x\rangle) \times \langle y\rangle$, where $\langle x\rangle \times \langle y\rangle = Q$, that is, G is of type (4).

If Q has a non-cyclic maximal subgroup, then it is easy to see that there is a unique non-cyclic maximal subgroup Q^* in Q and $Q^*P = Q^* \times P$. Hence we may assume that $Q = Q_1 \times Q_2$, where Q_1 is a cyclic maximal subgroup of Q and Q_2 is of prime order. Obviously, we have that $Q_2 < Q^*$ and that Q_1P is a non-nilpotent NSN-group. By Lemma 2.5(2), we have $p > q$ and $\Phi(Q_i) \leq Z(G)$. Then $G = C_q \times (P \rtimes C_{q^{n-1}})$ and $|Z(G)| = q^{n-1}$, that is, G is of type (5).

If Q is cyclic, then there exists a proper subgroup $X < Q$ such that X is neither normal in G nor self-normalizing since G is not an NSN-group. By hypothesis, PX is an NSN-group. Obviously, X is the maximal subgroup of Q (otherwise, X would be normal in G) and $p > q$. Moreover, since $\Phi(X) \leq Z(PX)$ by Lemma 2.5(2), we have $\Phi(X) \leq Z(G)$. Let $|Q| = q^n$. Then $n > 1$ and $|Z(G)| = q^{n-2}$. Now we can conclude that G is of type (6) from the lemma.

Suppose that $|P| > p$. We assert that $\Omega_1(P) = P$. Otherwise, assume that $\Omega_1(P) < P$. Then $\Omega_1(P)Q < G$ and hence $\Omega_1(P)Q$ is an NSN-group. By Lemma 2.5(2), we have $p > q$ and every subgroup of $\Omega_1(P)$ is normalized by Q, Q is cyclic and $\Phi(Q) \leq C_G(\Omega_1(P))$. It follows by Lemma 2.2 that $\Phi(Q) \leq Z(G)$. Thus G is an NSN-group, a contradiction. Hence $\Omega_1(P) = P$. Therefore, P is an elementary abelian group.

Let $P = \langle v_1\rangle \times \langle v_2\rangle \times \cdots \times \langle v_n\rangle$, $n > 1$. Suppose that Q acts reducibly on P. Then by Lemma 2.3, there exist two Q-invariant proper subgroups A and B of P such that $P = A \times B$. Now, both AQ and BQ are proper subgroups of G and Q centralizes neither A nor B since $N_P(Q) = 1$. By Lemma 2.5(2), we have that every subgroup of A and of B is Q-invariant, that is, each $\langle v_i\rangle \trianglelefteq G$. So $Q\langle v_i\rangle$ is an NSN-group. One has that Q is cyclic and $\Phi(Q) \leq C_G(v_i)$ for every i satisfying $v_i \notin C_G(Q)$

An element a is said to act on V by scalars if there exists an integer m such that $a^{-1}va = v^m$ for all v in V. We claim that not every element Q acts by scalars on P. Assume that Q acts by scalars on P. Then every subgroup of P is normal in G and hence $\Phi(Q) = Z(G)$. Hence G is an NSN- group, a contradiction. Thus our claim holds. Let a be an element of Q not acting by scalars on P. Then we may choose v_1 and v_2 so that $a^{-1}v_1a = v_1^{m_1}$ and $a^{-1}v_2a = v_2^{m_2}$, where $m_1 \not\equiv m_2 \pmod p$. Thus $\langle v_1v_2\rangle$ is not a normal subgroup of $\langle a\rangle\langle v_1, v_2\rangle$. Hence $\langle a\rangle\langle v_1, v_2\rangle$ is not an NSN-group, so $G = \langle a\rangle\langle v_1, v_2\rangle$, $Q = \langle a\rangle$ is cyclic and $P = \langle v_1, v_2\rangle$ is of order p^2. That is, G is of type (7).

Suppose that Q acts irreducibly on P. We claim that Q is cyclic. Indeed, if Q has two maximal subgroups Q_1 and Q_2, then PQ_1 and PQ_2 are distinct NSN-groups. By Lemma 2.5, every subgroup of P is normalized by both Q_1 and Q_2 and hence normalized by Q, a contradiction to the action being irreducible. Thus Q has a unique maximal subgroup, i.e., Q is cyclic. It follows that G is a minimal non-NSN-group of type (2).

Claim 2. If $N_P(Q) > 1$, then G is isomorphic to one of the groups of types (3) or (7).

Let
$$\Omega(P) = \begin{cases} \Omega_1(P) & p > 2, \\ \Omega_2(P) & p = 2. \end{cases}$$
We assert that $\Omega(P) = P$. Otherwise, if $\Omega(P) \neq P$, then p'-group Q acts nontrivially on p-group $\Omega(P)$ by Lemma 2.2. Hence $\Omega(P)Q$ is an NSN-group by hypothesis and so we have $p > q$ by Lemma 2.5(2). Since $N_P(Q) > 1$, we have that $N_{\Omega_1(P)Q}(Q) = \Omega_1(P)Q$, so $\Omega_1(P)Q = \Omega_1(P) \times Q$, Q acts trivially on $\Omega(P)$, a contradiction. Thus $\Omega(P) = P$.

Now we claim that $C_P(Q) = N_P(Q) > 1$. In fact, let $N = N_P(Q) = P \cap N_G(Q) \trianglelefteq N_G(Q)$. Then $Q \leq N_G(N)$ and so $NQ = N \times Q$, which implies that $N \leq C_P(Q)$. Hence $C_P(Q) = N = N_P(Q)$. If $\Omega_1(P) \lneq P$, then $\Omega_1(P)Q$ is an NSN-group and so $\Omega_1(P) \leq C_P(Q)$ by the structure of NSN-groups.

Suppose that $P = Q_8 \times E$, where E is a nontrivial elementary abelian 2-subgroup. If $Q_8 \leq C_P(Q)$, then $P \leq C_P(Q)$ since $\Omega_1(P) \leq C_P(Q)$, a contradiction. Hence $Q_8 \lneq C_P(Q)$. Moreover, let Q^* be a maximal subgroup of Q. Then PQ^* is an NSN-groups. By Lemma 2.5, we have $Q^* \leq C_G(P)$, which implies that Q has a unique maximal subgroup, that is, Q is cyclic. We now prove that G is a minimal non-nilpotent group. Let H be a nontrivial subgroup of G. If H does not contain any Sylow q-subgroup of G, then H is obviously nilpotent since $\Omega_1(P) \times \Phi(Q) \leq Z(G)$. So we may assume that $Q \leq H$ without loss of generality. If is not nilpotent, then we have by Lemma 2.5 that $H = Q \rtimes D$ is an NSN-group, where D is a cyclic subgroup with order 4. Consider the group $F = (Q \rtimes D) \times E$. Obviously, F is a proper subgroup of G and so is an NSN-group. But on the other hand, D is neither normal in G nor self-normalizing, a contradiction. Thus H is nilpotent and so G is a minimal non-nilpotent group. By the structure of minimal non-nilpotent groups, we know that G is a minimal non-Dedekind-group, a contradiction.

If $P = Q_8$, then $G = Q_8 \rtimes Q$. Since $\mathrm{Aut}(Q_8) = S_4$, we have that Q is a 3-group. Now we claim that Q is cyclic. Otherwise, let Q_1 and Q_2 be two distinct maximal subgroups of Q. Then $Q_8Q_i, i = 1, 2$, are NSN-groups by hypothesis and hence $Q_8Q_i = Q_8 \times Q_i$ for $\Omega_1(Q_8) \leq C_G(Q_i), i = 1, 2$. It follows that Q is normal in G, a contradiction. Thus $G = Q_8 \rtimes C_{3^m}$ and $Z(G) = \Omega_1(Q_8)\Phi(C_{3^m})$. That is, G is of type (3).

If $\Omega_1(P) = P$, then P is an elementary abelian p-subgroup. In this case, $C_P(Q)$ is a Q-invariant direct factor of P. Applying Lemma 2.3, we get that there exist two Q-invariant proper subgroups A and B of P such that $P = A \times B$. Now, both AQ and BQ are proper subgroups of G. Since Q is not normal in G, not both A and B can centralize Q. Without loss of generality, suppose that $A \nleq C_P(Q)$. Notice that $C_P(Q) = N_P(Q) > 1$. One has $AQ \lneq G$ and so AQ is a non-nilpotent NSN-group. By Lemma 2.5(2), $p > q$ and every subgroup of A is Q-invariant, Q is cyclic and $\Phi(Q)$ centralizes A. Let X be a minimal subgroup of A of order p such that $X \nleq C_P(Q)$ and Y be any minimal subgroup of $C_P(Q)$. Then $Y \times (X \rtimes Q)$ is not an NSN-group since $X < N_G(X) = X \times Y \times \Phi(Q) \lneq G$. Thus we get $G = Y \times (X \rtimes Q)$, $P = X \times Y$ is an elementary abelian p-subgroup of order p^2, and Q is cyclic. Suppose that $|Q| = q^n$. Then we have obviously that $|Z(G)| = pq^{n-1}$. It follows that G is of type (7). Thus the proof is completed. □

Lemma 3.7. *Let G be a minimal non-NSN-group. Suppose there exists exactly two distinct primes $p, q \in \pi(G)$ such that the Sylow p-subgroups and the Sylow q-subgroups of G are not normal in G. Then G is isomorphic to $G = C_r \rtimes (C_p \times C_q)$, where p, q and r are distinct primes, $r > q > p$ and $Z(G) = 1$.*

Proof. By Lemma 3.5, we may assume that $G = PQR$, where $P \in Syl_p(G), Q \in Syl_q(G)$ and $R \in Syl_r(G)$, P and Q are not normal in G and $R \trianglelefteq G$. Then by Lemma 2.5 and assumption, both P and Q are cyclic.

(1) Suppose that $PQ = P \times Q$. Then both PR and QR are non-nilpotent NSN-groups and $N_R(P) = N_R(Q) = 1$. By Lemma 2.5(2), we may choose a maximal subgroup R^* of R. The subgroup $K = R^* \rtimes (P \times Q)$ is an NSN-group

by hypothesis and $P \ntrianglelefteq K$ since $N_R(P) = 1$. However, $N_K(P) \geq PQ > P$, a contradiction. Thus R is cyclic of prime order and $G = C_r \rtimes (P \times Q)$. Since $\Phi(P) \leq Z(G)$, we know that $L = C_r \rtimes (\Phi(P) \times Q) = \Phi(P) \times (C_r \rtimes Q)$ is an NSN-group. In this case, $Q \ntrianglelefteq L$, but $N_L(Q) \geq Q\Phi(P)$. Hence $\Phi(P) = 1$ and P is cyclic of prime order. By the same argument, we have that Q is also of prime order. Thus $G = C_r \rtimes (C_p \times C_q)$, where p, q and r are distinct primes, $p, q < r$ and $Z(G) = 1$.

(2) Suppose that $PQ = P \rtimes Q$. Then $PR = R \rtimes P$ and $N_R(P) = 1$. Choose a maximal subgroup R^* of R and let $T = R^* \rtimes (P \rtimes Q)$. Then T is an NSN-group by hypothesis and $P \ntrianglelefteq T$ if $R^* \neq 1$. However, $N_T(Q) \geq PQ > Q$, hence $R^* = 1$ and R is cyclic of prime order. We have the following two cases:

(i) Suppose $RQ = R \rtimes Q$. Let Q^* be the maximal subgroup of Q. Then $U = C_r \rtimes (P \rtimes Q^*)$ is an NSN-group and $P \ntrianglelefteq U$. On the other hand, $N_U(P) \geq PQ^* > P$, a contradiction. Hence $Q^* = 1$ and Q is of prime order and so $G = C_r \rtimes (C_{p^m} \rtimes C_q)$. Let $V = C_{p^m} \rtimes C_q$. Then $V/C_V(C_r) \leq Aut(C_r)$ is a cyclic group, which forces that $C_{p^m} \leq C_V(C_r)$. Then $C_{p^m} \trianglelefteq G$, a contradiction.

(ii) Suppose $RQ = R \times Q$. Let P^* be the maximal subgroup of P. Then $W = C_r \rtimes (P^* \rtimes Q)$ is an NSN-group and $Q \ntrianglelefteq W$. On the other hand, $N_W(Q) \geq C_rQ > Q$, a contradiction. Hence $P^* = 1$ and P is of prime order. Hence $G = C_r \rtimes (C_p \rtimes C_{q^n})$. By the same argument as that in the above paragraph, we come to a contradiction. Thus the proof is completed. □

Proof of Main Theorem. It follows from Lemma 3.1, Lemma 3.2, Lemma 3.6 and Lemma 3.7. □

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References

[1] R. Brandl, *Groups with few non-normal subgroups*, Comm. Algebra **23** (1995), no. 6, 2091–2098.
[2] J. Buckley, J. C. Lennox, and J. Wieglod, *Generalizations of Hamiltonian groups*, Ricerche Mat. **41** (1992), no. 2, 369–376.
[3] S. Chen, *Classification of finite groups whose maximal subgroups are Dedekind groups*, J. Zhejiang Univ. Sci. Ed. **29** (2002), no. 2, 121–124.
[4] K. Doerk, *Minimal nicht überauflösbare endliche Gruppen*, Math. Z. **91** (1966), 198–205.
[5] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups*, An Introduction, New York: Springer-Verlag, 2004.
[6] T. J. Laffey, *A lemma on finite p-group and some consequences*, Proc. Cambr. Phil. Soc. **75** (1974), 133–137.
[7] G. A. Miller and H. C. Moreno, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc. **4** (1903), no. 4, 398–404.
[8] Gh. Pic, *On the structure of quasi-Hamiltonian groups*, Acad. Repub. Pop. Române Bul. Sti. A. **1** (1949), 973–979.
[9] D. J. S. Robinson, *A Course in the Theory of Groups*, New York, Springer, 1993.
[10] J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), 383–437.

[11] G. L. Walls, *Groups with maximal subgroups of Sylow subgroups normal*, Israel J. Math. **43** (1982), no. 2, 166–168.

[12] Q. Zhang, *Groups with only normal and self-normalzing subgroups*, J. Shanxi Normal University (Science Edition) **3** (1989), no. 2, 11–13(in Chinese).

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