J. Korean Math. Soc. **50** (2013), No. 3, pp. 509–527
http://dx.doi.org/10.4134/JKMS.2013.50.3.509

EXPLICIT EXPRESSION OF THE KRAWTCHOUK POLYNOMIAL VIA A DISCRETE GREEN'S FUNCTION

Gil Chun Kim and Yoonjin Lee

Abstract. A Krawtchouk polynomial is introduced as the classical MacWilliams identity, which can be expressed in weight-enumerator-free form of a linear code and its dual code over a Hamming scheme. In this paper we find a new explicit expression for the p-number and the q-number, which are more generalized notions of the Krawtchouk polynomial in the P-polynomial schemes by using an extended version of a discrete Green's function. As corollaries, we obtain a new expression of the Krawtchouk polynomial over the Hamming scheme and the Eberlein polynomial over the Johnson scheme. Furthermore, we find another version of the MacWilliams identity over a Hamming scheme.

1. Introduction

Let $\mathcal{C}$ be a linear code over a finite field $\mathbb{F}_q$ with q elements of length d. The MacWilliams identity for linear codes over $\mathbb{F}_q$ is one of the most important identities in coding theory, which expresses the Hamming weight enumerator of the dual code $\mathcal{C}^\perp$ of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ in terms of the Hamming weight enumerator of $\mathcal{C}$. Let $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ (respectively, $\mathbf{b} = (b_0, b_1, \ldots, b_d)$) be the weight distributions of $\mathcal{C}$ (respectively, $\mathcal{C}^\perp$). Then MacWilliams identity can be expressed in weight-enumerator-free form as [5, 8, 9]:

$$\mathbf{a} = \frac{1}{|\mathcal{C}^\perp|}\ \mathbf{b}\ (p_j(i)),$$

where $p_j(i) = p_j(i; d, q)$ is the *Krawtchouk polynomial* defined by

$$p_j(i) = \sum_{l=0}^{j} (-1)^l (q-1)^{j-l} \binom{i}{l}\binom{d-i}{j-l}\ (i, j = 0, 1, \ldots, d).$$

Received April 25, 2012; Revised December 18, 2012.

2010 *Mathematics Subject Classification.* Primary 58B34, 58J42, 81T75.

Key words and phrases. discrete Green's function, P-polynomial scheme, p-number, q-number, Krawtchouk polynomial, Eberlein polynomial.

The second named author was supported by Priority Research Centers Program through the NRF funded by the Ministry of Education, Science and Technology(2009-0093827) and by the NRF grant funded by the Korea government(MEST)(2012-0005432).

Delsarte introduced the association scheme in [5]. Association schemes are important notions in algebraic graph theory and in coding theory. The classes of strongly regular graphs, distance regular graphs and symmetric circulants are instances of association schemes. In particular, the Hamming scheme and the Johnson scheme are the most important association schemes [1, 5, 6]. The Hamming scheme is mainly concerned with the distance of a code. In other words, the linear programming method produces upper bounds for the size of a code with given minimum distance, and lower bounds for the size of a design with given strength.

More generalized notions of the Krawtchouk polynomial in association schemes are the p-number and the q-number of the association schemes. One of the most important association schemes is a P-polynomial scheme [5, 6, 8]. In fact, the Hamming scheme and the Johnson scheme are the P-polynomial schemes, and the p-number and the q-number in the P-polynomial schemes are applied to finding some universal bounds for codes and designs [5, 6, 8].

A Green's function is introduced in a famous essay by George Green in 1728. In [4], a discrete Green's function is defined on graphs. The Green function is closely associated with the normalized Laplacian $\mathcal{L}_\beta$ and is useful for solving discrete Laplace equations with boundary conditions. In [2, 3], F. Chung introduced the relationship between the PageRank and a discrete Green's function $\mathcal{G}_\beta$ with a positive real number β. A Green's function $\mathcal{G}_\beta$ can be explained with an inverse relation of the β-normalized Laplacian $\mathcal{L}_\beta$ represented by an adjacency matrix.

In this paper we find a new explicit expression (Theorem 4) for the p-number and the q-number by using an extended version of a discrete Green's function, called a *normalized Green's function* $\mathcal{G}_{\beta,\mathcal{N}}$; this is expressed by a basis of a nullspace of some $d \times (d+1)$ matrix L_{sub} (associated with the P-polynomial scheme and $\mathcal{G}_{\beta,\mathcal{N}}$) and the adjacency matrices of the P-polynomial scheme for $\beta \in \mathbb{R}$. As corollaries, we obtain a new expression of the Krawtchouk polynomial over the Hamming scheme and the Eberlein polynomial over the Johnson scheme. Furthermore, we find another version of the MacWilliams identity over a Hamming scheme (Corollary 5.6), and we also obtain another expression of the Eberlein polynomial $E_i(j)$ as p-number over the Johnson scheme $J(v, d)$ (Corollary 6.2).

In more detail, for some $\beta_j \in \mathbb{R}$, we show that the j-th column vector of the second eigenmatrix $\mathbf{Q} = (q_j(i))$ is the vector $(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})$ which is contained in a nullspace of some $d \times (d+1)$ matrix $L_{sub}^{(\beta_j)}$ (Proposition 4.3). Furthermore, we obtain the p-number by using $u_i^{(j)}$ and the relations with the q-number over the association schemes (Corollary 4.4). As a main result, we show that the p-number $p_j(i)$ and the q-number $q_j(i)$ can be explicitly expressed by a determinant of a submatrix $L_i^{(\beta_j)}$ of $d \times (d+1)$ matrix $L_{sub}^{(\beta_j)}$ with $\beta_j = \frac{p_1(j)}{k_1} - 1$ ($j = 0, 1, \ldots, d$, k_1 is a valency of R_1) (Theorem 4.1). In Corollary 5.1, we show that the Krawtchouk polynomial $p_j(i)$ over the Hamming scheme $H(d, q)$ can be explicitly expressed by a determinant of a submatrix $L_i^{(\beta_j)}$ of $d \times (d+1)$ matrix $L_{sub}^{(\beta_j)}$ with $\beta_j = -\frac{qj}{k_1}$ $(j = 0, 1, \ldots, d)$ and $k_1 = d(q-1)$.

This paper is organized as follows. In Section 2, we introduce basic facts of the P-polynomial scheme and a Green's function $\mathcal{G}_\beta$. In Section 3, we introduce a $d \times (d+1)$ matrix L_{sub}, and we define a normalized Green's function $\mathcal{G}_{\beta,\mathcal{N}}$. In Section 4, for $\beta_j \in \mathbb{R}$, we find the relationship between a normalized Green's function $\mathcal{G}_{\beta_j,\mathcal{N}}$ and the p-number $p_j(i)$ (or the q-number $q_j(i)$). Moreover, we obtain the relationship between determinants of submatrices $L_i^{(\beta_j)}$ of $L_{sub}^{(\beta_j)}$ and the p-number (or the q-number). In Section 5, we obtain the relationship between determinants of submatrices $L_i^{(\beta_j)}$ of $L_{sub}^{(\beta_j)}$ and the Krawtchouk polynomial $p_j(i)$. We thus obtain another version of the MacWilliams identity over $H(d,q)$. Finally, in Section 6, we obtain the Eberlein polynomial $E_i(j)$ as the p-number over the Johnson scheme $J(v,d)$ using the determinants of submatrices $L_i^{(\beta_j)}$ of $L_{sub}^{(\beta_j)}$.

2. Preliminaries

In this section we introduce basic facts on association schemes and the discrete Green's function.

Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be an association scheme and $d_M(x,y)$ be a metric over X. We describe the relations by their adjacency matrices A_i $(i = 0, 1, \ldots, d)$ which are the $|X| \times |X|$ matrices defined by

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } d_M(x,y) = i, \\ 0, & \text{otherwise.} \end{cases}$$

A Bose Mesner algebra $\mathcal{A}$ is generated by the adjacency matrices A_i, that is, $\mathcal{A} = \{\sum t_i A_i \mid t_0, t_1, \ldots, t_d \in \mathbb{C}\}$. A Bose-Mesner algebra $\mathcal{A}$ has a unique basis of primitive idempotent matrices $E_0, E_1, \ldots, E_d$, that is,

$$(1)\ E_k E_l = \delta_{kl} E_k\ (k, l = 0, 1, \ldots, d),\ (2)\ \sum_{i=0}^{d} E_i = I,$$

where δ_{kl} is the Kronecker delta function. A Bose-Mesner algebra $\mathcal{A}$ have two basis $\{A_i\}$ and $\{E_i\}$. For A_i and E_i, we express one in terms of the other and we obtain

$$A_j = \sum_{i=0}^{d} p_j(i) E_i,\ E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_j(i) A_i$$

for $j = 0, 1, \ldots, d$. The $(d+1) \times (d+1)$ matrix $\mathbf{P} = (p_j(i))$ (respectively, $\mathbf{Q} = (q_j(i))$) is called *the first eigenmatrix* (respectively, *the second eigenmatrix*) of the association scheme. Then $\mathbf{P} = (p_j(i))$ and $\mathbf{Q} = (q_j(i))$ satisfy that $q_j(i)/m_j = \overline{p_i(j)}/k_i$, where $m_j = \text{rank}(E_j)$, k_i is the valency of A_i, and $\overline{p_i(j)}$

is the complex conjugate of $p_i(j)$. The adjacency matrices A_i satisfy

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

for all i, j, where for $(x,y) \in R_k$, p_{ij}^k is the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$. The non-negative integers p_{ij}^k are called the *intersection numbers* of $\mathfrak{X}$. Let B_i be a matrix with (j,k)-entries p_{ij}^k, and let $\mathcal{B}$ be an algebra spanned by $B_0, B_1, \ldots, B_d$. Then B_i is called the i-th intersection matrix of $\mathfrak{X}$ and $\mathcal{B}$ is called the intersection algebra of $\mathfrak{X}$. In fact, the Bose-Mesner algebra $\mathcal{A}$ of $\mathfrak{X}$ is isomorphic to $\mathcal{B}$ by the map $A_i \to B_i$. In particular, A_i and B_i have the same minimal polynomials.

Now, we introduce definitions of the P-polynomial schemes and basic facts on the P-polynomial schemes [1, 5, 6].

Definition. A symmetric association scheme $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, \ldots, d)$ is called a P-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exists some complex coefficient polynomial $\upsilon_i(x)$ of degree i $(i = 0, 1, \ldots, d)$ such that $A_i = \upsilon_i(A_1)$, where A_i is the adjacency matrix with respect to R_i.

Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a symmetric association scheme, and Γ_1 be the graph whose vertex and edge sets are X and R_1 respectively. Then the following (1), (2), (3) and (4) are equivalent to each other.

(1) Γ_1 is a distance regular graph.

(2) The first intersection matrix B_1 is a tridiagonal matrix with non-zero off-diagonal entries,

$$B_1 = \begin{pmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 \\ 1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & \cdots & \cdots & 0 & c_d & a_d \end{pmatrix} \quad (b_i \neq 0, c_i \neq 0),$$

(i) $a_i + b_i + c_i = k_1$ $(i = 0, 1, \ldots, d), c_0 = b_d = 0$,

(ii) $k_i = \dfrac{k_1 b_1 b_2 \cdots b_{i-1}}{c_2 c_3 \cdots c_i}$ $(i = 2, 3, \ldots, d)$,

(iii) $k_1 \geq b_1 \geq \cdots \geq b_{d-1}$,

(iv) $1 \leq c_2 \leq \cdots \leq c_d$.

(3) $\mathfrak{X}$ is a P-polynomial scheme with respect to $R_0, R_1, \ldots, R_d$, that is, $A_i = \upsilon_i(A_1)$ $(i = 0, 1, \ldots, d)$ for some polynomial $\upsilon_i(x)$ of degree i.

(4) First eigenmatrix $\mathbf{P} = (p_j(i))$ satisfies $p_j(i) = \upsilon_i(\theta_j)$ for some polynomial $\upsilon_i(x)$ of degree i, where $\theta_j = p_1(j)$ $(i, j = 0, 1, \ldots, d)$.

Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme. Now, define a transition probability matrix P over $\mathfrak{X}$ by

$$P = \frac{1}{k_1} A_1,$$

where k_1 is the valency of A_1. For a function $f : X \to \mathbb{R}$, we define a *Laplace operator* Δ by

$$\Delta f(x) = \frac{1}{k_1} \sum_{(x,y)\in R_1} (f(x) - f(y)).$$

Then, we have $\Delta = I - P = I - \frac{1}{k_1} A_1$. For all $i = 0, 1, \ldots, d$, a Laplace operator Δ is symmetric since A_1 is symmetric. Then, Δ is a matrix representation of $\mathcal{L}$. For $j = 0, 1, \ldots, d$ and orthogonal eigenfunctions ϕ_j^*, we have

$$\mathcal{L} = \sum_{j=0}^{d} \lambda_j \phi_j^* \phi_j,$$

where λ_j is an eigenvalue of $\mathcal{L}$. Let $\mathcal{L}_\beta$ be the β-normalized Laplacian by $\beta I + \mathcal{L}$. For $\beta > 0$, let a discrete Green's function $\mathcal{G}_\beta$ denote the symmetric matrix satisfying $\mathcal{L}_\beta \mathcal{G}_\beta = I$. Then we have

$$\mathcal{G}_\beta = \sum_{j=0}^{d} \frac{1}{\beta + \lambda_j} \phi_j^* \phi_j.$$

For $\beta > 0$, we have

$$\mathcal{G}_\beta(\beta I + I - P) = I,$$

that is,

$$\mathcal{G}_\beta = ((\beta + 1)I - P)^{-1}.$$

Thus, this implies that

$$\begin{aligned}\mathcal{L}_\beta &= (\beta + 1)I - P = (\beta + 1)I - \frac{1}{k_1} A_1 \\ &= (\beta + 1)I - \frac{1}{k_1} \sum_{j=0}^{d} p_1(j) E_j.\end{aligned}$$

Since $I = E_0 + E_1 + \cdots + E_d$, we have

$$\begin{aligned}\mathcal{L}_\beta &= (\beta + 1)(E_0 + E_1 + \cdots + E_d) - \frac{1}{k_1} (p_1(0)E_0 + p_1(1)E_1 + \cdots + p_1(d)E_d) \\ &= \sum_{j=0}^{d} \left(\beta + 1 - \frac{1}{k_1} p_1(j)\right) E_j,\end{aligned}$$

where $\beta+1-\frac{1}{k_1}p_1(j)$ is an eigenvalue of $\mathcal{L}_\beta$. Hence, a Green's function $\mathcal{G}_\beta$ can be expressed by

$$\mathcal{G}_\beta = \sum_{j=0}^{d} \left(\frac{k_1}{(\beta+1)k_1 - p_1(j)} \right) E_j.$$

Since $E_j = (1/|X|)\sum q_j(i)A_i$, the Green function $\mathcal{G}_\beta$ is a linear combination of adjacency matrices A_i as follows:

$$\mathcal{G}_\beta = r_0A_0 + r_1A_1 + \cdots + r_dA_d \tag{1}$$

for some r_i $(i = 0, 1, \ldots, d)$.

The following notations are used throughout this paper.

- $\mathbf{P}$: the first eigenmatrix of the P-polynomial scheme.
- $\mathbf{Q}$: the second eigenmatrix of the P-polynomial scheme.
- $p_j(i)$: (i, j)-component of $\mathbf{P}$.
- $q_j(i)$: (i, j)-component of $\mathbf{Q}$.
- k_i : i-th the valency of the P-polynomial scheme.
- m_j : j-th the multiplicity of the P-polynomial scheme.
- $\mathcal{N}(A)$: a nullspace of a matrix A.
- Γ_1 : a graph with respect to R_1 of a P-polynomial scheme.

3. A reduction matrix L_{sub} on $\mathcal{L}_\beta$ of the P-polynomial scheme

In this section we first introduce a $d \times (d+1)$ matrix L_{sub} obtained from $\mathcal{L}_\beta$. Then we show that L_{sub} is closely related to the discrete Green's function $\mathcal{G}_\beta$ in the following lemma.

Lemma 3.1. *For $\beta > 0$, a Green's function $\mathcal{G}_\beta$ is expressed as*

$$\mathcal{G}_\beta = cu_0A_0 + cu_1A_1 + \cdots + cu_dA_d \tag{2}$$

for some nonzero $c \in \mathbb{R}$, where $(u_0, u_1, \ldots, u_d)$ is the unique basis of the nullspace $\mathcal{N}(L_{sub})$ of L_{sub} with $u_d = 1$.

Proof. Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme with respect to a matric $d_M(x, y)$ for $x, y \in X$. Let L be a $(|X|-1) \times |X|$ matrix obtained by the removal of the first row of $\mathcal{L}_\beta = (\beta+1)I - \frac{1}{k_1}A_1$. Then we have the rank of L is $|X|-1$, and the nullity is 1 since $\mathcal{G}_\beta$ has the inverse matrix. A basis of the nullspace of L can be induced from r_k's which are coefficients of A_k in $\mathcal{G}_\beta$. Let $\mathcal{G}_\beta^{(1)}$ be the first column vector of $\mathcal{G}_\beta$ which is arranged in the order r_0, r_1, …, r_d. Then $\mathcal{G}_\beta^{(1)}$ is a $|X| \times 1$ matrix, and we have

$$L\mathcal{G}_\beta^{(1)} = O \tag{3}$$

since $\mathcal{G}_\beta$ is orthogonal. Since the Bose-Mesner algebra $\mathcal{A}$ is isomorphic to the intersection algebra $\mathcal{B}$ of $\mathfrak{X}$, $\mathcal{G}_\beta^{-1} = (\beta+1)I - \frac{1}{k_1}A_1$ is corresponding with $(\beta+1)I - \frac{1}{k_1}B_1$. Let L' be a $(d+1) \times (d+1)$ matrix as $-k_1((\beta+1)I - \frac{1}{k_1}B_1)$, that is, $L' = B_1 - k_1(\beta+1)I$. Let L_{sub} be a $d \times (d+1)$ matrix obtained by the removal of the first row of L'. Then, we obtain L_{sub} as follows:

$$L_{sub} = \begin{pmatrix} c_1 & s_1 & b_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & s_2 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & s_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{d-1} & s_{d-1} & b_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & c_d & s_d \end{pmatrix}, \tag{4}$$

where $s_i = a_i - k_1(\beta+1)$ for $i = 1, 2, \ldots, d$ and a_i, b_i, c_i are as the entries of B_1 in Section 2. Therefore, by (3), we have

$$L_{sub} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_d \end{pmatrix} = O.$$

Since $1 = c_1 \leq c_2 \leq \cdots \leq c_d$, rank$(L_{sub})$=$d$ and dim$\mathcal{N}(L_{sub})$=1. Then from Eq. (1), we obtain the result as desired. □

A Green's function $\mathcal{G}_\beta$ is defined only for $\beta > 0$, and $\mathcal{G}_\beta$ is expressed as a linear combination of adjacency matrices A_i such as in Eq.(2). But, for $\beta \leq 0$, $\mathcal{G}_\beta$ may be a singular matrix, so there is no Green's function notion for this case. We, however, still have rank$(L_{sub}) = d$ for $\beta \in \mathbb{R}$, so we can obtain a unique basis $(u_0, u_1, \ldots, u_d)$ of $\mathcal{N}(L_{sub})$ with $u_d = 1$. In this context, we extend a notion of a Green's function $\mathcal{G}_\beta$ associated with any real number β as follows. The following definition plays an important role for computation of the p-number and the q-number as we will see in Section 4.

Definition. For $\beta \in \mathbb{R}$, let $(u_0, u_1, \ldots, u_d) \in \mathcal{N}(L_{sub})$ with $u_d = 1$ and let $\mathcal{G}_{\beta,\mathcal{N}} = cu_0A_0 + cu_1A_1 + \cdots + cu_dA_d$, where c is some nonzero $\in \mathbb{R}$ if $\beta > 0$ and $c = 1$ if $\beta \leq 0$. Then $\mathcal{G}_{\beta,\mathcal{N}}$ is called the normalized Green's function.

The P-polynomial schemes are defined by Delsarte. We know the two most important examples, namely the Hamming scheme $H(d, q)$ and the Johnson scheme $J(v, d)$. In the following example, we show a normalized Green's function over a Hamming scheme $H(5, 3)$.

Example 3.2. Let $H(5, 3)$ be a Hamming scheme over $\mathbb{F}_3^5$. Choosing $\beta = -\frac{1}{10}$, we obtain a 5×6 matrix L_{sub} as follows:

$$\begin{pmatrix} 1 & -8 & 8 & 0 & 0 & 0 \\ 0 & 2 & -7 & 6 & 0 & 0 \\ 0 & 0 & 3 & -6 & 4 & 0 \\ 0 & 0 & 0 & 4 & -5 & 2 \\ 0 & 0 & 0 & 0 & 5 & -4 \end{pmatrix}.$$

Also, a basis of $\mathcal{N}(L_{sub})$ is $(-\frac{40}{3}, -\frac{26}{15}, -\frac{1}{15}, \frac{1}{2}, \frac{4}{5}, 1)$. Thus we obtain a normalized Green's function $\mathcal{G}_{\beta,\mathcal{N}}$ as follows:

$$\mathcal{G}_{\beta,\mathcal{N}} = -\frac{40}{3}A_0 - \frac{26}{15}A_1 = \frac{1}{15}A_2 + \frac{1}{2}A_3 + \frac{4}{5}A_4 + A_5,$$

where A_i $(i = 0, 1, \ldots, 5)$ are the adjacency matrices of $H(5, 3)$.

4. Relation between the *p*-number (or the *q*-number) and the normalized Green's function

In this section, for $\beta \in R$, we find the relationship between a normalized Green's function $\mathcal{G}_{\beta,\mathcal{N}}$ and the p-number $p_j(i)$ (or the q-number $q_j(i)$) over the P-polynomial scheme. In fact, the goal of this section is to prove the following main result, which shows a new expression of the p-number (or the q-number) over the P-polynomial scheme.

Theorem 4.1. *Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme, and let $\mathbf{P} = (p_j(i))$ (respectively, $\mathbf{Q} = (q_j(i))$) be the first eigenmatrix (respectively, the second eigenmatrix) of $\mathfrak{X}$. Then, for $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \ldots, d)$, we have*

$$p_0(j) = k_0,\ q_j(0) = m_j,$$

$$p_i(j) = (-1)^i k_i c_1 c_2 \cdots c_i \frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})}\ (i = 1, 2, \ldots, d),$$

$$q_j(i) = (-1)^i m_j c_1 c_2 \cdots c_i \frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})}\ (i = 1, 2, \ldots, d),$$

where $m_j = \mathrm{rank}(E_j)$ and k_i is a valency of R_i.

For the proof of Theorem 4.1, we need Proposition 4.3, Corollary 4.4 and Lemma 4.5.

We need the following lemma for Proposition 4.3.

Lemma 4.2. *Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme and let L_{sub} be a matrix of $\mathfrak{X}$ as in Section 3. Let $u = (u_0, u_1, \ldots, u_d)$ be a $(d+1)$- vector. Then $u \in \mathcal{N}(L_{sub})$ if and only if*

$$u' := (\underbrace{u_0}_{k_0}, \underbrace{u_1, \ldots, u_1}_{k_1}, \ldots, \underbrace{u_d, \ldots, u_d}_{k_d}) \in \mathcal{N}(L),$$

where L is a $(|X| - 1) \times |X|$ matrix on (3) *in Section 3.*

Proof. ($\Rightarrow$) Let u be a basis of $\mathcal{N}(L_{sub})$, and let l_i be the i-th row vector of L_{sub} for $i = 1, 2, \ldots, d$. Then l_i is $(0, \ldots, 0, i, s_i, t_i, 0, \ldots 0)$. Since $u \cdot l_i = 0$,

$$(u_{i-1})(c_i) + (u_i)(s_i) + (u_{i+1})(t_i) = 0$$
$$\Leftrightarrow\ -\frac{1}{k_1}((u_{i-1})(c_i) + (u_i)(s_i) + (u_{i+1})(t_i)) = 0.$$

Thus, the row vector of L is orthogonal to u'. That is, $u' \in \mathcal{N}(L)$.

($\Leftarrow$) It is clear by the previous process of obtaining L_{sub} from L in Section 3. $\square$

For some $\beta_j \in \mathbb{R}$, let $L_{sub}^{(\beta_j)}$ be a $d \times (d+1)$ matrix as L_{sub} in Section 3, and let $(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})$ be a basis of $\mathcal{N}(L_{sub}^{(\beta_j)})$.

Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme, and let $P = (p_j(i))$ (respectively, $Q = (q_j(i))$) be the first eigenmatrix (respectively, the second eigenmatrix) of $\mathfrak{X}$. In the following theorem, we show that the j-th column vector of the second eigenmatrix $\mathbf{Q} = (q_j(i))$ belongs to $\mathcal{N}(L_{sub}^{(\beta_j)})$ for $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \ldots, d)$. That is, we find the relationship between $u_i^{(j)}$ and component $q_j(i)$ of the second eigenmatrix $\mathbf{Q} = (q_j(i))$ over the P-polynomial scheme.

Proposition 4.3. *Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme. For $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \ldots, d)$, let $\mathcal{G}_{\beta_j, \mathcal{N}} = u_0^{(j)} A_0 + u_1^{(j)} A_1 + \ldots + u_d^{(j)} A_d$ be a normalized Green's function with $u_d^{(j)} = 1$. Then $q_j(i)$ satisfy $q_j(i) = m_j \frac{u_i^{(j)}}{u_0^{(j)}}$ $(i = 0, 1, \ldots, d)$. That is, the j-th column of the second eigenmatrix $\mathbf{Q} = (q_j(i))$ is equal to $\frac{m_j}{u_0^{(j)}}(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})^T$, where m_j is the j-th multiplicity of $\mathfrak{X}$.*

Proof. Since $\beta_j I + \mathcal{L} = \sum((\beta_j + 1) - \frac{1}{k_1} p_1(j)) E_j = \sum[((\beta_j + 1)k_1 - p_1(j))/k_1] E_j$, $((\beta_j + 1)k_1 - p_1(j))/k_1$ are eigenvalues of $\beta_j I + \mathcal{L}$, where $\beta_j = \frac{p_1(i)}{k_1} - 1$ satisfies $(\beta_j + 1)k_1 - p_1(j) = 0$. So we have $(\beta_j I + \mathcal{L})E_j = O$. Therefore, every row vector of L in Section 3 is orthogonal to every column vector of E_j. Since $E_j = \frac{1}{|X|} \sum q_j(i) A_i$, the first column vector of E_j is a component of $\mathcal{N}(L)$ and can be written as

$$\frac{1}{|X|}(q_j(0), q_j(1), \ldots, q_j(1), q_j(2), \ldots, q_j(2), \ldots, q_j(d))^T.$$

Thus, by Lemma 4.2, $(q_j(0), q_j(1), \ldots, q_j(d)) \in \mathcal{N}(L_{sub}^{(\beta_j)})$. Since $\dim(\mathcal{N}(L_{sub}^{(\beta_j)})) = 1$, for $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \ldots, d)$, the j-th column vectors of the second eigenmatrix $\mathbf{Q} = (q_j(i))$ are equal to $q_j(d)(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})^T$ with respect to a basis $(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})$ of $\mathcal{N}(L_{sub}^{(\beta_j)})$ with $u_d^{(j)} = 1$. Moreover, we have $q_j(i) = q_j(d) u_i^{(j)}$. Since $q_j(0) = m_j$, we have $q_j(d) = \frac{m_j}{u_0^{(j)}}$. $\square$

Corollary 4.4. *Let $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ be a P-polynomial scheme and for $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \ldots, d)$, let $\mathcal{G}_{\beta_j, \mathcal{N}} = u_0^{(j)} A_0 + u_1^{(j)} A_1 + \cdots + u_d^{(j)} A_d$ be a normalized Green's function. Then $p_i(j)$ and $u_i^{(j)}$ satisfy $p_i(j) = k_i \frac{u_i^{(j)}}{u_0^{(j)}}$.*

Proof. By Proposition 4.3, we know $q_j(i) = m_j \frac{u_i^{(j)}}{u_0^{(j)}}$. Since $\mathfrak{X}$ is a symmetric association scheme, we have $\overline{p_i(j)} = p_i(j)$. Also, we know the p-number and the q-number satisfy $q_j(i)/m_j = \overline{p_i(j)}/k_i$. Thus we have

$$\frac{m_j}{k_i} p_i(j) = m_j \frac{u_i^{(j)}}{u_0^{(j)}} \;\Leftrightarrow\; p_i(j) = k_i \frac{u_i^{(j)}}{u_0^{(j)}},$$

where $m_j = \operatorname{rank}(E_j)$ and k_i is a valency of R_i. □

Let $(u_0, u_1, \ldots, u_d)$ be a basis of $\mathcal{N}(L_{sub})$ with $u_d = 1$. In the following Lemma, for $\beta \in \mathbb{R}$, we find an explicit expression of u_i by a determinant of a submatrix L_i of L_{sub} as in Section 3. Thus, by Proposition 4.3 and Corollary 4.4, the p-number $p_j(i)$ and the q-number $q_j(i)$ are expressed by a determinant of a submatrix of $L_{sub}^{(\beta_j)}$.

Lemma 4.5. *For $\beta \in \mathbb{R}$, let L_0 be a $d \times d$ matrix obtained by the removal of the first column of L_{sub}. Let L_i be a $(d-i) \times (d-i)$ matrix obtained by the removal from the first row (respectively, column) to the i-th row (respectively, column) of L_0, and let $(u_0, u_1, \ldots, u_d)$ be a basis of $\mathcal{N}(L_{sub})$ with $u_d = 1$. Then*

$$u_i = (-1)^{d-i} \frac{\det(L_i)}{c_{i+1} c_{i+2} \cdots c_d}, \; i = 0, 1, \ldots, d-1,$$

where L_{sub} is defined as in Section 3 and $\det(L_d) = 1$.

Proof. Since L_{d-1} is a 1×1 matrix, thus by (4) in Section 3, we have

$$\det(L_{d-1}) = s_d.$$

Thus, we have

$$u_{d-1} = -\frac{s_d}{c_d} = (-1)^{d-d+1} \frac{\det(L_{d-1})}{c_d}.$$

Similarly, we have $u_{d-2} = (-1)^{d-d+2} \frac{\det(L_{d-2})}{c_{d-1} c_d}$.
Next, we show the following: for $i = 3, \ldots, d$,

$$u_{d-i} = (-1)^i \frac{\det(L_{d-i})}{c_{d-i+1} c_{d-i+2} \cdots c_d}.$$

We consider an $i \times i$ submatrix T which is obtained from L_{sub} by applying some elementary operations as follows:

$$\begin{pmatrix} A & B & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & -u_{d-i+1} \\ 0 & 1 & 0 & \cdots & -u_{d-i+2} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -u_{d-1} \end{pmatrix},$$

where $A = \frac{s_{d-i+1}}{c_{d-i+1}}$ and $B = \frac{t_{d-i+1}}{c_{d-i+1}}$. Then $\det(T) = (-1)^i(-u_{d-i+1})A - (-1)^{i+1}(-u_{d-i+2})B$, and $\det(L_{d-i})$ is as follows:

$$\begin{aligned}\det(L_{d-i}) &= c_{d-i+1}\cdots c_{d-1}c_d \det(T)\\ &= (-1)^{i+1}c_{d-i+1}\cdots c_{d-1}c_d(u_{d-i+1}A + u_{d-i+2}B)\\ &= (-1)^{i+1}c_{d-i+1}\cdots c_{d-1}c_d\ (-u_{d-i})\\ \Leftrightarrow\ u_{d-i} &= (-1)^i\frac{\det(L_{d-i})}{c_{d-i+1}\cdots c_{d-1}c_d}.\end{aligned}$$

Therefore, we have

$$u_i = (-1)^{d-i}\frac{\det(L_i)}{c_{i+1}c_{i+2}\cdots c_d}\ (i = 0, 1, \dots, d-1,\ \det(L_d) = 1).$$

□

For $\beta_j = \frac{p_1(j)}{k_1} - 1$, let $\mathcal{G}_{\beta,\mathcal{N}} = u_0^{(j)}A_0 + u_1^{(j)}A_1 + \cdots + u_d^{(j)}A_d$ be a normalized Green's function. Then, by Proposition 4.3, Corollary 4.4 and Lemma 4.5, the p-number $p_j(i)$ and the q-number $q_j(i)$ are expressed by a determinant of a submatrix of $L_{sub}^{(\beta_j)}$. Let $L_i^{(\beta_j)}$ be a $(d-i)\times(d-i)$ submatrix of $L_{sub}^{(\beta_j)}$ for $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \dots, d)$ as L_i in Lemma 4.5. Then we obtain Theorem 4.1, the main result of this paper, by Proposition 4.3, Corollary 4.4 and Lemma 4.5.

The proof of Theorem 4.1. By Proposition 4.3 and Corollary 4.4, $p_i(j) = k_i\frac{u_i^{(j)}}{u_0^{(j)}}$ and $q_j(i) = m_j\frac{u_i^{(j)}}{u_0^{(j)}}$. Thus we have

$$p_0(j) = k_0,\ q_j(0) = m_j.$$

Also, by Lemma 4.5, we have $(i = 1, 2, \dots, d)$

$$p_i(j) = k_i\frac{u_i^{(j)}}{u_0^{(j)}} = k_i\frac{(-1)^{d-i}\frac{\det(L_i^{(\beta_j)})}{c_{i+1}c_{i+2}\cdots c_d}}{(-1)^d\frac{\det(L_0^{(\beta_j)})}{c_1c_2\cdots c_d}} = (-1)^i k_i c_1c_2\cdots c_i\frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})},$$

$$q_j(i) = m_j\frac{u_i^{(j)}}{u_0^{(j)}} = m_j\frac{(-1)^{d-i}\frac{\det(L_i^{(\beta_j)})}{c_{i+1}c_{i+2}\cdots c_d}}{(-1)^d\frac{\det(L_0^{(\beta_j)})}{c_1c_2\cdots c_d}} = (-1)^i m_j c_1c_2\cdots c_i\frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})},$$

where $\beta_j = \frac{p_1(j)}{k_1} - 1$ $(j = 0, 1, \dots, d)$. □

In [7], we have a method of computation for determinants of tridiagonal matrices, which we apply here to $L_i^{(\beta_j)}$.

Remark 4.6. Since $L_i^{(\beta_j)}$ is a tridiagonal matrix, the determinant of $L_i^{(\beta_j)}$ can be evaluated by multiplication of 2×2 matrices. Let $L_i^{(\beta_j)}$ be a $(d-i)\times(d-i)$

matrix as follows : $(i = 0, 1, \ldots, d-2)$

$$\begin{pmatrix} s_{i+1} & b_{i+1} & & \\ c_{i+2} & \ddots & \ddots & \\ & \ddots & \ddots & b_{d-1} \\ & & c_d & s_d \end{pmatrix},$$

where $s_k = a_k - k_1(\beta_j + 1)$ $(k = 1, \ldots, d)$ and a_k, b_k, c_k are as in (4). Thus, the determinant of $L_i^{(\beta_j)}$ is

$$\begin{aligned} & tr\Big[\begin{pmatrix} s_d & -b_{d-1}c_d \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} s_{i+2} & -b_{i+1}c_{i+2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} s_{i+1} & 0 \\ 1 & 0 \end{pmatrix}\Big] \\ &= tr\Big[\prod_{l=0}^{d-i-2}\begin{pmatrix} s_{d-l} & -b_{d-l-1}c_{d-l} \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} s_{i+1} & 0 \\ 1 & 0 \end{pmatrix}\Big] \\ &= \det(L_i^{(\beta_j)}). \end{aligned}$$

5. The Krawtchouk polynomial of the Hamming scheme on the normalized Green's function

Let $H(d,q)$ be a Hamming scheme over $\mathbb{F}_q^d$. Since the Hamming scheme is a self-dual scheme, $\mathbf{P} = (p_j(i))$ is equal to $\mathbf{Q} = (q_j(i))$. The p-number $p_j(i)$ of a Hamming scheme $H(d,q)$ is defined by the Krawtchouk polynomial. Thus, we have

$$\begin{aligned} p_1(j) =& (-1)^0(q-1)\binom{j}{0}\binom{d-j}{1} + (-1)(q-1)^{1-1}\binom{j}{1}\binom{d-j}{1-1} \\ =& d(q-1) - qj. \end{aligned}$$

Also, $k_1 = d(q-1)$. Then $\beta_j = \frac{p_1(j)}{k_1} - 1 = \frac{d(q-1)-qj-d(q-1)}{d(q-1)} = -\frac{qj}{d(q-1)}$. Thus, for $\beta_j = -\frac{qj}{d(q-1)}$, we have a matrix $L_{sub}^{(\beta_j)}$ as follows:

$$L_{sub}^{(\beta_j)} = \begin{pmatrix} 1 & s_1 & t_1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & s_2 & t_2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & s_3 & t_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & d-1 & s_{d-1} & t_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & d & s_d \end{pmatrix},$$

where $s_i = i(q-2) - d(q-1)(\beta_j + 1)$ for $i = 1, 2, \ldots, d$, and $t_k = (d-k)(q-1)$ for $k = 1, 2, \ldots, d-1$.

In the following Corollary 5.1, we show that the Krawtchouk polynomial $p_j(i)$ can be explicitly computed by determinants of submatrices of $L_{sub}^{(\beta_j)}$ associate with a Green's function.

Corollary 5.1. *Let $H(d,q)$ be a Hamming scheme over $\mathbb{F}_q^d$. Let $L_i^{(\beta_j)}$ be a $(d-i)\times(d-i)$ submatrix of $L_{sub}^{(\beta_j)}$ for $\beta_j = -\frac{qj}{d(q-1)}$ $(j=0,1,\ldots,d)$ as L_i in Lemma 4.5. Then, we have*

$$p_j(i) = p_j(d)u_i^{(j)} = (-1)^{d+j-i}\binom{d}{j}\frac{i!\,\det(L_i^{(\beta_j)})}{d!}$$

for $i,j=0,1,\ldots,d$.

Proof. The Hamming scheme is a self-dual scheme. That is, $\mathbf{P}=\mathbf{Q}$ $(p_j(i)=q_j(i))$. Thus, by proof of Proposition 4.3, we have $p_j(i)=p_j(d)u_i^{(j)}$. Since $p_j(d)=(-1)^j\binom{d}{j}$ and by Lemma 4.5,

$$p_j(i) = (-1)^{d+j-i}\binom{d}{j}\frac{i!\,\det(L_i^{(\beta_j)})}{d!}.$$

Moreover, by Theorem 4.1 and Corollary 5.1,

$$\det(L_0^{(\beta_j)}) = (-1)^{d+j}d!\frac{k_j}{\binom{d}{j}}.$$

□

Example 5.2. Let $H(5,3)$ be a Hamming scheme over $\mathbb{F}_3^5$. Then, the first eigenmatrix $\mathbf{P}=(p_j(i))$ is as follows:

$$\begin{pmatrix} 1 & 10 & 40 & 80 & 80 & 32 \\ 1 & 7 & 16 & 8 & -16 & -16 \\ 1 & 4 & 1 & -10 & -4 & 8 \\ 1 & 1 & -5 & -1 & 8 & -4 \\ 1 & -2 & -2 & 8 & -7 & 2 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix}.$$

Let $j=1$. Then we obtain the p-numbers $p_1(i)$ as the entries of the second column of the first eigenmatrix $\mathbf{P}$.
Let $L_{sub}^{(\beta_1)}$ be a 5×6 matrix over $\mathbb{F}_3^5$ for $\beta_1 = \frac{(3)(1)}{5(3-1)} = -\frac{3}{10}$ as follows:

$$\begin{pmatrix} 1 & -6 & 8 & 0 & 0 & 0 \\ 0 & 2 & -5 & 6 & 0 & 0 \\ 0 & 0 & 3 & -4 & 4 & 0 \\ 0 & 0 & 0 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 5 & -2 \end{pmatrix}.$$

Then, the matrices $L_0^{(\beta_1)}$, $L_1^{(\beta_1)}$, $L_2^{(\beta_1)}$, $L_3^{(\beta_1)}$, and $L_4^{(\beta_1)}$ are

$$L_0^{(\beta_1)} = \begin{pmatrix} -6 & 8 & 0 & 0 & 0 \\ 2 & -5 & 6 & 0 & 0 \\ 0 & 3 & -4 & 4 & 0 \\ 0 & 0 & 4 & -3 & 2 \\ 0 & 0 & 0 & 5 & -2 \end{pmatrix}, L_1^{(\beta_1)} = \begin{pmatrix} -5 & 6 & 0 & 0 \\ 3 & -4 & 4 & 0 \\ 0 & 4 & -3 & 2 \\ 0 & 0 & 5 & -2 \end{pmatrix},$$

$$L_2^{(\beta_1)} = \begin{pmatrix} -4 & 4 & 0 \\ 4 & -3 & 2 \\ 0 & 5 & -2 \end{pmatrix}, L_3^{(\beta_1)} = \begin{pmatrix} -3 & 2 \\ 5 & -2 \end{pmatrix}, L_4^{(\beta_1)} = \begin{pmatrix} -2 \end{pmatrix}.$$

Also, $\det(L_0^{(\beta_1)}) = 240$, $\det(L_1^{(\beta_1)}) = -168$, $\det(L_2^{(\beta_1)}) = 48$, $\det(L_3^{(\beta_1)}) = -4$, $\det(L_4^{(\beta_1)}) = -2$. Therefore, $p_1(i)$ $(i = 0, 1, 2, 3, 4, 5)$ are

$$\begin{aligned}
p_1(0) =& (-1)^{5+1-0}\binom{5}{1}\frac{0!\det(L_0^{(\beta_1)})}{5!} = \frac{\det(L_0^{(\beta_1)})}{24} = \frac{240}{24} = 10,\\
p_1(1) =& (-1)^{5+1-1}\binom{5}{1}\frac{1!\det(L_1^{(\beta_1)})}{5!} = -\frac{\det(L_1^{(\beta_1)})}{24} = -\frac{-168}{24} = 7,\\
p_1(2) =& (-1)^{5+1-2}\binom{5}{1}\frac{2!\det(L_2^{(\beta_1)})}{5!} = \frac{\det(L_2^{(\beta_1)})}{12} = \frac{48}{12} = 4,\\
p_1(3) =& (-1)^{5+1-3}\binom{5}{1}\frac{3!\det(L_3^{(\beta_1)})}{5!} = -\frac{\det(L_3^{(\beta_1)})}{4} = -\frac{-4}{4} = 1,\\
p_1(4) =& (-1)^{5+1-4}\binom{5}{1}\frac{4!\det(L_4^{(\beta_1)})}{5!} = \frac{\det(L_4^{(\beta_1)})}{1} = \frac{-2}{1} = -2,\\
p_1(5) =& (-1)^{5+1-5}\binom{5}{1}\frac{5!\det(L_5^{(\beta_1)})}{5!} = -5,
\end{aligned}$$

respectively. Thus, $(10, 7, 4, 1, -2, -5)^T$ is the first column vector of the first eigenmatrix $\mathbf{P}$.

In the following corollary, we explain a result over the Hamming scheme $H(d, q)$ as Proposition 4.3.

Corollary 5.3. *For $\beta_j = -\frac{qj}{d(q-1)}$ $(j = 0, 1, \ldots, d)$, let $\mathcal{G}_{\beta_j,\mathcal{N}} = u_0^{(j)}A_0 + u_1^{(j)}A_1 + \cdots + u_d^{(j)}A_d$ be a normalized Green's function with $u_d^{(j)} = 1$. Let $\mathbf{N}$ be a matrix with the j-th column vector as $(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})^T$ for any $j = 0, 1, \ldots, d$ and $\beta_j = -\frac{qj}{d(q-1)}$. Then $u_i^{(j)}$ is a (i, j)-component of $\mathbf{N}$ and $\mathbf{N}$ is equal to $\mathbf{PD}$, where $\mathbf{D}$ is a diagonal matrix with the (j, j)-diagonal entries $\frac{(-1)^j}{\binom{d}{j}}$.*

Proof. Since by Proposition 4.3, for $\beta_j = -\frac{qj}{d(q-1)}$ $(j = 0, 1, \ldots, d)$, the j-th column vector of the first eigenmatrix $\mathbf{P}$ is equal to $p_j(d)(u_0^{(j)}, u_1^{(j)}, \ldots, u_d^{(j)})^T$ $(u_d^{(j)} = 1)$. Therefore,

$$\mathbf{N}\begin{pmatrix} p_0(d) & 0 & 0 & \cdots & 0 \\ 0 & p_1(d) & 0 & \cdots & 0 \\ 0 & 0 & p_2(d) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_d(d) \end{pmatrix} = \mathbf{P}.$$

Since $p_j(d) = \sum_l (-1)^l (q-1)^{j-l} \binom{d}{l}\binom{0}{j-l} = (-1)^j \binom{d}{j}$, we have $\mathbf{N} = \mathbf{PD}$, where $\mathbf{D}$ is a diagonal matrix with the (j,j)-diagonal entries $\frac{(-1)^j}{\binom{d}{j}}$ $(j = 0, 1, \ldots, d)$. □

The following theorem is a matrix version of MacWilliams identity over $H(n,q)$.

Theorem 5.4. *Let $\mathcal{C}$ (respectively, $\mathcal{C}^\perp$) be a linear code (respectively, a dual code of a linear code $\mathcal{C}$) over $\mathbb{F}_q$ of length d, and let $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ (respectively, $\mathbf{b} = (b_0, b_1, \ldots, b_d)$) be a weight distribution of $\mathcal{C}$ (respectively, $\mathcal{C}^\perp$). Then, $\mathbf{a} = \frac{1}{|\mathcal{C}^\perp|}\mathbf{b}(p_j(i))$ is expressed by*

$$\mathbf{aD} = \frac{1}{|\mathcal{C}^\perp|}\ \mathbf{bN},$$

where $\mathbf{N}$ is a $(d+1)\times(d+1)$ matrix with the j-th column vectors as $(u_0^{(j)}, \ldots, u_d^{(j)})^T$ for $\beta_j = -\frac{qj}{d(q-1)}$, and $\mathbf{D}$ is a $(d+1)\times(d+1)$ diagonal matrix with the (j,j)-diagonal entries $\frac{(-1)^j}{\binom{d}{j}}$ for $j = 0, 1, \ldots, d$.

Proof. Since $\mathbf{a} = \frac{1}{|\mathcal{C}^\perp|}\ \mathbf{bP}$ and by Corollary 5.3, $\mathbf{N} = \mathbf{PD}$, so the result follows. □

Example 5.5. Let $\mathbf{P} = (p_j(i))$ be a first eigenmatrix of the Hamming scheme $H(5,3)$. Choosing $\beta = -\frac{3j}{10}$ $(j = 0, 1, \ldots, 5)$, we have

j	a basis of $\mathcal{N}(L_{sub}^{(\beta_j)})$	$p_j(5)$
0	$(1,\ 1,\ 1,\ 1,\ 1,\ 1)$	1
1	$(-2, -\frac{7}{5}, -\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}\ 1)$	-5
2	$(4, \frac{8}{5}, \frac{1}{10}, -\frac{1}{2}, -\frac{1}{5}\ 1)$	10
3	$(-8, -\frac{4}{5}, 1, \frac{1}{10}, -\frac{4}{5}, 1)$	-10
4	$(16, -\frac{16}{5}, -\frac{4}{5}, \frac{8}{5}, -\frac{7}{5}, 1)$	5
5	$(-32, 16, -8, 4, -2, 1)$	-1

Thus, the matrices $\mathbf{N}$ and $\mathbf{D}$ are as follows:

$$\mathbf{N} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 & -32 \\ 1 & -\frac{7}{5} & \frac{8}{5} & -\frac{4}{5} & -\frac{16}{5} & 16 \\ 1 & -\frac{4}{5} & \frac{1}{10} & 1 & -\frac{4}{5} & -8 \\ 1 & -\frac{1}{5} & -\frac{1}{2} & \frac{1}{10} & \frac{8}{5} & 4 \\ 1 & \frac{2}{5} & -\frac{1}{5} & -\frac{4}{5} & -\frac{7}{5} & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},\quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{-5} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-10} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $\mathbf{N} = \mathbf{PD}$, we have $\mathbf{ND}^{-1} = \mathbf{P} = (p_j(i))$ as follows:

$$\begin{pmatrix} 1 & 10 & 40 & 80 & 80 & 32 \\ 1 & 7 & 16 & 8 & -16 & -16 \\ 1 & 4 & 1 & -10 & -4 & 8 \\ 1 & 1 & -5 & -1 & 8 & -4 \\ 1 & -2 & -2 & 8 & -7 & 2 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix}.$$

Corollary 5.6. *Let* $\mathcal{C}$ (*respectively,* $\mathcal{C}^{\perp}$) *be a linear code* (*respectively, a dual code of a linear code* $\mathcal{C}$) *over* $\mathbb{F}_q$ *of length* d*, and let* $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ (*respectively,* $\mathbf{b} = (b_0, b_1, \ldots, b_d)$) *be a weight distribution of* $\mathcal{C}$ (*respectively,* $\mathcal{C}^{\perp}$). *Then we have*

$$\sum_{i=0}^{d} \left((-1)^{d-i} \frac{i! \ \det(L_i^{(\beta_j)})}{d!} \right) b_i = (-1)^j \ \frac{|\mathcal{C}^{\perp}|}{\binom{d}{j}} \ a_j$$

for $\beta_j = -\frac{qj}{d(q-1)}$, $(i, j = 0, 1, \ldots, d)$, *where* $\det(L_i^{(\beta_j)})$ *is given as in Corollary 5.1.*

Proof. Since $\mathbf{a} = \frac{1}{|\mathcal{C}^{\perp}|}\mathbf{b}(p_j(i))$, the result follows immediately from Corollary 5.1. □

6. The Eberlein polynomial of the Johnson scheme on the normalized Green's function

In fact, the Eberlein polynomial $E_i(j)$ is the p-number of the Johnson scheme $J(v, d)$. In this section we show that the Eberlein polynomial $E_i(j)$ can be explicitly computed by determinants of submatrices of $L_{sub}^{(\beta_j)}$ associated with a normalized Green's function $\mathcal{G}_{\beta_j, \mathcal{N}}$.

Now, we introduce the Johnson scheme $J(v, d)$ as a P-polynomial scheme.

Johnson scheme. Let S be a set of cardinality v and $X = \{T \subset S : \ |T| = d\}$ $(d \le v/2)$. Define the distance of $T_1, T_2 \in X$ as $d - |T_1 \cap T_2|$ and let R_i be the i-th distance relation on X, that is,

$$R_i = \{(T_1, T_2) : |T_1 \cap T_2| = d - i\}.$$

Then $\mathfrak{X} = (X, \{R_i\})$ $(i = 0, 1, \ldots, d)$ is a symmetric association scheme and is called *the Johnson scheme* $J(v, d)$.

The Johnson scheme $J(v, d)$ is a P-polynomial scheme. Thus, the intersection matrix B_1 of $J(v, d)$ is a tridiagonal matrix with non-zero off-diagonal entries as follows:

$$B_1 = \begin{pmatrix} 0 & k_1 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & c_3 & a_3 & \ddots & \\ & & & \ddots & \ddots & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix},$$

where, $a_i = i(v-2i)$, $b_i = (d-i)(v-d-i)$ and $c_i = i^2$ $(i = 1, 2, \ldots, d)$. Thus, for $\beta_j = \frac{p_1(j)}{k_1} - 1$, we obtain $L_{sub}^{(\beta_j)}$ as follows:

$$\begin{pmatrix} c_1 & a_1 - k_1(\beta_j+1) & b_1 & & & \\ & c_2 & a_2 - k_1(\beta_j+1) & b_2 & & \\ & & c_3 & a_3 - k_1(\beta_j+1) & \ddots & \\ & & & \ddots & \ddots & b_{d-1} \\ & & & & c_d & a_d - k_1(\beta_j+1) \end{pmatrix}.$$

Remark 6.1. Let $J(v,d)$ be a Johnson scheme and let k_i and m_j be the valencies and multiplicities of $J(v,d)$. Then

$$k_i = \binom{d}{i}\binom{v-d}{i}, \ m_j = \frac{v-2j+1}{v-j+1}\binom{v}{j}.$$

The following is a corollary to Theorem 4.1, and this shows that the Eberlein polynomial $E_i(j)$ (that is, the p-number over the Johnson scheme) can be explicitly computed by determinants of submatrices of $L_{sub}^{(\beta_j)}$ associated with a Green's function.

Corollary 6.2. *Let $J(v,d)$ be a Johnson scheme. For $\beta_j = \frac{j(j-v-1)}{d(v-d)}$ $(j = 0, 1, \ldots, d)$, let $\mathcal{G}_{\beta_j,\mathcal{N}} = u_0^{(j)}A_0 + u_1^{(j)}A_1 + \cdots + u_d^{(j)}A_d$ be a normalized Green's function of $J(v,d)$. Then, the Eberlein polynomial $E_i(j)$ $(i = 0, 1, \ldots, d)$ is*

$$E_i(j) = p_i(j) = k_i \frac{u_i^{(j)}}{u_0^{(j)}} = (-1)^i \binom{d}{i}\binom{v-d}{i}(i!)^2 \frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})},$$

where $L_i^{(\beta_j)}$ is defined as L_i in Lemma 4.5.

Proof. We know that the Eberlein polynomial $E_i(j)$ is

$$E_i(j) = \sum_{t=0}^{i} (-1)^{i-t}\binom{d-t}{i-t}\binom{d-j}{t}\binom{v-d+t-j}{t}.$$

Thus, we have

$$E_1(j) = -d + (d-j)(v-d-j+1).$$

Since $k_1 = d(v-d)$, we have

$$\beta_j = \frac{p_1(j)}{k_1} - 1 = \frac{E_1(j) - k_1}{k_1} = \frac{j(j-v-1)}{d(v-d)}.$$

Since $c_i = i^2$, by Remark 6.1, the i-th valency k_i of R_i $(i = 0, 1, \ldots, d)$ is

$$k_i = \binom{d}{i}\binom{v-d}{i}.$$

By Theorem 4.1, the Eberlein polynomial $E_i(j)$ is as follows:

$$E_i(j) = p_i(j) = (-1)^i \binom{d}{i}\binom{v-d}{i}(i!)^2 \frac{\det(L_i^{(\beta_j)})}{\det(L_0^{(\beta_j)})} \ (i = 0, 1, \ldots, d).$$

□

Example 6.3. Let $J(8,4)$ be a Johnson scheme over $\mathbb{F}_2^8$. Then, the order of the set X is 70. Let $L_{sub}^{(\beta_2)}$ be a 4×5 matrix over $\mathbb{F}_2^8$ for $\beta_2 = -\frac{14}{16}$ as follows:

$$\begin{pmatrix} 1 & 4 & 9 & 0 & 0 \\ 0 & 4 & 6 & 4 & 0 \\ 0 & 0 & 9 & 4 & 1 \\ 0 & 0 & 0 & 16 & -2 \end{pmatrix}.$$

Then, the matrices $L_0^{(\beta_2)}$, $L_1^{(\beta_2)}$, $L_2^{(\beta_3)}$ and $L_3^{(\beta_2)}$ are

$$L_0^{(\beta_2)} = \begin{pmatrix} 4 & 9 & 0 & 0 \\ 4 & 6 & 4 & 0 \\ 0 & 9 & 4 & 1 \\ 0 & 0 & 16 & -2 \end{pmatrix}, L_1^{(\beta_2)} = \begin{pmatrix} 6 & 4 & 0 \\ 9 & 4 & 1 \\ 0 & 16 & -2 \end{pmatrix},$$

$$L_2^{(\beta_3)} = \begin{pmatrix} 4 & 1 \\ 16 & -2 \end{pmatrix}, L_3^{(\beta_2)} = \begin{pmatrix} -2 \end{pmatrix}.$$

Also, $\det(L_0^{(\beta_2)}) = 576$, $\det(L_1^{(\beta_2)}) = -72$, $\det(L_2^{(\beta_2)}) = -24$, $\det(L_3^{(\beta_2)}) = -2$, $\det(L_4^{(\beta_2)}) = 1$. Therefore, $E_i(2)$ $(i = 0, 1, 2, 3, 4)$ are as follows:

$$\begin{aligned}
E_0(2) =& (-1)^0 \binom{4}{0}\binom{8-4}{0}\frac{(0!)^2(576)}{576} = 1, \\
E_1(2) =& (-1)^1 \binom{4}{1}\binom{8-4}{1}\frac{(1!)^2(-72)}{576} = 2, \\
E_2(2) =& (-1)^2 \binom{4}{2}\binom{8-4}{2}\frac{(2!)^2(-24)}{576} = -6, \\
E_3(2) =& (-1)^3 \binom{4}{3}\binom{8-4}{3}\frac{(3!)^2(-2)}{576} = 2, \\
E_4(2) =& (-1)^4 \binom{4}{4}\binom{8-4}{4}\frac{(4!)^2(1)}{576} = 1.
\end{aligned}$$

Thus, a vector $(1, 2, -6, 2, 1)$ is the second row vector of the first eigenmatrix $\mathbf{P}$ of $J(8, 4)$. In fact, the first eigenmatrix $\mathbf{P}$ of $J(8, 4)$ is as follows:

$$\begin{pmatrix} 1 & 16 & 36 & 16 & 1 \\ 1 & 8 & 0 & -8 & -1 \\ 1 & 2 & -6 & 2 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

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Gil Chun Kim
Department of Mathematics
Ewha Womans University
Seoul 120-750, Korea
E-mail address: higen@ewha.ac.kr

Yoonjin Lee
Department of Mathematics
Ewha Womans University
Seoul 120-750, Korea
E-mail address: yoonjinl@ewha.ac.kr