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SURFACES OF GENERAL TYPE WITH $p_g = 1$ AND $q = 0$

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Abstract. We construct a new family of simply connected minimal complex surfaces of general type with $p_g = 1$, $q = 0$, and $K^2 = 3, 4, 5, 6, 8$ using a $\mathbb{Q}$-Gorenstein smoothing theory.

1. Introduction

In the geography of minimal complex surfaces of general type, one of the fundamental problems is to find a new family of simply connected minimal surfaces with given topological invariants such as p_g, q, and K^2. This geography problem has been studied extensively by algebraic geometers and topologists for a long time, and various families of surfaces of general type have been constructed ([1], Chapter VII). Nevertheless, in the case of $p_g = 1$ and $q = 0$, only a few examples are known (see below) and there are no such simply connected examples with $K^2 \geq 3$ previously known. Note that this class of surfaces has drawn attention because they provide counterexamples to the Torelli problems.

On the other hand, complex surfaces of general type with $p_g = 1$ and $q = 0$ also provide exotic smooth structures on the topological 4-manifolds $3\mathbb{CP}^2\sharp n\overline{\mathbb{CP}}^2$. In fact, many examples of exotic $3\mathbb{CP}^2\sharp n\overline{\mathbb{CP}}^2$ for $n = 5$ or $7 \leq n \leq 19$ have been constructed via various surgery techniques. For example, Gompf [6] constructed exotic $3\mathbb{CP}^2\sharp n\overline{\mathbb{CP}}^2$ for $14 \leq n \leq 18$ by a symplectic sum, B. D. Park [12] constructed exotic $3\mathbb{CP}^2\sharp n\overline{\mathbb{CP}}^2$ for $10 \leq n \leq 13$ via knot surgery and symplectic sum, and Stipsicz and Szabó [16], and the second author of [13] constructed exotic $3\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\sharp 8\overline{\mathbb{CP}}^2$ using rational blowdowns, respectively. However, it is not known whether those exotic $3\mathbb{CP}^2\sharp n\overline{\mathbb{CP}}^2$'s constructed admit complex structures.

Some examples of surfaces of general type with $p_g = 1$ and $q = 0$ have been constructed. Kynev [9] constructed a surface with $K^2 = 1$ as a quotient of the Fermat sextic in $\mathbb{CP}^3$ by a suitable action of a group of order 6. According

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to Catanese [2], all minimal surfaces of general type with $p_g = 1$ and $K^2 = 1$ are diffeomorphic and simply connected. Catanese and Debarre [3] completely classified surfaces with $K^2 = 2$ and $p_g = 1$. Indeed, they classified such surfaces into five classes according to the degree and the image of the bicanonical map. Four of them are simply connected and the remaining one has a torsion $\mathbb{Z}/2\mathbb{Z}$. Todorov [17] also constructed non-simply connected surfaces with $2 \leq K^2 \leq 8$ by considering double covers of K3 surfaces.

The main result of this paper is the following.

Theorem. *There are simply connected minimal complex surfaces of general type with* $p_g = 1$*,* $q = 0$*, and* $K^2 = 3, 4, 5, 6, 8$.

In order to construct such surfaces with $K^2 = 3, 4, 5, 6$, we take a similar strategy as by Lee-Park [10]. We blow up an elliptic K3 surface in a suitable set of points so that we obtain a surface with a very special configuration of rational curves. Inside this configuration, we find some disjoint chains that can be contracted to special quotient singularities. These singularities admit a local $\mathbb{Q}$-Gorenstein smoothing, which is a smoothing whose relative canonical class is $\mathbb{Q}$-Cartier. Then we prove that these local smoothings can be glued to a global $\mathbb{Q}$-Gorenstein smoothing of the whole singular surface by showing that the obstruction space of a global smoothing is zero. Finally, it is not difficult to show that a general fiber of the smoothing is the desired surface.

For constructing a simply connected surface with $K^2 = 8$, we blow up a K3 surface $\overline{Y}$ in a suitable set of points so that we obtain a surface with some special disjoint linear chains of rational curves that can be contracted to singularities class T on a singular surface $\overline{X}$ with $H^2(\mathcal{T}_{\overline{X}}) \neq 0$. In order to prove the existence of a global $\mathbb{Q}$-Gorenstein smoothing of $\overline{X}$, we apply the cyclic covering trick developed by Lee-Park [11]. The cyclic covering trick says that if a cyclic covering $\pi : V \to W$ of singular surfaces satisfies certain conditions and the base W has a $\mathbb{Q}$-Gorenstein smoothing, then the cover V has also a $\mathbb{Q}$-Gorenstein smoothing. The main ingredient is that we construct an unramified double covering $\overline{\pi} : \overline{X} \to X$ to a singular surface X constructed in a recent paper by Keum-Lee-Park [7]. One of the main results of Keum-Lee-Park [7] is that the singular surface X has a global $\mathbb{Q}$-Gorenstein smoothing and that a general fiber X_t of the smoothing of X is a surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. We show that the double covering $\overline{\pi} : \overline{X} \to X$ satisfies all the conditions of the cyclic covering trick; hence there is a global $\mathbb{Q}$-Gorenstein smoothing of $\overline{X}$. Then it is not difficult to show that a general fiber $\overline{X}_t$ of the smoothing of $\overline{X}$ is the desired surface.

This paper is organized as follows. In Section 2 we construct a simply connected minimal complex surface with $p_g = 1$, $q = 0$ and $K^2 = 3$. We prove in Section 3 that the obstruction spaces of global smoothings of the singular surfaces constructed in Section 2 are zero. In Section 4 and Section 5, we construct various examples of simply connected surfaces with $p_g = 1$, $q = 0$ and $K^2 = 3, 4, 5, 6, 8$.

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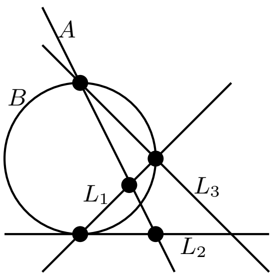
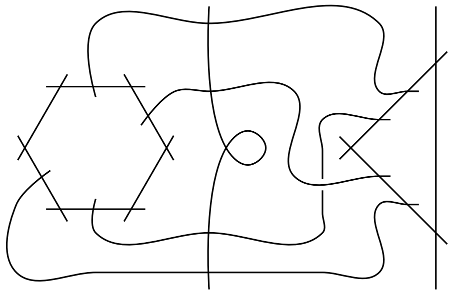
2. A simply connected surface with $p_g = 1$, $q = 0$, and $K^2 = 3$

In this section, we construct a simply connected minimal surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 3$.

Let A, L_i $(i = 1, 2, 3)$ be lines on the projective plane $\mathbb{CP}^2$ and B a nonsingular conic on $\mathbb{CP}^2$ which intersect as in Figure 1(A). Consider a pencil of cubics generated by the two cubics $A + B$ and $L_1 + L_2 + L_3$. Blow up the five base points • of the pencil of cubics including infinitely near base-points at each point. Then we obtain a rational elliptic surface $E(1)$ with an I_6-singular fiber, an I_3-singular fiber, three nodal singular fibers, and five sections which intersect as in Figure 1(B), where we omit one nodal singular fiber and one section that are not used in the following construction.



(A) A pencil



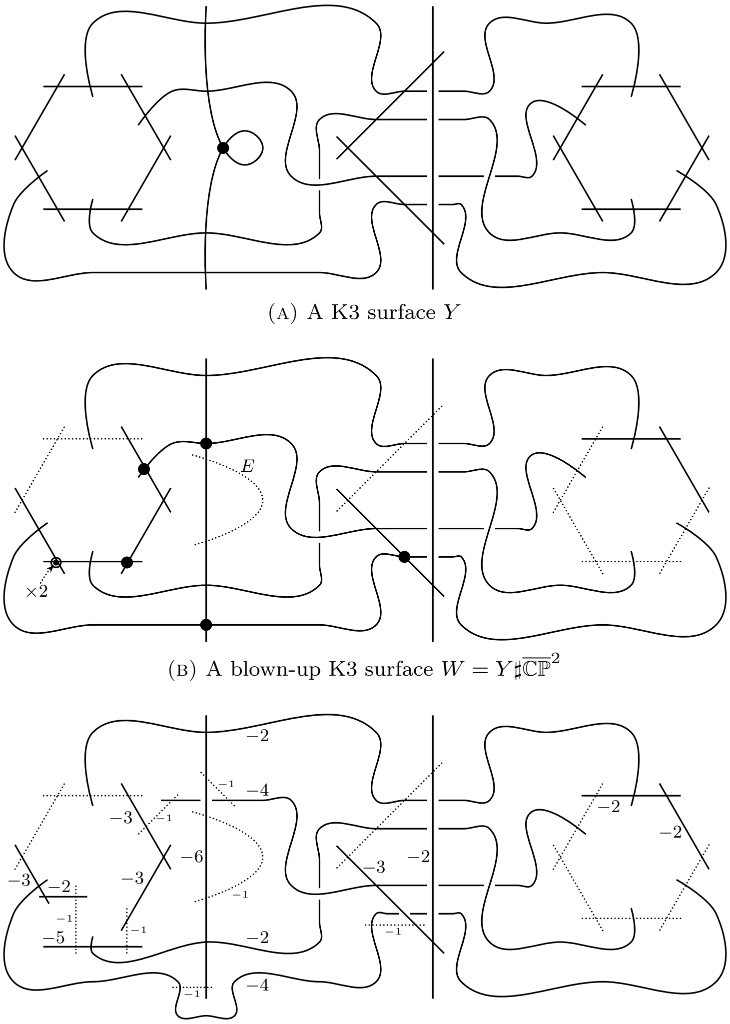
(B) A rational elliptic surface $E(1)$

FIGURE 1. A rational elliptic surface $E(1)$ for $K^2 = 3$

Let Y be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers. Then Y is an elliptic K3 surface with two I_6-singular fibers, two I_3-singular fibers, six nodal singular fibers, and five sections. We use only two I_6-singular fibers, one I_3-singular fibers, one nodal singular fibers, and four sections; see Figure 2(A). We blow up once the K3 surface Y at the node of the nodal singular fiber which is marked by •; see Figure 2(B). Let W be the blown-up K3 surface and let E be the exceptional divisor. We further blow up

W five times at the five marked points $\bullet$ and twice at the marked point $\odot$. We then get a surface Z; see Figure 2(B). There exist four disjoint linear chains of $\mathbb{CP}^1$'s:

$$\overset{-5}{\circ} - \overset{-2}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}, \quad \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-4}{\circ}, \quad \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-4}{\circ}, \quad \overset{-3}{\circ} - \overset{-3}{\circ}.$$



(A) A K3 surface Y

(B) A blown-up K3 surface $W = Y \sharp \overline{\mathbb{CP}}^2$

(C) $Z = Y \sharp 8\overline{\mathbb{CP}}^2$

FIGURE 2. An example with $K^2 = 3$

Main construction. By applying $\mathbb{Q}$-Gorenstein smoothing theory to the surface Z as by Lee-Park [10] and the authors of [14, 15], we construct a complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 3$. That is, we first contract the three chains of $\mathbb{CP}^1$'s from the surface Z so that it produces a normal projective surface X with four permissible singular points. In Section 3, we will show that the singular surface X has a global $\mathbb{Q}$-Gorenstein smoothing. Let X_t be a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of X. Since X is a (singular) surface with $p_g = 1$, $q = 0$, and $K^2 = 3$, by applying general results of complex surface theory and $\mathbb{Q}$-Gorenstein smoothing theory, one may conclude that a general fiber X_t is a complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 3$. Furthermore, it is not difficult to show that a general fiber X_t is minimal and simply connected by using a similar technique as by Lee-Park [10] and the authors of [14, 15].

3. The existence of a $\mathbb{Q}$-Gorenstein smoothing

This section is devoted to the proof of the following theorem.

Theorem 3.1. *The singular surface X constructed as in the main construction from Section 2 has a global $\mathbb{Q}$-Gorenstein smoothing.*

Since the singular surface X has only singularities of class T, it is enough to show that the obstruction space to local-to-global deformations of X vanish. That is, we will prove:

Proposition 3.2. $H^2(X, \mathcal{T}_X) = 0$.

Recall that the contraction map $Z \to X$ is the minimal resolution of the singular surface X. Then the vanishing of the obstructions space can be proved by the vanishing of the second cohomology of certain logarithmic tangent sheaf on the minimal resolution Z.

Proposition 3.3 ([10, Theorem 2]). *If $\pi : \widetilde{S} \to S$ is the minimal resolution of a normal projective singular surface S and E is the reduced exceptional divisor of the resolution π, then $H^2(S, \mathcal{T}_S) = H^2(\widetilde{S}, \mathcal{T}_{\widetilde{S}}(-\log E))$.*

Therefore, for proving Proposition 3.2, it is enough to show that

$$H^2(Z, \mathcal{T}_Z(-\log A)) = 0, \tag{3.1}$$

where A is the divisor on Z consisting of the four linear chains of $\mathbb{CP}^1$'s contracted to the four singular points of X. On one hand, we have the following well-known result.

Proposition 3.4 ([5, Lemma 1.5]). *Let $\widetilde{S}$ be a nonsingular surface and let E be a simple normal crossing divisor in $\widetilde{S}$. Let $f : \widetilde{S}' \to \widetilde{S}$ be a blowing up of $\widetilde{S}$ at a point p of E. Set $E' = f^{-1}(E)_{red}$. Then $h^2(\widetilde{S}', \mathcal{T}_{\widetilde{S}'}(-\log E')) = h^2(\widetilde{S}, \mathcal{T}_{\widetilde{S}}(-\log E))$.*

Let $D = \sum_{i=1}^{13} D_i$ be the reduced divisor in $W = Y\sharp\overline{\mathbb{CP}}^2$ consisting of the rational curves D_i whose proper transforms in the blown-up surface $Z = W\sharp 7\overline{\mathbb{CP}}^2$ are contracted to singular points of Y. In Figure 2(B), the thirteen solid curves are the components D_i of D. By Proposition 3.4, the vanishing of $H^2(Z, \mathcal{T}_Z(-\log A)) = 0$ in (3.1) and Proposition 3.2 follow from the next proposition.

Proposition 3.5. $h^2(W, \mathcal{T}_W(-\log D)) = h^0(W, \Omega^1_W(\log D)(K_W)) = 0$.

We will use the following exact sequences.

Proposition 3.6 ([4, 2.3])**.** *Let $D = \sum_{i=1}^{n} D_i$ be a simple normal crossing divisor on a smooth surface S. Then one has the following exact sequences*:

(a) $0 \to \Omega^1_S \to \Omega^1_S(\log D) \to \bigoplus_{i=1}^n \mathcal{O}_{D_i} \to 0,$
(b) $0 \to \Omega^1_S(\log D) \to \Omega^1_S(\log(D - D_1))(D_1) \to \Omega^1_{D_1}(D|_{D_1}) \to 0.$

Proof of Proposition 3.5. Note that $K_W = E$. We need to show that

$$H^0(W, \Omega^1_W(\log D)(E)) = 0.$$

By Proposition 3.6(b), we have the following exact sequence

$$0 \to H^0(W, \Omega^1_W(\log(D+E))) \to H^0(W, \Omega^1_W(\log D)(E)) \to H^0(E, \Omega^1_E((D+E)|_E)).$$

Since $H^0(E, \Omega^1_E((D+E)|_E)) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-1)) = 0$, we have

$$H^0(W, \Omega^1_W(\log D)(E)) = H^0(W, \Omega^1_W(\log(D+E))).$$

On the other hand, since $H^0(W, \Omega^1_W) = 0$, we have the following exact sequence by Proposition 3.6(a):

$$0 \to H^0(W, \Omega^1_W(\log(D+E))) \to \bigoplus_{i=1}^{13} H^0(D_i, \mathcal{O}_{D_i}) \oplus H^0(E, \mathcal{O}_E) \xrightarrow{\delta} H^1(W, \Omega^1_W).$$

Recall that the connecting homomorphism

$$\delta : \bigoplus_{i}^{13} H^0(D_i, \mathcal{O}_{D_i}) \oplus H^0(E, \mathcal{O}_E) \to H^1(W, \Omega^1_W)$$

is the first Chern class map. One can check easily that the curves $D_1, \ldots, D_{13}, E$ are linearly independent in the Picard group $\operatorname{Pic} W$ by showing that the intersection matrix of the curves $D_1, \ldots, D_{13}, E$ is invertible. Therefore, the connecting homomorphism δ is injective. Hence we have

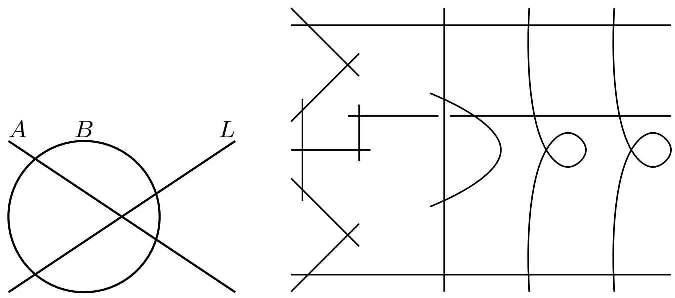
$$\ker \delta = H^0(W, \Omega^1_W(\log(D+E))) = 0. \qquad \square$$

4. Simply connected surfaces with $p_g = 1$, $q = 0$, and $K^2 = 4, 5, 6$

In this section, we construct various examples of simply connected minimal surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 4, 5, 6$. Since all proofs are similar as for the case of the main construction, we describe only complex surfaces Z that allow us to obtain singular surfaces X.

4.1. An example with $K^2 = 4$

Let A and L be lines on the projective plane $\mathbb{CP}^2$ and B a nonsingular conic on $\mathbb{CP}^2$ which intersect as in Figure 3(A). Consider a pencil of cubics generated by the two cubics $A+B$ and $3L$. By resolving the base points of the pencil, we obtain a rational elliptic surface $E(1)$ with an $\tilde{E}_6$-singular fiber, an I_2-singular fiber, two nodal singular fibers, and three sections; see Figure 3(B).



(A) A pencil (B) $E(1)$

FIGURE 3. A rational elliptic surface $E(1)$ for $K^2 = 4$

Let Y be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers. Then Y is an elliptic K3 surface with two $\tilde{E}_6$-singular fibers, two I_2-singular fibers, four nodal singular fibers, and three sections. We use only two $\tilde{E}_6$-singular fibers, two I_2-singular fibers, one nodal singular fibers, and three sections; see Figure 4(A).

We blow up the surface Y totally 16 times at the marked points. We then get a surface Z; see Figure 4(B). There exist six disjoint linear chains of $\mathbb{CP}^1$'s:

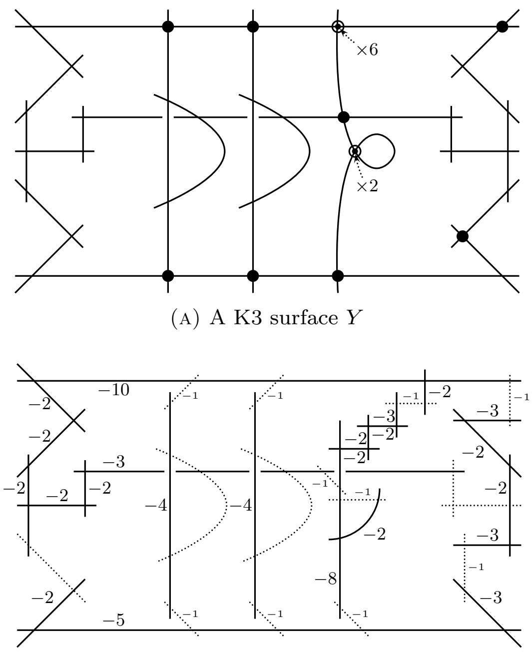
$$\overset{-2}{\circ} - \overset{-10}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}, \quad \overset{-2}{\circ} - \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$$

$$\overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}, \quad \overset{-2}{\circ} - \overset{-5}{\circ} - \overset{-3}{\circ}, \quad \overset{-4}{\circ}, \quad \overset{-4}{\circ}.$$

4.2. An example with $K^2 = 5$

Let A, L_i $(i = 1, 2, 3)$ be lines on the projective plane $\mathbb{CP}^2$ and B a nonsingular conic on $\mathbb{CP}^2$ which intersect as in Figure 5(A). Consider a pencil of cubics generated by the two cubics $A + B$ and $L_1 + L_2 + L_3$. By resolving the base points of the pencil of cubics including infinitely near base-points, we obtain a rational elliptic surface $E(1)$ with an I_7-singular fiber, an I_2-singular fiber, a cusp singular fiber, a nodal singular fiber, and five sections as in Figure 5(B).

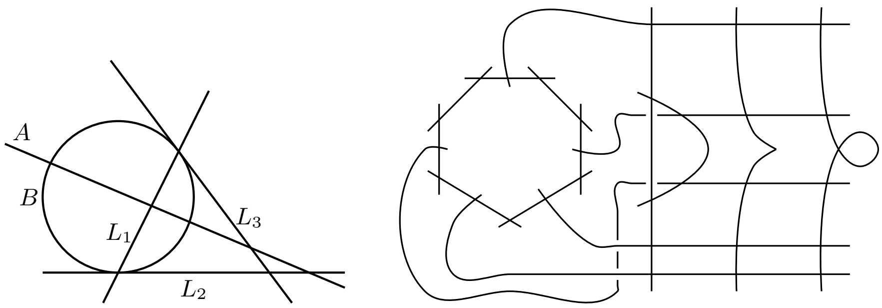
Let Y be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers F_1 and F_2 near the cusp singular fiber. Then Y is an elliptic K3 surface with two I_7-singular fibers, two I_2-singular fibers, two cusp singular fibers, two nodal singular fibers, and five sections. We use only two I_7-singular



(A) A K3 surface Y

(B) $Z = Y \sharp 16\overline{\mathbb{CP}}^2$

FIGURE 4. An example with $K^2 = 4$



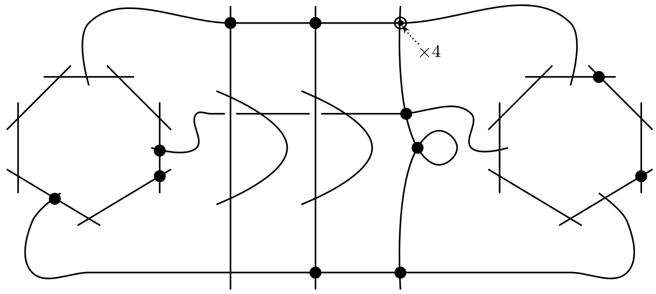
(A) A pencil

(B) $E(1)$

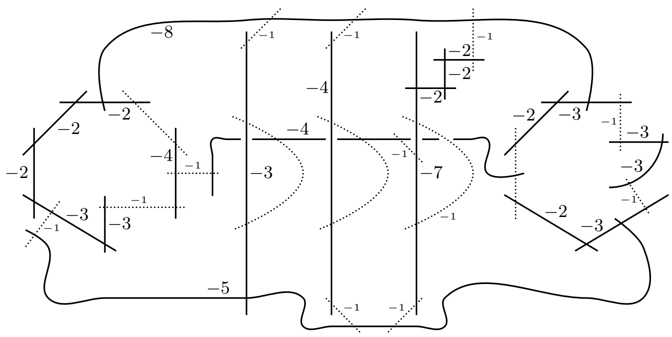
FIGURE 5. A rational elliptic surface $E(1)$ for $K^2 = 5$

fibers, two I_2-singular fibers, one nodal singular fibers, and three sections; see Figure 6(A).

We blow up the surface Y totally 15 times at the marked points. We then get a surface Z; see Figure 6(B). There exist seven disjoint linear chains of



(A) A K3 surface Y



(B) $Z = Y \sharp 15\overline{\mathbb{CP}}^2$

Figure 6. An example with $K^2 = 5$

$\mathbb{CP}^1$'s:

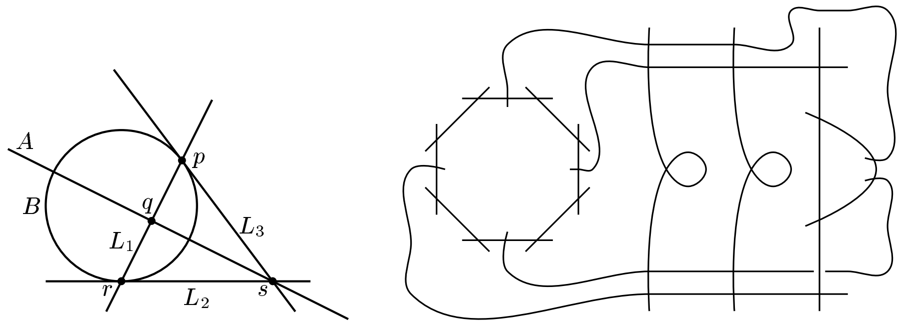
$$\overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-3}{\circ}, \quad \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

$$\overset{-3}{\circ} - \overset{-5}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ}, \quad \overset{-3}{\circ} - \overset{-3}{\circ}, \quad \overset{-4}{\circ}, \quad \overset{-4}{\circ}, \quad \overset{-4}{\circ}.$$

4.3. An example with $K^2 = 6$

Let L_1, L_2, L_3, and A be lines in $\mathbb{CP}^2$ and let B be a smooth conic in $\mathbb{CP}^2$ intersecting as in Figure 7(A). By resolving all base points (including infinitely near base points) of the pencil, we get a rational elliptic surface $E(1)$ with an I_8-singular fiber, an I_2-singular fiber, two nodal singular fibers, and four sections; see Figure 7(B).

Let Y be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers. Then the surface Y is an elliptic K3 surface with two I_8-singular fibers, two I_2-singular fibers, four nodal singular fibers, and four sections; see Figure 8(A). We use only two I_8-singular fibers, two I_2-singular fibers, one nodal singular fibers, and three sections; see Figure 8(A).



(A) A pencil (B) $E(1)$

FIGURE 7. A rational elliptic surface $E(1)$ for $K^2 = 6$

We blow up the surface Y totally 18 times at the marked points. We then get a surface Z; see Figure 8(B). There exist five disjoint linear chains of $\mathbb{CP}^1$'s:

$$\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-9}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-4}{\circ},\quad \overset{-7}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ},$$

$$\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-7}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-3}{\circ},\quad \overset{-4}{\circ}-\overset{-3}{\circ}-\overset{-2}{\circ},\quad \overset{-4}{\circ}.$$

5. A simply connected surface with $p_g = 1$, $q = 0$, and $K^2 = 8$

In this section, we construct a simply connected minimal complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 8$. According to Kondo [8], there is an Enriques surface Y with an elliptic fibration over $\mathbb{CP}^1$ which has an I_9-singular fiber, a nodal singular fiber F, and two bisections S_1 and S_2; see Figure 9. Again by Kondo [8], there is an unbranched double covering $\pi : \overline{Y} \to Y$ of Y, where $\overline{Y}$ is an elliptic K3 surface which has two I_9-singular fiber, two nodal singular fiber $\overline{F}_1$ and $\overline{F}_2$, and four sections $\overline{S}_1, \ldots, \overline{S}_4$ such that $\pi(\overline{F}_1) = \pi(\overline{F}_2) = F$, $\pi(\overline{S}_1) = \pi(\overline{S}_3) = S_1$, and $\pi(\overline{S}_2) = \pi(\overline{S}_4) = S_2$; see Figure 10.

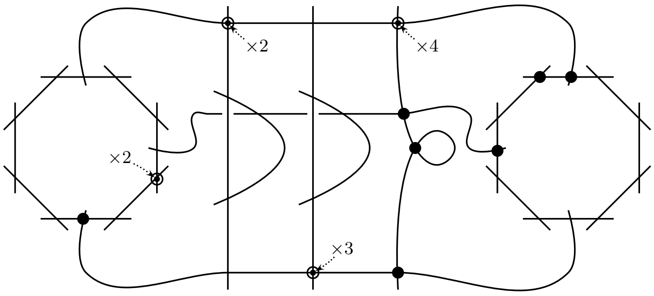
We blow up the K3 surface $\overline{Y}$ totally 30 times at the marked points $\bullet$ and $\odot$. We then get a surface $\overline{Z} = \overline{Y}\sharp 30\overline{\mathbb{CP}}^2$; see Figure 11. There exist four disjoint linear chains of $\mathbb{CP}^1$'s in $\overline{Z}$:

$$C_{19,6} : \overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-9}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ},$$

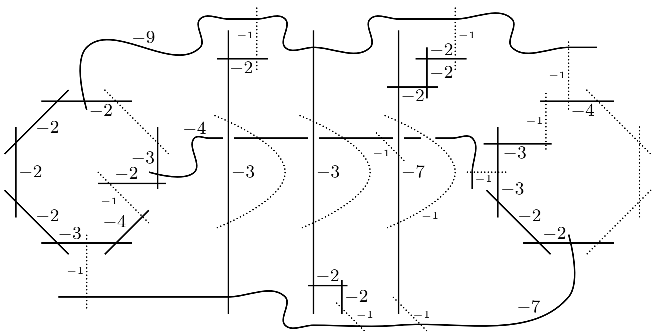
$$C_{19,6} : \overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-9}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ},$$

$$C_{73,50} : \overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-7}{\circ}-\overset{-6}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ},$$

$$C_{73,50} : \overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-7}{\circ}-\overset{-6}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ}.$$



(A) A K3 surface Y



(B) $Z = Y \sharp 18\overline{\mathbb{CP}}^2$

FIGURE 8. An example with $K^2 = 6$

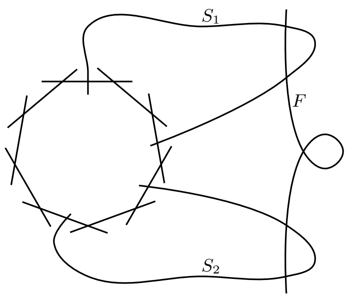


FIGURE 9. An Enriques surface Y

We contract these four chains of $\mathbb{CP}^1$'s from the surface $\overline{Z}$ so that it produces a normal projective surface $\overline{X}$ with four singular points of class T. It is not difficult to show that $H^2(\overline{X}, \mathcal{T}_{\overline{X}}) \neq 0$.

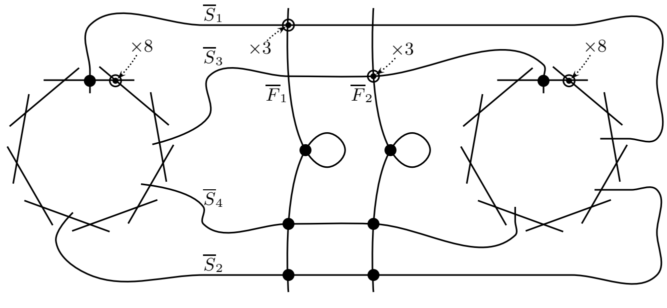


FIGURE 10. A K3 surface $\overline{Y}$

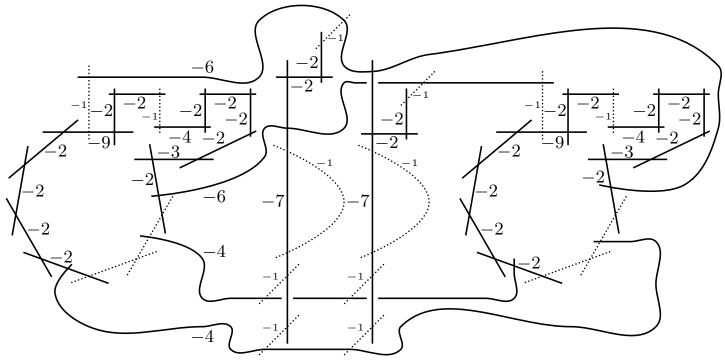


FIGURE 11. A surface $\overline{Z} = \overline{Y} \sharp 30 \overline{\mathbb{CP}}^2$

Theorem 5.1. *The singular surface $\overline{X}$ has a global $\mathbb{Q}$-Gorenstein smoothing. A general fiber $\overline{X}_t$ of the smoothing of $\overline{X}$ is a simply connected minimal complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 8$.*

In order to prove Theorem 5.1, we apply the following proposition.

Proposition 5.2 ([11])**.** *Let V be a normal projective surface with singularities of class T. Assume that a cyclic group G acts on X such that*

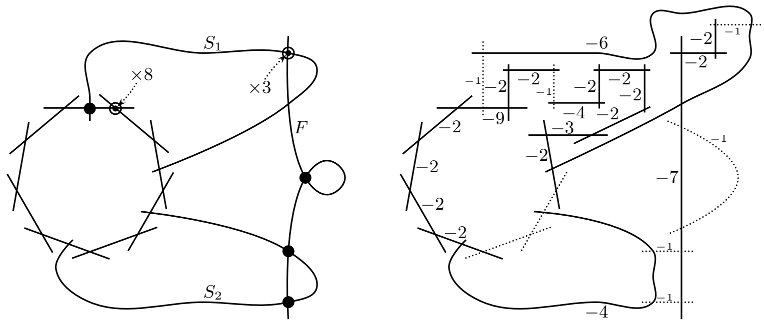
(1) *$W = V/G$ is a normal projective surface with singularities of class T,*
(2) *$p_g(W) = q(W) = 0$,*
(3) *W has a $\mathbb{Q}$-Gorenstein smoothing,*
(4) *the map $\sigma : V \to W$ induced by a cyclic covering is flat, and the branch locus D (resp., the ramification locus) of the map $\sigma : V \to W$ is a nonsingular curve lying outside the singular locus of W (resp., of V), and*
(5) *$H^1(W, \mathcal{O}_W(D)) = 0$.*

Then there exists a $\mathbb{Q}$-Gorenstein smoothing of V that is compatible with a $\mathbb{Q}$-Gorenstein smoothing of W. Furthermore, the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing.

We now construct an unramified double covering from the singular surface $\overline{X}$ to another singular surface. We begin with the Enriques surface Y in Figure 9. We blow up totally 15 times at the marked points $\bullet$ and $\odot$; see Figure 12(A). We then get a surface $Z = Y \sharp 15 \overline{\mathbb{CP}}^2$; see Figure 12(B). There exist two disjoint linear chains of $\mathbb{CP}^1$'s in Z:

$$C_{19,6} : \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-9}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ},$$

$$C_{73,50} : \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}.$$



(A) An Enriques Y (B) $Z = Y \sharp 15 \overline{\mathbb{CP}}^2$

FIGURE 12. An example with $K^2 = 8$

We contract the two chains of $\mathbb{CP}^1$'s from the surface Z so that it produces a normal projective surface X with two singular points class T. It is clear that there is an unbranched double covering $\overline{\pi} : \overline{X} \to X$. The singular surface X satisfies the third condition of Proposition 5.2.

Proposition 5.3 ([7])**.** *The singular surface X has a global $\mathbb{Q}$-Gorenstein smoothing. A general fiber X_t of the smoothing of X is a minimal complex surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1(X_t) = \mathbb{Z}/2\mathbb{Z}$.*

Proof of Theorem 5.1. It is easy to show that the covering $\overline{\pi} : \overline{X} \to X$ satisfies all conditions of Proposition 5.2. Therefore, the singular surface $\overline{X}$ has a global $\mathbb{Q}$-Gorenstein smoothing. Let $\overline{X}_t$ be a general fiber of the smoothing of $\overline{X}$. Since $p_g(\overline{X}) = 1$, $q(\overline{X}) = 0$, and $K^2_{\overline{X}} = 8$, by applying general results of complex surface theory and $\mathbb{Q}$-Gorenstein smoothing theory, one may conclude that a general fiber $\overline{X}_t$ is a complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 8$. Furthermore, it is not difficult to show that a general fiber X_t is minimal by using a similar technique as by Lee-Park [10] and the authors of [14, 15].

Claim. A general fiber $\overline{X}_t$ is simply connected: By Proposition 5.2, there is an induced unbranched double covering $\overline{X}_t \to X_t$, and hence a general fiber $\overline{X}_t$ is simply connected as $\pi_1(X_t) = \mathbb{Z}/2\mathbb{Z}$ by Proposition 5.3. □

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