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SOBOLEV ESTIMATES FOR THE LOCAL EXTENSION OF BOUNDARY HOLOMORPHIC FORMS ON REAL HYPERSURFACES IN $\mathbb{C}^n$

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Abstract. Let M be a smooth real hypersurface in complex space of dimension n, $n \geq 3$, and assume that the Levi-form at z_0 on M has at least $(q+1)$-positive eigenvalues, $1 \leq q \leq n-2$. We estimate solutions of the local $\bar\partial$-closed extension problem near z_0 for (p,q)-forms in Sobolev spaces. Using this result, we estimate the local solution of tangential Cauchy-Riemann equation near z_0 in Sobolev spaces.

1. Introduction

For a set $D \subset \mathbb{C}^n$, we denote the vector space of smooth (p,q)-forms on D by $\bigwedge^{p,q}(D)$. Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$ with a smooth defining function ρ, and let $\mathcal{B}^{p,q}(\mathcal{M})$ be the restriction of $\bigwedge^{p,q}(\mathbb{C}^n)$ to $\mathcal{M}$ which are pointwise orthogonal to the ideal generated by $\bar\partial\rho$. In the sequel, we let $z_0 \in \mathcal{M}$ be a fixed point and V be a neighborhood of z_0 in $\mathbb{C}^n$ where ρ is defined. For each open set $U \subset V$, $z_0 \in U$, we set $U^- = \{z \in U; \rho(z) \leq 0\}$ and $U^+ = \{z \in U; \rho(z) \geq 0\}$.

If there exists a neighborhood $U \subset V$, $z_0 \in U$, such that for any $\alpha \in \mathcal{B}^{p,q}(\mathcal{M} \cap U)$ with $\bar\partial_b \alpha = 0$ on $\mathcal{M} \cap U$, there exists a smooth (p,q)-form $\tilde\alpha \in \bigwedge^{p,q}(U^-)$ with $\bar\partial\tilde\alpha = 0$ in U^- and $(\tilde\alpha - \alpha) \wedge \bar\partial\rho = 0$ on $\mathcal{M} \cap U$, then we say one-sided weak $\bar\partial$-closed extension problem is locally solvable.

The $\bar\partial$-closed extension problem and the local solvability of the tangential Cauchy-Riemann equation for functions were first introduced in two papers by Hans Lewy [17, 18]. For the case when $\mathcal{M}$ is the boundary of a smoothly bounded domain Ω in $\mathbb{C}^n$, the global $\bar\partial$-closed extension problem for forms from $\mathcal{M}$ to the domain Ω was studied by J. J. Kohn and H. Rossi [14], who first introduced the $\bar\partial_b$-complex. They showed that a global $\bar\partial$-closed extension exists for any (p,q)-form from the boundary $\mathcal{M} = b\Omega$ to the domain Ω in a complex manifold if Ω satisfies the condition $Z(n-q-1)$ at all points of $b\Omega$. Analogous

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result was obtained by Henkin and Leiterer [13] using kernel methods. For the case when Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$, Shaw and Boas [19, 3] constructed a two-sided $\bar\partial$-closed extension for $\bar\partial_b$-closed forms near $b\Omega$ using the L^2-Cauchy problem for $\bar\partial$, and solved $\bar\partial_b$-problem on the boundary.

For the local extension problem, Andreotti and Hill [1] solved the local weak $\bar\partial$-closed extension problem when the Levi-form at $z_0 \in \mathcal{M}$ satisfies the condition $Y(q)$. Under the same assumption, Boggess and Shaw [4] proved the same result using integral kernel method. Recall that we say $\mathcal{M}$ satisfies condition $Y(q)$ at z_0 if the Levi form of $\mathcal{M}$ at z_0 has either $\max\{n-q, q+1\}$ eigenvalues of the same sign or $\min\{n-q, q+1\}$ positive and $\min\{n-q, q+1\}$ negative eigenvalues. Thus when $(n-q) > (q+1)$, we need $(q+1)$ mixed (positive and negative) eigenvalues for the condition $Y(q)$ to be satisfied. In [8], Cho and Choi proved the one-sided smooth extension problem (without estimates) when the Levi-form at $z_0 \in \mathcal{M}$ has at least $(q+1)$ positive eigenvalues.

Note that the estimates of the solutions of these extension problems in various spaces, such as C^k, L^p, Lipschitz or Sobolev spaces, have many applications in the study of complex analysis. For example, function theories on a bounded domain $D \subset \mathbb{C}^n$ or the embeddability of abstract CR structures [5, 6, 15, 21].

For a set $W \subset \mathbb{C}^n$, we denote the Sobolev norm of order s on W by $\|\cdot\|_{s,W}$. In [10], the author proved the local extension problem, with estimates in Sobolev spaces, for $\bar\partial_b$-closed $(0,1)$-forms on real hypersurfaces $\mathcal{M}$ in $\mathbb{C}^n$ when the Levi-form at $z_0 \in \mathcal{M}$ has two positive eigenvalues. Therefore, it is natural to ask the local extension problem, with estimates in Sobolev spaces, for (p,q)-forms when the Levi-form at $z_0 \in \mathcal{M}$ has at least $(q+1)$ positive eigenvalues (not mixed). In this case, the condition $Y(q)$ is not satisfied when $n-q > q+1$. The following theorem answers this problem.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function ρ and suppose that the Levi-form at $z_0 \in \mathcal{M}$ has at least $(q+1)$ positive eigenvalues, $1 \leq q \leq n-2$. Then there is a neighborhood U_0 of z_0 such that for any $\alpha \in \mathcal{B}^{p,q}(\mathcal{M} \cap U_0)$, satisfying $\bar\partial_b\alpha = 0$ on $\mathcal{M} \cap U_0$, there exists $\tilde\alpha \in \bigwedge^{p,q}(U_0^-)$ such that $\bar\partial\tilde\alpha = 0$ on U_0^- and $(\tilde\alpha - \alpha) \wedge \bar\partial\rho = 0$ on $\mathcal{M} \cap U_0$. Also, if $U \subset\subset U_0$ is a neighborhood of z_0 and if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U, then for each real $s \geq 0$, $\tilde\alpha$ satisfies the estimate*:

$$\|\tilde\alpha\|_{s+\frac{1}{2},U^-} \leq C_s \|\chi\alpha\|_{s+\frac{q+1}{2},\mathcal{M}}. \tag{1.1}$$

We note that the estimate (1.1) is comparable to the case when $q = 1$ in [10]. We also note that there are well-known non-solvability results of tangential Cauchy-Riemann equation for $n = 2$ [18] and for $q = n-1$ [11]. Note, however, that the local $\bar\partial$-closed extension problem and the local solvability of $\bar\partial_b$ equation are closely related [19, 3]. Using the results of Theorem 1.1, we solve the local $\bar\partial_b$-equation in Sobolev spaces.

Theorem 1.2. *Let $\mathcal{M}$, $z_0 \in \mathcal{M}$ and U_0 be as in Theorem 1.1. Also, assume that $\mathcal{M}$ is pseudoconvex near $z_0 \in \mathcal{M}$. Then there is a neighborhood W of*

z_0, $W \subset\subset \mathcal{M} \cap U_0$, such that for any $\alpha \in \mathcal{B}^{p,q}(\mathcal{M} \cap U_0)$ satisfying $\bar\partial_b \alpha = 0$ on $\mathcal{M} \cap U_0$ and for each real $s \geq 0$, there exists $u_s \in \mathcal{B}^{(p,q-1)}(W)$ such that $\bar\partial_b u_s = \alpha$ on W and satisfies the estimate:

$$\|u_s\|_{s,W} \leq C_s \|\alpha\|_{s+\frac{q+1}{2},\mathcal{M}\cap U_0^-}.$$

Remark 1.3. In Theorem 1.1, the differentiability assumption $\alpha \in C^\infty$ can be weakened to $\alpha \in H^{s+\frac{q+1}{2}}(\mathcal{M} \cap U_0)$ to get $\tilde\alpha \in H^{s+\frac{1}{2}}(M \cap U_0)$, and similarly for Theorem 1.2.

Note that the weak extension problem is a Cauchy problem to preserve the boundary values in tangential direction. This means that we have to solve $\bar\partial^*$-equation instead of $\bar\partial$-equation. Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^k$ and let $\alpha \in \mathcal{B}^{p,q}(bD)$, where $0 \leq p \leq k$ and $1 \leq q \leq k-1$. Note that a necessary and sufficient condition for the extension problem to be solved is:

$$\int_{bD} \alpha \wedge \psi = 0 \tag{1.2}$$

for every $\psi \in \bigwedge^{k-p,k-q-1}(D) \cap \mathrm{Ker}(\bar\partial)$ ([7], Theorem 9.2.1). We also note that (1.2) is equivalent to the condition $\bar\partial_b \alpha = 0$ for $q \leq k-2$.

2. Preliminaries

Let Ω be a domain in $\mathbb{C}^k$ with smooth boundary and let $\bar\partial$ be the Cauchy-Riemann operator on Ω and let $N_{(p,q)}$ denote the Neumann operator for (p,q)-forms. We also let $\mathcal{C}^{p,q}(\Omega)$ be the collection of forms $\phi \in \bigwedge^{p,q}(\Omega)$ such that $\phi \wedge \bar\partial r = 0$ on $b\Omega$, where r is a smooth defining function for Ω. Let I be an open ball in $\mathbb{R}^d$ and let $|I|$ denote the diameter of I, and let $H_{s,l}(\Omega \times I)$ be the Sobolev space of order s on Ω and of order l in I with the norm denoted by $\|\cdot\|_{s,l}$. We state a theorem (Theorem 1.7 in [9]) for the smooth dependence of solutions of the $\bar\partial$-Neumann problem with respect to a parameter $\tau \in I$.

Theorem 2.1. *Let $\{\Omega_\tau\}_{\tau\in I}$ be a smooth family of diffeomorphic strongly pseudoconvex domains in $\mathbb{C}^k$ and suppose that $\{\alpha_\tau\}_{\tau\in I}$ is a family of (p,q)-forms on $\{\Omega_\tau\}_{\tau\in I}$ such that $\alpha_\tau \in R(\bar\partial_\tau)$, the range of $\bar\partial_\tau$, for each $\tau \in I$, where $|I|$ is sufficiently small. Then for each real number $s \geq -1/2$ and for each nonnegative integer l, there is $C_{s,l} > 0$ such that the Neumann solution U_τ of $\Box U_\tau = \alpha_\tau$ and the canonical solution $u_\tau = \bar\partial^* U_\tau$ of $\bar\partial u_\tau = \alpha_\tau$ on each Ω_τ, $\tau \in I$, satisfy*

$$\|U\|_{s+1,l}, \|u\|_{s+1/2,l} \leq C_{s,l} \sum_{r=0}^{l} \|\alpha\|_{s+l-r,r}, \tag{2.1}$$

where $\alpha \in H_{s+l-r,r}(\Omega \times I)$, $0 \leq r \leq l$, and where $U := \{u_\tau\}_{\tau\in I}$ and $\alpha := \{\alpha_\tau\}_{\tau\in I}$.

Let $(\mathcal{M},\, \rho)$ be as in Section 1, and assume that the Levi-form at $z_0 \in \mathcal{M}$ has $k \geq 1$ positive eigenvalues. To prove the solvability of the local extension problem, we need to construct a smooth family of strongly convex domains near $z_0 \in \mathcal{M}$ which are foliated in the side $\rho \leq 0$ and make up a neighborhood U_0^- of $z_0 \in \mathcal{M}$. We first prove the following lemma, which describes the local geometry of $\mathcal{M}$ near z_0 in terms of local coordinates.

Lemma 2.2. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$ and assume that the Levi-form at $z_0 \in \mathcal{M}$ has $k \geq 1$ positive eigenvalues. There is a special coordinate $z = (z_1, \dots, z_n)$ defined in a neighborhood of z_0 and new defining function ρ of $\mathcal{M}$ which can be written, in new coordinates, by*

$$\rho(z) = |z_n|^2 - 1 + \sum_{i=1}^{k} |z_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |z_i|^2 + \mathcal{O}(|z - z_0|^3). \tag{2.2}$$

Proof. Let ρ be a smooth defining function of $\mathcal{M}$. By a standard method of holomorphic coordinate changes, we have special coordinates $u = (u', u'')$, $u' = (u_1, \dots, u_k)$, $u'' = (u_{k+1}, \dots, u_n)$, $u(z_0) = 0$ and the Taylor expansion near $z_0 = 0$ can be written as:

$$\rho(u) = u_n + \bar{u}_n + \sum_{i=1}^{k} |u_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |u_i|^2 + \sum_{k=1}^{n} c_k u_k \bar{u}_n + \sum_{k=1}^{n} \bar{c}_k \bar{u}_k u_n + \mathcal{O}(|u|^3),$$

where each λ_i is a real number and $\mathcal{O}(|u|^3)$ is the remainder whose first and second derivatives vanishes at 0. Set $r(u) = \rho(u) \cdot (1 - \sum_{k=1}^{n} c_k u_k - \sum_{k=1}^{n} \bar{c}_k \bar{u}_k)$, where $(1 - 2\mathrm{Re} \sum_{k=1}^{n} c_k u_k) > 1/2$ in a neighborhood of the origin. Therefore, r is a new local defining function of $\mathcal{M}$ near $z_0 \in \mathcal{M}$ and can be written as

$$r(u) = (u_n - \sum_{k=1}^{n} c_k u_k u_n) + (\bar{u}_n - \sum_{k=1}^{n} \bar{c}_k \bar{u}_k \bar{u}_n) + \sum_{i=1}^{k} |u_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |u_i|^2 + \mathcal{O}(|u|^3).$$

Set $w_j = u_j$ for $j < n$, and $w_n = u_n - \sum_{k=1}^{n} c_k u_k u_n$. In w coordinates, $r(w)$ has the representation:

$$r(w) = w_n + \bar{w}_n + \sum_{i=1}^{k} |w_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |w_i|^2 + \mathcal{O}(|w|^3).$$

Set $\tilde{r}(w) = r(w) \cdot (1 + (w_n + \bar{w}_n)/2)$, where $1 + (w_n + \bar{w}_n)/2) > 1/2$ near $z_0 = 0$. Then $\tilde{r}(w)$ can be written as:

$$\tilde{r}(w) = w_n + \bar{w}_n + \frac{1}{2}(w_n^2 + \bar{w}_n^2) + |w_n|^2 + \sum_{i=1}^{k} |w_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |w_i|^2 + \mathcal{O}(|w|^3).$$

By setting

$$\tilde{u}_n = w_n + \frac{1}{2} w_n^2 \quad \text{and} \quad \tilde{u}_j = w_j,\ j < n,$$

$\tilde{r}$ can be written, in $\tilde{u}$-coordinates, by:

$$\tilde{r}(\tilde{u}) = \tilde{u}_n + \overline{\tilde{u}}_n + |\tilde{u}_n|^2 + \sum_{i=1}^{k} |\tilde{u}_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |\tilde{u}_i|^2 + \mathcal{O}(|\tilde{u}|^3).$$

Finally, we set $z_n = \tilde{u}_n + 1$ and $z_j = \tilde{u}_j$ for $j < n$, and denote $\tilde{r}$ by ρ. Then $z(z_0) = (0, \ldots, 0, 1)$ and in z-coordinates, the local defining function ρ can be written as:

$$\rho(z) = |z_n|^2 - 1 + \sum_{i=1}^{k} |z_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |z_i|^2 + \mathcal{O}(|z - z_0|^3)$$

near $z_0 = z(z_0) = (0, \ldots, 0, 1)$. □

Set $z = (z', z'')$, where $z' = (z_1, \ldots, z_k)$ and $z'' = (z_{k+1}, \ldots, z_n)$. In the following proposition, we regard $z'' \in \mathbb{C}^{n-k}$ as a parameter variable near $t_0'' := z_0'' = (0, \ldots, 0, 1) \in \mathbb{C}^{n-k}$ and construct a family of strongly convex domains. In Section 3, we will apply Theorem 2.1 to this parameter family of domains.

Proposition 2.3. *Let $(\mathcal{M}, \rho)$ and z_0 be as in Lemma 2.2. Then there exist a small open ball $I := B_{\sigma_0}(t_0'') \subset \mathbb{C}^{n-k}$ for a small $\sigma_0 > 0$ and a family of bounded strongly convex domains $\{\Omega_{t''}\}_{t'' \in I}$ in $\mathbb{C}^k$, and $\Omega_{t''}$ is diffeomorphic to $\Omega_{t_0''}$ for each $t'' \in I$ with diameters being strictly bounded from below (say, by $\sigma_0^{17/48}$), and foliate into the part $\rho \leq 0$ making up a neighborhood U_0^- of z_0.*

Proof. For a sufficiently small $\sigma > 0$ to be determined, let $B_\sigma(t_0'') \subset \mathbb{C}^{n-k}$ be a ball of radius $\sigma > 0$ centered at t_0''. For any fixed $t'' = (t_{k+1}, \ldots, t_n) \in B_\sigma(t_0'')$ and for each $|z'| < \sigma^{1/4}$, set $\tilde{\rho}(z', t'') = \rho(z', t_{k+1}, \ldots, t_{n-1}, t_n - \sigma^{1/3} z_1)$. In view of (2.2), we can write

$$\begin{aligned} \tilde{\rho}(z', t'') &= (|t_n|^2 - 1) - 2\sigma^{1/3} \mathrm{Re}(z_1 \bar{t}_n) + \sigma^{2/3} |z_1|^2 + \sum_{i=1}^{k} |z_i|^2 \\ &\quad + \sum_{i=k+1}^{n-1} \lambda_i |t_i|^2 + \mathcal{O}(|z - z_0|^3). \end{aligned} \tag{2.3}$$

For $t'' \in B_\sigma(t_0'')$, we set

$$r_{t''}^\sigma(z') := (1 - |t_n|^2) + 2\sigma^{1/3} \mathrm{Re}(z_1 \bar{t}_n) - \sum_{i=k+1}^{n-1} \lambda_i |t_i|^2 + \mathcal{O}(|z - z_0|^3).$$

Then when $t'' = t_0'' = (0, \ldots, 0, 1)$, i.e., at the center of the ball $B_\sigma(t_0'')$, we have

$$r_{t_0''}^\sigma(z') = 2\sigma^{1/3} \mathrm{Re}(z_1) + \mathcal{O}(|z'|^3) > 0$$

for an appropriate z_1 (say, at $\mathrm{Re} z_1 = \sigma^{3/8}$) provided $\sigma > 0$ is sufficiently small. Therefore, it follows that

$$\Omega_{t''_0} := \{z' \in B_{\sigma^{1/4}}(z'_0); \sigma^{2/3}|z_1|^2 + \sum_{i=1}^{k} |z_i|^2 < r^{\sigma}_{t''_0}(z')\}$$

is a non-empty strongly convex domain contained in the side of $\rho \le 0$. Note that $\Omega_{t''_0}$ is a small deformation of a ball whose radius is bigger than or equal to $\sigma^{17/48}$. Also we see that $z_0 \in b\Omega_{t''_0} \subset \mathcal{M}$ and $\Omega_{t''_0}$ is the central slice of the side $\rho \le 0$.

For any $t'' = (t_{k+1}, \dots, t_n) \in B_\sigma(t''_0)$, we note that $|t_n - 1| < \sigma$. Hence $r^\sigma_{t''}$ is a small (of size less than σ) perturbation of $r^\sigma_{t''_0}$. Therefore, as for the $r^\sigma_{t''_0}$ case, it follows that $r^\sigma_{t''}(z') > 0$ for some z' and hence for each $t'' \in B_\sigma(t''_0)$,

$$\Omega_{t''} := \{z' \in \mathbb{C}^k; \sigma^{2/3}|z_1|^2 + \sum_{i=1}^{k} |z_i|^2 < r^{\sigma}_{t''}(z')\}$$

is a nonempty strongly convex domain in $\mathbb{C}^k$ contained in the side of $\{z; \rho(z) < 0\}$ and $b\Omega_{t''} \subset \mathcal{M}$, and the diameter of $\Omega_{t''}$ is bigger than or equal to $\sigma^{17/48}$ provided σ is sufficiently small. Let us fix $\sigma = \sigma_0$ satisfying the above conditions, and set $I := B_{\sigma_0}(t''_0) \subset \mathbb{C}^{n-k}$ and

$$U_0^- := \bigcup_{t'' \in I} \overline{\Omega}_{t''} \times \{t''\}. \tag{2.4}$$

This proves the proposition. □

Remark 2.4. Note that $\mathrm{Re} z_1 \lesssim \sigma^{1/3}$ if $z' \in \Omega_{t''}$, which forces that $|z'| \lesssim \sigma^{1/3}$, that is, $\Omega_{t''} \subset B_{\sigma^{7/24}}(z_0) \subset B_{\sigma^{1/4}}(z_0)$ provided σ is sufficiently small. Also, in view of our construction, we may take σ_0 sufficiently small so that Theorem 2.1 holds for $I = B_{\sigma_0}$ and $U_0^- \subset B_{\sigma_0^{1/4}}(z_0)$.

Remark 2.5. With the special coordinates $z = (z', t'')$ defined in (2.3), set $\tilde{t}_n = 1 + \frac{1}{2}\sigma^{7/24}$, and for each $|t_j| < \sigma^{1/3}$, $j = k+1, \dots, n-1$, set $\tilde{t}'' = (t_{k+1}, \dots, t_{n-1}, \tilde{t}_n)$. For each $|z'| < \sigma^{1/3}$, we then have $r^\sigma_{\tilde{t}''}(z') \approx -\sigma^{7/24}$, and hence $\tilde{\rho}(z', \tilde{t}'') > 0$. Set $\tilde{D}''_\sigma := \{t'' \in \mathbb{C}^{n-k}; |t_j| < \sigma^{1/3},\ k+1 \le j \le n-1,\ |t_n - \tilde{t}_n| < \sigma\}$. Then for each $t'' \in \tilde{D}''_\sigma$ and $|z'| < \sigma^{1/3}$, it follows that $\tilde{\rho}_{t''}(z') > 0$, and hence $\Omega_{t''}$ is an empty set when $t'' \in \tilde{D}''_\sigma$.

3. $\bar{\partial}$-closed extension for (p, q)-forms

To prove the local extension theorem, we use the local decomposition of the set U_0^- considered in (2.4) and use Proposition 2.3 with $k = q+1$. We will solve the ϑ-equation to correct the terms and use the estimates (2.1) on parameter variables $z'' = t'' \in I$, where I is defined as in (2.4). Set $\mathcal{K} = \{1, \ldots, q+1\}$ and $\mathcal{K}^c = \{q+2, \ldots, n\}$. For a smooth function f defined in $\mathbb{C}^n$, we define

$$\bar\partial_{\mathcal{K}} f = \sum_{j=1}^{q+1} \frac{\partial f}{\partial \bar z_j} d\bar z_j \quad \text{and} \quad \bar\partial_{\mathcal{K}^c} f = \sum_{j=q+2}^{n} \frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

We can extend this definition for arbitrary smooth forms.

Since p does not play any important role in the estimates, we set $p = 0$, i.e., we consider only the cases of $\bigwedge^{0,q}(W)$, where W is an appropriate set. We recall that $\|\cdot\|_{s,k,W}$ is the Sobolev space of order s in $z' \in \mathbb{C}^{q+1}$ variables and of order k in $z'' \in I \subset \mathbb{C}^{n-q-1}$ variables. We also note that $\bigwedge^{p,q}(\Omega_{z''})$ and $\bigwedge^{p,q}(b\Omega_{z''})$ are defined on $\Omega_{z''} \subset \mathbb{C}^{q+1}$, and that every summation will be over strictly increasing indices. In the sequel, the constants, such as C_s or $C_{s,k}$, depend only on s or k and can vary line-to-line while we estimate.

Proposition 3.1. *Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function ρ defined in a neighborhood V of $z_0 \in \mathcal{M}$, and suppose that the Levi-form at z_0 has at least $(q+1)$ positive eigenvalues, $1 \leq q \leq n-2$. Then there is a neighborhood U_0, $z_0 \in U_0$, such that for any $\alpha \in \mathcal{B}^{0,q}(\mathcal{M} \cap U_0)$, there are $\tilde\alpha_j \in \bigwedge^{0,q}(U_0^-)$, $0 \leq j \leq q+1$, such that $(\tilde\alpha_j - \alpha) \wedge \bar\partial\rho = 0$ on $\mathcal{M} \cap U_0$ and $\bar\partial\tilde\alpha_j$ can be written as*

$$\bar\partial\tilde\alpha_j = \sum_{\substack{I \subset \mathcal{K}, J \subset \mathcal{K}^c \\ |I|+|J| = q+1, |J| \geq j}} \alpha^j_{IJ} d\bar z^I \wedge d\bar z^J \tag{3.1}$$

on U_0^- for some smooth functions α^j_{IJ}. Also, if $U \subset\subset U_0$ is a neighborhood of z_0 and if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U, then for each real $s \geq 0$, $\tilde\alpha_j$ satisfy the estimate

$$\|\tilde\alpha_j\|_{s+\frac{1}{2}, U^-} \leq C_s \|\chi\alpha\|_{s+\frac{j}{2}, \mathcal{M}} \tag{3.2}$$

for each $0 \leq j \leq q+1$.

Proof. Let us take $U_0 \subset B_{\sigma_0^{1/4}} \subset V$ as defined in (2.4) where special frames are defined on V. By shrinking I if necessary, we may assume that Theorem 2.1 holds on U_0. Using Theorem 2.1, we shall construct $\tilde\alpha_j$ inductively satisfying (3.1) and (3.2). From Lemma 9.3.3 in [7], there is $\tilde\alpha_0 := E\alpha \in \bigwedge^{0,q}(U_0)$, $E\alpha = \alpha$ and $\bar\partial E\alpha = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U_0$ such that for each real s, we have

$$\|\tilde\alpha_0\|_{s, U_0^-} \leq C_s \|\chi\alpha\|_{s-1/2, \mathcal{M}}. \tag{3.3}$$

Thus (3.1) and (3.2) hold for $j = 0$.

Let $U \subset\subset U_0$ be a neighborhood of z_0 and choose a smooth cut-off function $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U. Replacing $\tilde\alpha_0$ by $\chi\tilde\alpha_0$, we may assume that

$\tilde{\alpha}_0 \in C_0^\infty(U_0)$. Let us write

$$\begin{aligned}\bar{\partial}\tilde{\alpha}_0 &= \alpha^0_{\mathcal{K}} d\bar{z}^{\mathcal{K}} + \sum_{j=q+2}^{n} \sum_{\substack{|I|=q\\ I\subset\mathcal{K}}} \beta^1_{Il} d\bar{z}^I \wedge d\bar{z}^j + \sum_{\substack{|I|+|J|=q+1,\ |J|\geq 2\\ I\subset\mathcal{K}, J\subset\mathcal{K}^c}} E^2_{IJ} d\bar{z}^I \wedge d\bar{z}^J \\ &:= \alpha^0 + \beta^1 + E^2,\end{aligned}$$

and set

$$g^0 = \alpha^0 \quad \text{on} \quad U_0^-, \quad \text{and} \quad g^0 = 0 \quad \text{on} \quad U_0^+.$$

Since $\bar{\partial}\tilde{\alpha}_0 = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U_0$, it follows that $g^0 \in C^\infty(U_0)$. Similarly, if we define $\beta^1_{Il} = E^2_{IJ} = 0$ on U_0^+, it follows that $\beta^1_{Il}, E^2_{IJ} \in C^\infty(U_0)$. Note that g^0 comes from the components of $\bar{\partial}_{\mathcal{K}}\tilde{\alpha}_0$. By (3.3), for each real $s \geq -1$ and nonnegative integer k, there are $\tilde{C}_{s,k}$ and $C_{s,k}$ such that

$$\|g^0\|_{s,k,U^-} \leq \tilde{C}_{s,k} \|\tilde{\alpha}_0\|_{s+1,k,U^-} \leq C_{s,k} \|\chi\alpha\|_{s+k+1/2,\mathcal{M}}. \tag{3.4}$$

To remove α^0 term in $\bar{\partial}\tilde{\alpha}_0$, we try to solve $\bar{\partial}_{\mathcal{K}} u_0(\cdot, z'') = g^0(\cdot, z'')$ in $\Omega_{z''}$ and set $\tilde{\alpha}_1(\cdot, z'') = \tilde{\alpha}_0(\cdot, z'') - u_0(\cdot, z'')$ for each $z'' \in I$. However, to preserve the boundary condition, it is required that $u_0(\cdot, z'') \in \mathcal{C}^{0,q}(\Omega_{z''})$ for each $z'' \in I$. This means that we have to solve $\bar{\partial}^*_{\mathcal{K}}$-equation rather than $\bar{\partial}_{\mathcal{K}}$-equation. Since $g^0(\cdot, z'')$ is a $(0, q+1)$-form in $\Omega_{z''} \subset \mathbb{C}^{q+1}$, it becomes a top degree problem in $\mathbb{C}^{q+1}$ and hence it is required to satisfy (1.2), that is,

$$F_h(z'') := \int_{\Omega_{z''}} g^0(\cdot, z'') \wedge h(\cdot) = 0, \quad \text{for every} \quad h \in C^\infty_{(q+1,0)}(\mathbb{C}^{q+1}) \cap \ker(\bar{\partial}_{\mathcal{K}}). \tag{3.5}$$

To prove (3.5), we consider the coefficients of the terms of $d\bar{z}^{\mathcal{K}} \wedge d\bar{z}_l$ in $\bar{\partial}^2\tilde{\alpha}_0 = 0$, where $l \in \mathcal{K}^c$. Note that these terms are coming from $\bar{\partial}_{\mathcal{K}^c}\alpha^0 + \bar{\partial}_{\mathcal{K}}\beta^1$. Thus we obtain that

$$\frac{\partial \alpha^0_{\mathcal{K}}}{\partial \bar{z}_l} = \sum_{j=1}^{q+1} \sum_{\substack{|I|=q\\ I\subset\mathcal{K}}} (-1)^{q+j-1} \frac{\partial \beta^1_{Il}}{\partial \bar{z}_j}, \quad l = q+2, \ldots, n. \tag{3.6}$$

In view of (3.5) and (3.6), it follows, for $l \in \mathcal{K}^c$, that

$$\begin{aligned}\frac{\partial F_h}{\partial \bar{z}_l}(z'') &= \frac{\partial}{\partial \bar{z}_l} \int_{\mathbb{C}^{q+1}} g^0(\cdot, z'') \wedge h \\ &= \int_{\Omega_{z''}} \frac{\partial}{\partial \bar{z}_l} \alpha^0_{\mathcal{K}}(\cdot, z'') \wedge h \\ &= \int_{\Omega_{z''}} \sum_{j=1}^{q+1} \sum_{\substack{|I|=q\\ I\subset\mathcal{K}}} (-1)^{q+j-1} \frac{\partial \beta^1_{Il}}{\partial \bar{z}_j} \wedge h \\ &= \int_{\Omega_{z''}} \sum_{j=1}^{q+1} \sum_{\substack{|I|=q\\ I\subset\mathcal{K}}} (-1)^{q+j} \beta^1_{Il} \wedge \frac{\partial h}{\partial \bar{z}_j} = 0\end{aligned}$$

because $h(\cdot)$ is holomorphic in $\Omega_{z''}$. Here, the first and the second equalities hold because g^0 is supported in $\Omega_{z''} \subset \mathbb{C}^{q+1}$, and the fourth equality holds because $\beta^1_{Il} = 0$ on $b\Omega_{z''}$ (and hence we can perform integration by parts). Therefore, $F_h(z'')$ is holomorphic in $\mathbb{C}^{n-q-1}$. Moreover, in view of Remark 2.5, it follows that $F_h(z'') = 0$ for $z'' \in \tilde{D}''_\sigma$ (since $\Omega_{z''}$ becomes the empty set), provided σ is sufficiently small. Here, $\tilde{D}''_\sigma$ is the tube defined in Remark 2.5. Thus we see that (3.5) holds.

Set $u_0(\cdot, z'') = - *_\mathcal{K} \bar{\partial}_\mathcal{K} N^\mathcal{K}_{(q+1,0)} *_\mathcal{K} g^0(\cdot, z'')$, $z'' \in I$, where $N^\mathcal{K}_{(r,s)}$ is the Neumann operator for (r, s)-forms and $*_\mathcal{K}$ is the Hodge star operator on $\Omega_{z''}$ for each $z'' \in I$. Then we have $\overline{\partial}_\mathcal{K} u_0(\cdot, z'') = g^0(\cdot, z'')$ and $u_0(\cdot, z'') \in \mathcal{C}^{0,q}(\Omega_{z''})$ for each $z'' \in I$. We also note that $u_0(\cdot, z'')$ depends smoothly on $z'' \in I$ and satisfies the estimate (2.1) including the parameter variable $z'' \in I$. Thus for each real $s \geq -1/2$ and nonnegative integer k, it follows from (2.1) and (3.4) that

$$(3.7) \quad \|u_0\|_{s+\frac{1}{2},k,U^-} \lesssim \sum_{r=0}^{k} \|g^0\|_{s+k-r,r,U^-} \lesssim \|\tilde{\alpha}_0\|_{s+1+k,U^-} \lesssim \|\chi\alpha\|_{s+\frac{1}{2}+k,\mathcal{M}}.$$

Note that $u_0 \wedge \bar{\partial}_\mathcal{K}\rho = 0$ on $\mathcal{M} \cap U_0$ because $u_0(\cdot, z'') \in \mathcal{C}^{0,q}(\Omega_{z''})$ for each $z'' \in I$. We have to correct u_0 so that the corrected one, $\tilde{u}_0$, belongs to $\mathcal{C}^{0,q}(\Omega)$, that is, $\tilde{u}_0 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$. Since $u_0 \wedge \bar{\partial}_\mathcal{K}\rho = 0$ on $\mathcal{M} \cap U_0$, we can write

$$u_0(\cdot, z'') = \delta^0(\cdot, z'') \wedge \bar{\partial}_\mathcal{K}\rho(\cdot, z'') + \rho(\cdot, z'')\gamma^0(\cdot, z'')$$

for some $\delta^0(\cdot, z'') \in \mathcal{C}^{0,q}(\Omega_{z''})$ and $\gamma^0(\cdot, z'') \in \mathcal{C}^{0,q+1}(\Omega_{z''})$, $z'' \in I$. We may assume that $\delta^0 \wedge \bar{\partial}_\mathcal{K}\rho$ and $\rho\gamma^0$ are disjoint, and hence it follows from the estimate in (3.7) that

$$(3.8) \qquad \|\delta^0\|_{s+\frac{1}{2},k,U^-} \lesssim \|u_0\|_{s+\frac{1}{2},k,U^-} \lesssim \|\chi\alpha\|_{s+\frac{1}{2}+k,\mathcal{M}}$$

for each real $s \geq -\frac{1}{2}$ and nonnegative integer k.

Set $\tilde{u}_0 = u_0 + \delta^0 \wedge \bar{\partial}_{\mathcal{K}^c}\rho$. Then one obtains that

$$\tilde{u}_0 \wedge \bar{\partial}\rho = \left(\delta^0 \wedge \bar{\partial}_\mathcal{K}\rho + \rho\gamma^0 + \delta^0 \wedge \bar{\partial}_{\mathcal{K}^c}\rho\right) \wedge \bar{\partial}\rho = 0,$$

on $\mathcal{M} \cap U_0$, and hence $\tilde{u}_0 \in \mathcal{C}^{0,q}(U_0^-)$. Now we set $\tilde{\alpha}_1 = \tilde{\alpha}_0 - \tilde{u}_0$. Then it follows that $(\tilde{\alpha}_1 - \alpha) \wedge \partial\rho = -\tilde{u}_0 \wedge \partial\rho = 0$ on $U \cap \mathcal{M}$, and we can write

$$(3.9) \qquad \bar{\partial}\tilde{\alpha}_1 = \sum_{l=q+2}^{n} \sum_{\substack{|I|=q \\ I\subset\mathcal{K}}} \tilde{\beta}^1_{Il} d\bar{z}^I \wedge d\bar{z}^l + \sum_{\substack{|I|+|J|=q+1,\ |J|\geq 2 \\ I\subset\mathcal{K}, J\subset\mathcal{K}^c}} \tilde{\beta}^2_{IJ} d\bar{z}^I \wedge d\bar{z}^J$$

for some smooth functions $\tilde{\beta}^1_{Il}$ and $\tilde{\beta}^2_{IJ}$. In view of (3.3), (3.7) and (3.8), one obtains that

$$(3.10) \qquad \|\tilde{\alpha}_1\|_{s+\frac{1}{2},k,U^-} \lesssim \|\chi\alpha\|_{s+\frac{1}{2}+k,\mathcal{M}}$$

for each real $s \geq -\frac{1}{2}$ and nonnegative integer k.

By induction, assume that there are $\tilde{\alpha}_j \in \bigwedge^{0,q}(U_0^-)$, $j \geq 0$, satisfying (3.1) and (3.2), and that for each real $s \geq 0$, the estimate

$$\|\tilde{\alpha}_j\|_{s+\frac{1}{2},U^-} \leq C_s \|\chi\alpha\|_{s+\frac{j}{2},\mathcal{M}}, \tag{3.11}$$

holds and $(\tilde{\alpha}_j - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$. By (3.7), (3.10) and the Sobolev interpolation theorem, we see that (3.1), (3.2) and (3.11) hold for $j = 1$.

If we replace $\tilde{\alpha}_j$ by $\chi\tilde{\alpha}_j$, we may assume that $\tilde{\alpha}_j \in C_0^\infty(U_0)$ as before. Let us write:

$$\begin{aligned}\bar{\partial}\tilde{\alpha}_j &= \sum_{\substack{|I|=q-j+1,|J|=j\\ I\subset\mathcal{K},J\subset\mathcal{K}^c}} \beta^j_{IJ} d\bar{z}^I \wedge d\bar{z}^J + \sum_{\substack{|I|+|J|=q+1,|J|\geq j+1\\ I\subset\mathcal{K},J\subset\mathcal{K}^c}} E^{j+1}_{IJ} d\bar{z}^I \wedge d\bar{z}^J \\ &:= \beta^j + E^{j+1}.\end{aligned}$$

For each fixed J with $|J| = j \geq 1$, set

$$\beta^j_J = \sum_{\substack{|I|=q-j+1\\ I\subset\mathcal{K}}} \beta^j_{IJ} d\bar{z}^I.$$

If we consider the terms in $\bar{\partial}^2\tilde{\alpha}_j$ with $|J| = j \geq 1$, $J \subset \mathcal{K}^c$, we see that $\beta^j_J|_{\bar{\Omega}_{z''}} = \beta^j_J(\cdot, z'')$ is a smooth $\bar{\partial}_{\mathcal{K}}$-closed $(0, q-j+1)$-form in $\Omega_{z''} \subset \mathbb{C}^{q+1}$. Since $\vartheta_{\mathcal{K}} = -*_{\mathcal{K}}\bar{\partial}_{\mathcal{K}}*_{\mathcal{K}}$, it follows that $*_{\mathcal{K}}\beta^j_J$ is a $\vartheta_{\mathcal{K}}$-closed $(q+1, j)$-form in $\Omega_{z''} \subset \mathbb{C}^{q+1}$ for each $z'' \in B_{\sigma_0}(z''_0) = I$.

Set

$$u^j_J(\cdot, z'') = -*_{\mathcal{K}}\bar{\partial}_{\mathcal{K}} N^{\mathcal{K}}_{(q+1,j)} *_{\mathcal{K}} \beta^j_J(\cdot, z''),$$

where $N^{\mathcal{K}}_{(q+1,j)}$ is the Neumann operator for $(q+1, j)$-forms on the strongly pseudoconvex domains $\Omega_{z''} \subset \mathbb{C}^{q+1}$. Thus $u^j_J(\cdot, z'') \in \mathcal{C}^{0,q-j}(\Omega_{z''})$, varying smoothly on $z'' \in I$, and for each real $s \geq 0$ and nonnegative integer k, it follows from (2.1) and (3.11) that

$$\|u^j_J\|_{s+\frac{1}{2},k,U^-} \lesssim \sum_{r=0}^{k} \|\beta^j_J\|_{s+k-r,r,U^-} \lesssim \|\tilde{\alpha}_j\|_{s+k+1,U^-} \lesssim \|\chi\alpha\|_{s+k+\frac{j+1}{2},\mathcal{M}}. \tag{3.12}$$

Here, we need $s \geq 0$ (rather than $s \geq -1/2$) because β^j_J may contain terms in $\bar{\partial}_{\mathcal{K}^c}\tilde{u}_{j-1}$ that can be estimated only in Sobolev s-norm in $\mathbb{C}^{q+1}$ for $s \geq 0$.

As for the $j = 1$ case, we have to correct u^j_J so that the corrected one, $\tilde{u}^j_J$, belongs to $\mathcal{C}^{0,q-j}(U_0)$, that is, $\tilde{u}^j_J \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$. Since $u^j_J \wedge \bar{\partial}_{\mathcal{K}}\rho = 0$ on $\mathcal{M} \cap U_0$, we can write

$$u^j_J(\cdot, z'') = \delta^j_J(\cdot, z'') \wedge \bar{\partial}_{\mathcal{K}}\rho(\cdot, z'') + \rho(\cdot, z'')\gamma^j_J(\cdot, z'') \quad z'' \in I$$

for some $\delta^j_J(\cdot, z'') \in \mathcal{C}^{0,q-j-1}(\Omega_{z''})$ and $\gamma^j_J(\cdot, z'') \in \mathcal{C}^{0,q-j}(\Omega_{z''})$, $z'' \in I$. Set $\tilde{u}^j_J = u^j_J + \delta^j_J \wedge \bar{\partial}_{\mathcal{K}^c}\rho$. Then it follows that

$$\tilde{u}^j_J \wedge \bar{\partial}\rho = \left(\delta^j_J \wedge \bar{\partial}_{\mathcal{K}}\rho + \rho\gamma^j_J + \delta^j_J \wedge \bar{\partial}_{\mathcal{K}^c}\rho\right) \wedge \bar{\partial}\rho = 0$$

on $\mathcal{M} \cap U_0$ for each $J \subset \mathcal{K}^c$ with $|J| = j$, and hence $\tilde{u}_j := \sum_{J\subset\mathcal{K}^c,|J|=j} \tilde{u}^j_J \wedge d\bar{z}^J \in \mathcal{C}^{0,q}(U_0 \cap \Omega)$. Also, one obtains that

$$\begin{aligned} \bar{\partial}\tilde{u}_j &= \sum_{J\subset\mathcal{K}^c,|J|=j} \left(\beta^j_J + \bar{\partial}_{\mathcal{K}^c}\tilde{u}^j_J + \bar{\partial}\left(\delta^j_J \wedge \bar{\partial}_{\mathcal{K}^c}\rho\right)\right) \wedge d\bar{z}^J \\ &:= \sum_{J\subset\mathcal{K}^c,|J|=j} \beta^j_J \wedge d\bar{z}^J + \tilde{E}^{j+1}. \end{aligned} \tag{3.13}$$

Set $\tilde{\alpha}_{j+1} = \tilde{\alpha}_j - \tilde{u}_j \in C^\infty(U_0)$. Note that we may assume that $\delta^j_J \wedge \bar{\partial}_{\mathcal{K}}\rho$ and $\rho\gamma^j_J$ are disjoint. Therefore, it follows, from the estimates in (3.11) and (3.12), and by the Sobolev interpolation theorem, that

$$\|\tilde{\alpha}_{j+1}\|_{s+\frac{1}{2},U^-} \lesssim \|\chi\alpha\|_{s+\frac{j+1}{2},\mathcal{M}} \tag{3.14}$$

for each real $s \geq 0$. Also, $(\tilde{\alpha}_{j+1} - \tilde{\alpha}_j) \wedge \bar{\partial}\rho = -\tilde{u}_j \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$ since $\tilde{u}_j \in \mathcal{C}^{0,q}(U_0 \cap \Omega)$. In view of (3.13), we can write:

$$\bar{\partial}\tilde{\alpha}_{j+1} = \sum_{\substack{|I|=q-j,|J|=j+1\\ I\subset\mathcal{K},J\subset\mathcal{K}^c}} \beta^{j+1}_{IJ} d\bar{z}^I \wedge d\bar{z}^J + \sum_{\substack{|I|+|J|=q+1,|J|\geq j+2\\ I\subset\mathcal{K},J\subset\mathcal{K}^c}} E^{j+2}_{IJ} d\bar{z}^I \wedge d\bar{z}^J$$

for some smooth functions β^{j+1}_{IJ} and E^{j+2}_{IJ}. This fact, together with the estimate in (3.14), proves the inductive step. If we proceed up to $j = q + 1$, then the proof of the proposition is completed. □

Remark 3.2. When $n = q+1+k$, $1 \leq k \leq q$, the above inductive step will stop at the $(k+1)$-th step, thus proving $\bar{\partial}\tilde{\alpha}_{k+1} = 0$. This proves Theorem 1.1 with better estimates.

Now we are ready to prove the weak $\bar{\partial}$-closed extension problem (Theorem 1.1). We recall that $U_0^- = \bigcup_{z''\in I} \overline{\Omega}_{z''} \times \{z''\}$ as defined in (2.4).

Proof of Theorem 1.1. In view of Proposition 3.1, there exists $\alpha_{q+1} \in \bigwedge^{0,q}(U_0^-)$ which can be written as:

$$\bar{\partial}\alpha_{q+1} = \sum_{\substack{J\subset\mathcal{K}^c\\ |J|=q+1}} H_J d\bar{z}^J \tag{3.15}$$

for some smooth functions H_J on U_0^-. If we consider the coefficients of $d\bar{z}_i \wedge d\bar{z}^J$ of $\bar{\partial}^2\alpha_{q+1} = 0$, we see that

$$\frac{\partial H_J}{\partial \bar{z}_i} = 0$$

for $1 \leq i \leq q + 1$. Hence $H_J(\cdot, z'')$ is a holomorphic function on $\Omega_{z''}$ for each $z'' \in I$.

Also note that $(\alpha_{q+1}-\alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$. Therefore, $\bar{\partial}(\alpha_{q+1}-\alpha) \wedge \bar{\partial}\rho = 0$ and hence $\bar{\partial}\alpha_{q+1} \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U_0$ since $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U_0$. Considering the coefficients of $d\bar{z}_i \wedge d\bar{z}^J$ of $\bar\partial\alpha_{q+1} \wedge \bar\partial\rho$, one obtains, from (3.15), that

$$\frac{\partial\rho}{\partial\bar{z}_i} H_J = 0$$

for $1 \le i \le q+1$ on $\mathcal{M} \cap U_0$. Since $\bar\partial_{\mathcal{K}}\rho \neq 0$ on $b\Omega_{z''}$, at least one of $\partial\rho/\partial\bar{z}_i$, for $1 \le i \le q+1$, is not equal to zero, and hence $H_J(\cdot, z'') = 0$ on $b\Omega_{z''}$. Thus it follows that $H_J(\cdot, z'') \equiv 0$ on $\Omega_{z''}$ because $H_J(\cdot, z'')$ is a holomorphic function on $\Omega_{z''}$ for each $z'' \in I$. In view of (3.15), we thus obtain that $\bar\partial\alpha_{q+1} = 0$. If we set $\tilde\alpha = \alpha_{q+1}$, then $\tilde\alpha$ satisfies the estimates (1.1) from the estimates in (3.2). This proves Theorem 1.1. □

Proof of Theorem 1.2. Let U_0^- be the neighborhood constructed in Theorem 1.1. Then there is a weak $\bar\partial$-closed extension $\tilde\alpha$ of α onto U_0^- satisfying the estimate (1.1). By the lemma in Section 4 of [2], we can construct a small pseudoconvex domain $B \subset\subset U_0^-$ with the property that $W := B \cap \mathcal{M}$ is a neighborhood of $z_0 \in \mathcal{M}$.

For each real $s \ge 0$, set $\tilde{u}_s = \bar\partial^* N_s^B \tilde\alpha$, where N_s^B denotes the weighted $\bar\partial$-Neumann operator in B with weight $e^{-t_s|z|^2}$ for sufficiently large $t_s > 0$ depending on s. Then we have $\bar\partial\tilde{u}_s = \tilde\alpha$ in B, and it follows that

$$\|\tilde{u}_s\|_{s,B} \le C_s \|\tilde\alpha\|_{s,B}. \tag{3.16}$$

Set $u_s = \tau\tilde{u}_s$, where τ is the projection in $\bigwedge^{p,q-1}(\mathcal{M})$ onto $\mathcal{B}^{p,q-1}(\mathcal{M})$ defined by first restricting a $(p, q-1)$-form ϕ in $\mathbf{C}^n$ to $\mathcal{M}$, then projecting the restriction to $\mathcal{B}^{p,q-1}(\mathcal{M})$. Then $\bar\partial_b u_s = \alpha$ on W and if we use the estimates (1.1) and (3.16), and the trace theorem in Sobolev spaces, then we obtain that

$$\|u_s\|_{s,W} \le C_s \|D\tilde{u}_s\|_{s-\frac{1}{2},B} \le C_s \|\tilde\alpha\|_{s+\frac{1}{2},B} \le C_s \|\chi\alpha\|_{s+\frac{q+1}{2},U_0^-\cap\mathcal{M}}$$

for each real $s \ge 0$. This completes the proof of Theorem 1.2. □

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