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Representation Theory of the Lie Group T_3 and Three Index Bessel Functions

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Abstract. The theory of generalized Bessel functions is reformulated within the framework of an operational formalism using the multiplier representation of the Lie group T_3 as suggested by Miller. This point of view provides more efficient tools which allow the derivation of generating functions of generalized Bessel functions. A few special cases of interest are also discussed.

1. Introduction

The Generalized Bessel functions (GBF), which are a multivariable extension of ordinary Bessel functions (BF), have been recently investigated by Dattoli et al.[3,4,5,6,7,8]. This research activity was caused by a number of physical problems where this type of functions plays an essential role in application to the radiation and optical problems. Many variable one index BF have been throughly studied in connection with non-dipolar scattering problems [2,15,9](Brown and Kibble, 1964; Reiss, 1962; Dattoli and Voykov, 1993). Many index many variable BF have been introduced and studied within the framework of a radiation problem involving moving charges in complex magnetic structures [9](Dattoli and Voykov, 1993).
The threee index Bessel function $J_n^{(p,q)}$(x) is specified by the series expansion [6; p.20(20)]

$$J_n^{(p,q)}(x) \;=\; \sum_{s=0}^{\infty} \frac{(-1)^{n+s}\left(\frac{x}{2}\right)^{n+2s}}{s!(q+s)!(n+s)!(p+n+s)!} \tag{1.1}$$

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The generating function for $J_n^{(p,q)}$(x) is given by [7;p.20(19)]

$$\sum_{n=-\infty}^{\infty} t^n J_n^{(p,q)}(x) \;=\; C_p\left(\frac{xt}{2}\right) C_q\left(\frac{-x}{2t}\right) \tag{1.2}$$

where the function [1]

$$C_n(x) \;=\; \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \tag{1.3}$$

is a Bessel like function known as Tricomi function and is characterized by the following link with the ordinary Bessel functions

$$C_n(x) \;=\; x^{\frac{-n}{2}} J_n(2\sqrt{x}) \tag{1.4}$$

where J_n(x) is the cylindrical Bessel function of first kind [1].

A routine calculation [6;p.20(21)] shows that $J_n^{(p,q)}$(x) satisfies the following pure and differential recrrence relations:

$$\frac{d}{dx} J_n^{(p,q)}(x) \;=\; -\frac{1}{2}\left[J_{n-1}^{(p+1,q)}(x) - J_{n+1}^{(p,q+1)}(x)\right]$$

$$nJ_n^{(p,q)}(x) \;=\; -\frac{x}{2}\left[J_{n-1}^{(p+1,q)}(x) - J_{n+1}^{(p,q+1)}(x)\right] \tag{1.5}$$

In a number of previous papers by Dattoli et al [4,5,6,7,8] it has been shown that by exploiting operational methods many properties of generalized Bessel functions, recently introduced by them are easily derived. Dattoli et al [6] presented an outline of the theory of Bessel like functions with more than one index and one or more variables. Their link with other types of functions is discussed and their use in applications is touched on. So it is instructive and convenient to develop them from a completely different approach that of generating functions by Miller's [11] method. This approach has indeed allowed the derivation (see, Pathan,Goyal and Shahwan [12] and Pathan and Shahwan [13]) of the GBF in a more general context.

In this paper, we obtain generating functions of three index Bessel functions $J_n^{(p,q)}$(x) by using a representation of the Lie group T_3. The principal interest in our results lies in the fact that a number of special cases would yield invitably to many new and known results of the theory of special functions. It is worth recalling that several fundamental identities of Miller [11;p.62-63] for cylindrical functions and Graf's addition theorem [10;p.44] are special cases of our results obtained in sections 2 and 3.

2. Multiplier representation of T_3 and generating functions

Let τ_3 be the Lie algebra of a three-dimensional complex local Lie group T_3, a multiplicative matrix group with elements (cf. Miller[11;p.10])

$$g(b,c,\tau) \;=\; \begin{pmatrix} 1\,0\,0\,\tau \\ 0\,e^{-\tau}0\,\tau \\ 0\,0\,e^{\tau}b \\ 0\,0\,0\,1 \end{pmatrix}, b,c,\tau\varepsilon C. \tag{2.1}$$

T_3 has the topology of C^3 and is simply connected (see Pontrjagin [14;chapter 8]). A basis for τ_3 is provided by the matrices [11;p.11]

$$j^+ \;=\; \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{pmatrix},\; j^- \;=\; \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix},\; j^3 \;=\; \begin{pmatrix} 0\,0\,0\,1 \\ 0\;-1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,0 \end{pmatrix} \tag{2.2}$$

with the commutation relations:

$$[j^3, j^{\pm}] \;=\; j^{\pm}, [j^+, j^-] \;=\; 0 \tag{2.3}$$

The machinery constructed in [11,chapters 1 and 2] will be applied to find a realization of the representation $Q(w,0)$ of τ_3, where w is a complex constant such that $w \neq 0$. The spectrum S of this set is $\{n : n\ integer\}$. In particular, we are looking for the functions $f_n(x,t) = Z_n e^{nz}$ such that

$$J^3 f_n = n f_n, J^+ f_n = w f_n, J^- f_n = w f_{n-1}, C_{0,0} f_n = j^+ j^- f_n = w^2 f_n, w \neq 0 \tag{2.4}$$

for all $n\epsilon S$, where the differential operators

$$J^3 \;=\; \frac{\partial}{\partial z},\; J^{\pm} \;=\; e^{\pm z}\left\{\mp\frac{\partial}{\partial x} + \frac{1}{x}\frac{\partial}{\partial z}\right\} \tag{2.5}$$

In terms of the functions $Z_n(x) \;=\; J_n^{(p,q)}(x)$, these relations become

$$\begin{aligned} \left[\frac{-\partial}{\partial x} + \frac{n}{x}\right] J_n^{(p,q)}(x) \;&=\; -\,J_n^{(p,q+1)}(x) \\ \left[\frac{\partial}{\partial x} + \frac{n}{x}\right] J_n^{(p,q)}(x) \;&=\; -\,J_{n-1}^{(p+1,q)}(x) \\ \left[x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n^2\right] J_n^{(p,q)}(x) \;&=\; -\,x^2 J_n^{(p+1,q+1)}(x) \end{aligned} \tag{2.6}$$

where $J_n^{(p,q)}(x)$ is given by (1.1).
If the functions $Z_n(x)$ defined for all $n\ \epsilon\ S$ satisfy (2.6) for $w \;=\; -1$, then the vectors

$f_n(x,t) = Z_n(x)e^{nz}$ form a basis for the realization of the representation $Q(-1,0)$ of τ_3. The differential operators (2.5) generate a Lie algebra which is the algebra of the generalized Lie derivatives of a multiplier representation T of τ_3. If Cl is the space of all functions analytic in some neighborhood of the point $(x_0,y_0) = (1,0)$, the Lie derivatives (2.5) define a local representation T of τ_3 on Cl.
A simple computation using [11,p.18] and (2.5) gives

$$\left[T(exp\,\tau j^3)f\right](x_0,t_0) = f(x_0,t_0e^{\tau})$$

$$\left[T(exp\,cj^-)f\right](x_0,t_0) = f\left(x_0\left(1+\frac{2c}{x_0t_0}\right)^{\frac{1}{2}},t_0\left(1+\frac{2c}{x_0t_0}\right)^{\frac{1}{2}}\right) \tag{2.7}$$

$$\left[T(exp\,bj^+)f\right](x_0,t_0) = f\left(x_0\left(1-\frac{2bt_0}{x_0}\right)^{\frac{1}{2}},t_0\left(1+\frac{2bt_0}{x_0}\right)^{\frac{1}{2}}\right)$$

If $g\varepsilon T_3$ is given by (2.1), we find

$$g = exp(bj^+)exp(cj^-)exp(\tau j^3)$$

and

$$\begin{aligned} &T\left[exp(bj^+)exp(cj^-)exp(\tau j^3)f\right](x,t)\\ &=\left[T(exp(bj^+))T(exp(cj^-))T(exp(\tau j^3))f\right](x,t)\\ &= f\left[x\sqrt{\theta\phi},te^{\tau}\sqrt{\frac{\phi}{\theta}}\right]\end{aligned} \tag{2.8}$$

defined for $\mid \frac{2bt}{x} \mid < 1$, $\mid \frac{2c}{tx} \mid < 1$, where $\theta = 1-\frac{2bt}{x}$, $\phi = 1+\frac{2c}{tx}$

According to Miller [11,see 2.2], our realization $Q(-1,0)$ of τ_3 on the space generated by the functions $f_n(x,t), n\,\varepsilon\, S$, can be extended to a local representation T_3, where the group action is given by (2.8).
The matrix elements of the local representation with respect to the basis f_n are uniquely determined by $Q(-1,0)$, and we obtain the relations

$$\left[T(g)f_k\right](x,t) = \sum_{-\infty}^{\infty} A_{lk}(g)f_l(x,t) \tag{2.9}$$

where $k = 0,\pm1,\pm2,...$

$$t^ke^{k\tau}\left(\frac{\phi}{\theta}\right)^{\frac{k}{2}}J_k^{(p,q)}(x\sqrt{\theta\phi}) = \sum_{l=-\infty}^{\infty} A_{lk}(g)J_n^{(p,q)}(x)t^l \tag{2.10}$$

and the matrix elements $A_{lk}(g)$ are given by [11,p.56(3.12)]

$$A_{lk}(g) = \frac{e^{k\tau}(-c)^{\frac{(k-l+|k-l|)}{2}}(-b)^{\frac{(l-k+|k-l|)}{2}}}{\mid k-l\mid!}{}_0F_1(\mid k-l\mid+1;bc)) \tag{2.11}$$

valid for all integral values of l and k. Since $J_n^{(p,q)}(x)$, $n \varepsilon S$, is analytic in x, for all nonzero values in x, the infinite series (2.10) conveereges absolutely for $\mid \frac{2bt}{x} \mid< 1$, $\mid \frac{2c}{tx} \mid< 1$. Thus our main generating function becomes

$$\left(\frac{\phi}{\theta}\right)^{\frac{k}{2}} J_k^{(p,q)}(x\sqrt{\theta\phi}) = \sum_{n=-\infty}^{\infty} (-c)^{\frac{(-n+|n|)}{2}} (b)^{\frac{(n+|n|)}{2}} {}_0F_1(\mid n \mid +1; bc)) J_n^{(p,q)}(x) t^n \tag{2.12}$$

defined for $\mid \frac{2bt}{x} \mid< 1$, $\mid \frac{2c}{tx} \mid< 1$, where $\theta = 1 - \frac{2bt}{x}$, $\phi = 1 + \frac{2c}{tx}$, which forms a generalization of a result of Bessel functions of Miller [11,p.62(3.29)]. If $bc \neq 0$, we can introduce the coordinates r, ν defined by $r = (ibc)^{\frac{1}{2}}$ and $\nu = \left(\frac{b}{ic}\right)^{\frac{1}{2}}$ such that $b = \frac{r\nu}{2}$, $c = \frac{-r}{2\nu}$. In this case equation (2.12) yields the generating function

$$\left(\frac{\beta}{\alpha}\right)^{\frac{k}{2}} J_k^{(p,q)}(x\sqrt{\alpha\beta}) = \sum_{n=-\infty}^{\infty} (-\nu)^n J_n(-r) J_{k+n}^{(p,q)}(x) t^n \tag{2.13}$$

where $\alpha = 1 - \frac{r\nu t}{x}$, $\beta = 1 - \frac{r}{\nu t x}$.

3. Applications

We shall mention a few special cases of (2.12)

I. If $c = 0$, $t = 1$, equation (2.12) becomes

$$\text{(3.1)} \qquad \left(1 - \frac{2b}{x}\right)^{\frac{-k}{2}} J_k^{(p,q)}\left[x\left(1 - \frac{2b}{x}\right)^{\frac{1}{2}}\right] = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} J_{k+n}^{(p,q)}(x), \mid \frac{2b}{x} \mid< 1$$

which forms a generalization of a result of Miller involving BF [11,p.62(3.30)].

II. If $b = 0$, $t = 1$, equation (2.12) becomes

$$\text{(3.2)} \qquad \left(1 + \frac{2c}{x}\right)^{\frac{-k}{2}} J_k^{(p,q)}\left[x\left(1 + \frac{2c}{x}\right)^{\frac{1}{2}}\right] = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} J_{k-n}^{(p,q)}(x), \mid \frac{2c}{x} \mid< 1$$

which forms a generalization of a result of Miller involving BF [11,p.62(3.31)]. Again setting $t = 1$ in (2.13), we get

$$\text{(3.3)} \qquad \left(1 - \frac{r\nu}{x}\right)^{\frac{-k}{2}} \left(1 - \frac{r}{\nu x}\right)^{\frac{k}{2}} J_k^{(p,q)}\left[x\left(1 - \frac{r\nu}{x}\right)^{\frac{1}{2}}\left(1 - \frac{r}{\nu x}\right)^{\frac{1}{2}}\right]$$
$$= \sum_{n=-\infty}^{\infty} (-\nu)^n J_n(-r) J_{k+n}^{(p,q)}(x)$$

which forms a generalization of Graf's addition theorem [10,p.44]

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