KYUNGPOOK Math. J. 53(2013), 135-141
http://dx.doi.org/10.5666/KMJ.2013.53.1.135

G-frames as Sums of Some g-orthonormal Bases

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Abstract. In this paper we show that a g-frame for a Hilbert space $\mathcal{H}$ can be written as a linear combination of two g-orthonormal bases for $\mathcal{H}$ if and only if it is a g-Riesz basis for $\mathcal{H}$. Also, we show that every g-frame for a Hilbert space $\mathcal{H}$ is a multiple of a sum of three g-orthonormal bases for $\mathcal{H}$.

1. Introduction

The concept of frame was introduced by Duffin and Schaeffer [5] in 1952 while studying some problems in nonharmonic Fourier analysis. Let $\mathcal{H}$ be a Hilbert space. A countable family $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \tag{1.1}$$

for all $f \in H$. If in (1.1), the right hand inequality holds for all $f \in \mathcal{H}$ then $\{f_i\}_{i=1}^{\infty}$ is called a Bessel sequence. In [9] the authors give a necessary and sufficient conditions on Bessel sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ and bounded operators L_1 and L_2 on a Hilbert space $\mathcal{H}$ such that $\{L_1 f_i + L_2 g_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, in other words, under some conditions sum of two Bessel sequences can be a frame. Also they show that, one can get a new frame by adding a frame to any of its dual frames. Casazza proved that every frame for a Hilbert space $\mathcal{H}$ can be written as a sum of three (but not two) orthonormal bases [1].

Frames in Hilbert spaces have several generalizations [2, 7, 6, 4]. But the concept of g-frames, the most recent generalization of frames, introduced by Sun [10].

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces, where I is a countable subset of $\mathbb{N}$.

Definition 1.1. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g-frame for $\mathcal{H}$ with

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Received May 14, 2011; accepted September 28, 2011.
2010 Mathematics Subject Classification: Primary 41A58, 42C15 .
Key words and phrases: g-Bessel sequence, g-frame, g-orthonormal basis, g-Riesz basis.

respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \tag{1.2}$$

for all $f \in \mathcal{H}$. We call A and B the lower and upper $g-$frame bounds, respectively. We call $\{\Lambda_i\}_{i\in I}$ a tight g-frame, if $A = B$ and Parseval g-frame, if $A = B = 1$. The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called the g-Bessel sequence if the right hand inequality in (1.2) holds for all $f \in \mathcal{H}$.

Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ be given for all $i \in I$. Let us define the set

$$\left(\sum_{i\in I} \oplus \mathcal{H}_i\right)_{l_2} = \left\{\{f_i\} : f_i \in \mathcal{H}_i, \sum_{i\in I} \|f_i\|^2 < \infty\right\}$$

with this inner product given by $\langle\{f_i\}, \{g_i\}\rangle = \sum\limits_{i\in I}\langle f_i, g_i\rangle$. It is clear that $\left(\sum_{i\in I} \oplus\mathcal{H}_i\right)_{l_2}$ is a Hilbert space with respect to the poitwise operations. It is proved in [8], if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$ then the operator

$$T : \left(\sum_{i\in I} \oplus \mathcal{H}_i\right)_{l_2} \to \mathcal{H}$$

defined by

$$T(\{f_i\}) = \sum_{i\in I} \Lambda_i^*(f_i) \tag{1.3}$$

is well defined and bounded and its adjoint is $T^*f = \{\Lambda_i f\}_{i\in I}$. Also, a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame if and only if the operator T defined in (1.3), is a bounded and onto operator. We call the operators T and T^*, synthesis and analysis operators, respectively. Also in [10], it is proved that, if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$, then the operator

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

is a positive bounded invertible operator and every $f \in \mathcal{H}$ has the expansions

$$f = \sum_{i\in I} S^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1} f.$$

The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in I}$. If $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence then $S = TT^*$.

Definition 1.2. A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called

(1) a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist two positive constants A and B such that for any finite subset $F \subseteq I$ and $g_i \in \mathcal{H}_i$

$$A\sum_{i\in F}\|g_i\|^2 \le \|\sum_{i\in F}\Lambda_i^* g_i\|^2 \le B\sum_{i\in F}\|g_i\|^2$$

and $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i) : i \in I\}$ is g-complete, i.e.,

$$\{f|\Lambda_i f = 0, i \in I\} = \{0\}.$$

(2) a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if for all $f \in \mathcal{H}$, $\sum_{i\in I}\|\Lambda_i f\|^2 = \|f\|^2$, and

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle, \quad g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j, \quad i,j \in I. \tag{1.4}$$

We call condition (1.4), the orthogonality condition of g-orthonormal basis $\{\Lambda_i\}_{i\in I}$.

2. Main results

The following theorem is proved in [10] and we use this result in the rest of paper.

Theorem 2.1. *Let $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i) : i \in I\}$ be a sequence of bounded operators and $\{g_{i,j}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ where J_i is a subset of $\mathbb{Z}$ and $i \in I$. Then $\{\Lambda_i\}_{i\in I}$ is a g-frame (resp. g-Riesz basis, g-orthonormal basis) for $\mathcal{H}$ if and only if $\{\Lambda_i^* g_{i,j}\}_{i\in I, j\in J_i}$ is a frame (resp. Riesz basis, orthonormal basis) for $\mathcal{H}$.*

Proposition 2.2. *$\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if there is a g-orthonormal basis $\{Q_i\}_{i\in I}$ and a bounded onto operator T on $\mathcal{H}$ such that $\Lambda_i = Q_i T^*$ for all $i \in I$.*

Proof. Let $\{g_{i,j}\}_{i\in J_i}$ be an orthonormal basis for Hilbert space $\mathcal{H}_i$. If $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ then by Theorem 2.1,$\{\Lambda_i^* g_{i,j}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ and we have

$$\Lambda_i f = \sum_{j\in J_i}\langle f, \Lambda_i^* g_{i,j}\rangle g_{i,j},$$

for all $f \in \mathcal{H}$. We can take an orthonormal basis $\{v_{i,j}\}_{i\in I, j\in J_i}$ for $\mathcal{H}$ and surjective bounded operator $T : \mathcal{H} \to \mathcal{H}$ so that $\Lambda_i^* g_{i,j} = T v_{i,j}$ [3, *Theorem 5.5.5*]. Let us define

$$Q_i : \mathcal{H} \to \mathcal{H}_i, \qquad Q_i f = \sum_{j\in J_i}\langle f, v_{i,j}\rangle g_{i,j}.$$

Then $\Lambda_i f = Q_i T^* f$ for all $f \in \mathcal{H}$ and $i \in I$ and again Theorem 2.1 implies that $\{Q_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. Conversely, if $\{Q_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ and $\Lambda_i = Q_i T^*$ for some bounded onto operator T, then we can find orthonormal basis $\{u_{i,j}\}_{i\in I, j\in J_i}$ for $\mathcal{H}$ such that $Q_i f = \sum_{j\in J_i}\langle f, u_{i,j}\rangle g_{i,j}$ for all $f \in \mathcal{H}$. Since

$$\Lambda_i f = Q_i T^* f = \sum_{j\in J_i}\langle T^* f, u_{i,j}\rangle g_{i,j} = \sum_{j\in J_i}\langle f, T u_{i,j}\rangle g_{i,j},$$

by Theorem 2.1, $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$. □

We mention that, if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ then the orthonormal bases for $\mathcal{H}$ are precisely the sets $\{Ue_n\}_{n=1}^{\infty}$, where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator [3]. Using this fact and Theorem 2.1, we have the same situation for g-frames.

Proposition 2.3. *Let $\{Q_i\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{H}$, with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ if and only if there is a unitary operator $T : \mathcal{H} \to \mathcal{H}$ such that $\Lambda_i = Q_i T$ for all $i \in I$.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. If $\{g_{i,j}\}_{i\in J_i}$ is an orthonormal basis for $\mathcal{H}_i$, then $\{\Lambda_i^* g_{i,j}\}_{i\in I, j\in J_i}$ and $\{Q_i^* g_{i,j}\}_{i\in I, j\in J_i}$ are orthonormal bases for $\mathcal{H}$. So $\Lambda_i^* g_{i,j} = T_1 Q_i^* g_{i,j}$, where T_1 is a unitary operator. We have

$$\begin{aligned}\Lambda_i f &= \sum_{j\in J_i}\langle f, \Lambda_i^* g_{i,j}\rangle g_{i,j} = \sum_{j\in J_i}\langle f, T_1 Q_i^* g_{i,j}\rangle g_{i,j}\\ &= \sum_{j\in J_i}\langle T_1^* f, Q_i^* g_{i,j}\rangle g_{i,j} = Q_i T_1^* f\end{aligned}$$

for all $f \in \mathcal{H}$ and $i \in I$. One can consider $T = T_1^*$. Then T is unitary and $\Lambda_i = Q_i T$ for all $i \in I$. For the inverse implication, assume that there is a unitary operator T and $\Lambda_i = Q_i T$, for $i \in I$. Then

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \langle T^* Q_i^* g_i, T^* Q_j^* g_j\rangle = \langle Q_i^* g_i, Q_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle$$

for all $i, j \in I$, and $g_i \in \mathcal{H}_i$, $g_j \in \mathcal{H}_j$. Also we have

$$\sum_{i\in I}\|\Lambda_i f\|^2 = \sum_{i\in I}\|Q_i T f\|^2 = \|Tf\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

So $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. □

Proposition 2.4. *A g-frame $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ can be written as a real linear combination of two g-orthonormal bases for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$.*

Proof. Let $\{\Lambda_i\}_{i\in I}$ be a g-Riesz basis for $\mathcal{H}$. Then there is a g-orthonormal basis $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ and a bounded invertible operator T on $\mathcal{H}$ such that $\Lambda_i = Q_i T$, for each $i \in I$ [10]. Since T is invertible, there are unitary operators U_1, U_2 such that $T = a_1 U_1 + a_2 U_2$ for some $a_1, a_2 \in \mathbb{R}$ [1]. So

$$\Lambda_i = a_1 Q_i U_1 + a_2 Q_i U_2, \qquad i \in I.$$

Proposition 2.3 implies that $\{Q_iU_1\}_{i\in I}$ and $\{Q_iU_2\}_{i\in I}$ are g-orthonormal bases. Conversely, let $\Lambda_i = a_1Q_i + a_2Q'_i$ for $i \in I$, where $\{Q_i\}_{i\in I}$ and $\{Q'_i\}_{i\in I}$ are g-orthonormal bases. Again by Proposition 2.3, there is a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $Q'_i = Q_iU$ and $\Lambda_i = Q_i(a_1I + a_2U)$ for $i \in I$ and for some $a_1, a_2 \in \mathbb{R}$. Since the operator $a_1I + a_2U$ is invertible, $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$. □

It is proved in [1], if $\{x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with upper frame bound B, then for every $\varepsilon > 0$ there are orthonormal bases $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ for $\mathcal{H}$ and a constant $a = B(1 + \varepsilon)$ so that

$$x_i = a(f_i + g_i + h_i),\ i \in I.$$

We show g-frames have the similar property.

Theorem 2.5. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with bounds $0 < A \le B$, with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then for every $\varepsilon > 0$, there are g-orthonormal bases $\{\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$, $\{\Phi_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ for $\mathcal{H}$ and a constant $a = B(1 + \varepsilon)$ such that*

$$\Lambda_i f = a(\Gamma_i f + \Phi_i f + \Theta_i f),\ i \in I,\ f \in \mathcal{H}.$$

Proof. Let $\{g_{i,j}\}_{i\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$. Then $\{\Lambda_i^* g_{i,j}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ and $\Lambda_i f = \sum_{j\in J_i}\langle f, \Lambda_i^* g_{i,j}\rangle g_{i,j}$, for every $f \in \mathcal{H}$. So for given $\varepsilon > 0$, there are orthonormal bases $\{f_{ij}\}_{i\in I, j\in J_i}$, $\{e_{ij}\}_{i\in I, j\in J_i}$ and $\{h_{ij}\}_{i\in I, j\in J_i}$ for $\mathcal{H}$ such that

$$\Lambda_i f = \sum_{j\in J_i}\langle f, \Lambda_i^* g_{i,j}\rangle g_{i,j} = \sum_{j\in J_i}\langle f, a(f_{ij} + e_{ij} + h_{ij})\rangle g_{i,j} \tag{2.1}$$

for each $f \in \mathcal{H}$ and $a = B(1+\varepsilon)$. For given $f \in \mathcal{H}$, we define $\Gamma_i f = \sum_{j\in J_i}\langle f, f_{ij}\rangle g_{i,j}$ and $\Phi_i f = \sum_{j\in J_i}\langle f, e_{ij}\rangle g_{i,j}$ and $\Theta_i f = \sum_{j\in J_i}\langle f, h_{ij}\rangle g_{i,j}$. Then (2.1) implies that

$$\Lambda_i f = a(\Gamma_i f + \Phi_i f + \Theta_i f),\ i \in I,\ f \in \mathcal{H}.$$

We show that $\{\Gamma_i\}_{i\in I}$, $\{\Phi_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are g-orthonormal bases for $\mathcal{H}$. If $g_i \in \mathcal{H}_i$, then $\Gamma_i^* g_i = \sum_{j\in J_i}\langle g_i, g_{i,j}\rangle f_{ij}$ and it is easy to show that

$$\langle \Gamma_i^* g_i, \Gamma_i^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle,\ g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j,\ i, j \in I$$

and

$$\sum_{i\in I}\|\Gamma_i f\|^2 = \sum_{i\in I}\|\sum_{j\in J_i}\langle f, f_{ij}\rangle g_{i,j}\|^2 = \sum_{i\in I}\sum_{j\in J_i}|\langle f, f_{ij}\rangle|^2 = \|f\|^2.$$

This completes the proof. □

As the following example shows, in general, a g-frame cannot be represented as a sum of two g-orthonormal bases.

Example 2.6. Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{N}}$ be a g-orthonormal for $\mathcal{H}$, with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{N}}$ and $\dim(\mathcal{H}_1) < \infty$. We consider the g-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{N}}$ by $\Lambda_1 = 0$ and $\Lambda_{i+1} = \Theta_i$ for $i \in \mathbb{N}$. Assume that there are g-orthonormal bases $\{\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{N}}$, $\{\Phi_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in\mathbb{N}}$ and scalars $c, d \neq 0$ such that $\Lambda_i = c\Gamma_i + d\,\Psi_i$ for all $i \in \mathbb{N}$. Then, $0 = \Lambda_1 = c\Gamma_1 + d\,\Psi_1$. Therefore

$$\operatorname{span}\{\Gamma_1^*(\mathcal{H}_1)\} = \operatorname{span}\{\Psi_1^*(\mathcal{H}_1)\}.$$

since $\{\Gamma_i\}_{i\in I}$ and $\{\Phi_i\}_{i\in I}$ are g-complete, a result of [8] shows that

$$\overline{\operatorname{span}}\{\Gamma_i^*(\mathcal{H}_i)\}_{i\in\mathbb{N}} = \overline{\operatorname{span}}\{\Psi_i^*(\mathcal{H}_i)\}_{i\in\mathbb{N}} = \mathcal{H}.$$

The orthogonality condition of g-orthonormal bases $\{\Gamma_i\}_{i\in\mathbb{N}}$ and $\{\Phi_i\}_{i\in\mathbb{N}}$ implies

$$\overline{\operatorname{span}}\{\Gamma_i^*(\mathcal{H}_i)\}_{i\geq 2} = \overline{\operatorname{span}}\{\Psi_i^*(\mathcal{H}_i)\}_{i\geq 2} \neq \mathcal{H}.$$

But

$$\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\geq 2} = \overline{\operatorname{span}}\{(\overline{c}\Gamma_i^* + \overline{d}\Psi_i^*)(\mathcal{H}_i)\}_{i\geq 2} = \mathcal{H}$$

and this is contradiction.

We recall that, a bounded operator $T : \mathcal{H} \to \mathcal{H}$ is a maximal partial isometry if T or its adjoint is an isometry. Next proposition shows that every g-frame can be written as a multiple of a sum of two Parseval g-frame. Note that if $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ and T is an isometry, then $\{\Theta_i T\}_{i\in I}$ is a Parseval g-frame.

Proposition 2.7. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then there are two Parseval g-frames $\{\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ and $\{\Psi_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ for $\mathcal{H}$ and a scalar a such that*

$$\Lambda_i f = a(\Gamma_i f + \Psi_i f),\ i \in I,\ f \in \mathcal{H}.$$

Proof. By Proposition 2.2, $\Lambda_i = \Theta_i T^*$ for $i \in I$, where T is a bounded onto operator on $\mathcal{H}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. Then

$$T = \frac{\|T\|U}{2}(W + W^*),$$

where W is a unitary and U ia a maximal partial isometry and $(UW)^*$ and $(UW^*)^*$ are isometries [1]. So

$$\Lambda_i = \frac{\|T\|}{2} \cdot \left[\Theta_i(UW)^* + \Theta_i(UW^*)^*\right],$$

for each $i \in I$. Here $\{\Theta_i(UW)^*\}_{i\in I}$ and $\{\Theta_i(UW^*)^*\}_{i\in I}$ are Parseval g-frames. □

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