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On Nearly Pairwise Compact Spaces

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Abstract. In this paper, we introduce the notion of near pairwise compactness which generalizes the notion of pairwise compactness.

1. Introduction

Singal and Mathur [10] introduced and studied the notion of near compactness by generalizing the concept of compactness of a topological space. Later the notion of near compactness studied and developed considerably by Carnahan [1], Singal and Mathur [8], Herrington [3], Joseph [4] and others. The notion of near compactness became an important meadow to topologists. Following these trends, Nandi [6] introduced the notion of near compactness in bitopological spaces: A bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is said to be ij-nearly compact if for each $(\mathscr{P}_i)$open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}V) \mid V \in \mathscr{V}\}$ covers X. X is said to be pairwise nearly compact if it is 12- and 21-nearly compact. The notion of pairwise near compactness is defined considering only $(\mathscr{P}_i)$open sets. As such, this notion of pairwise near compactness cannot be a generalization of pairwise compactness (Fletcher et al. [2]). In this paper, we introduce a generalized notion of pairwise compactness and we call it nearly pairwise compact (Definition 2.7). It is also a generalization of near compactness.

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2. Preliminaries

Unless or otherwise mentioned, X stands for the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$. We recall the following definitions.

Definition 2.1. A collection $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ is said to be pairwise open if for each $\alpha \in A$, U_α is $(\mathscr{P}_i)$open for some $i \in \{1, 2\}$ and for each $i \in \{1, 2\}$, $\mathscr{U} \cap \mathscr{P}_i \neq \emptyset$. A pairwise open collection covering X is called a pairwise open cover (Fletcher et al. [2]).

A collection $\mathscr{F} = \{F_\alpha \mid \alpha \in A\}$ of subsets of X is said to be pairwise closed (Pahk and Choi [7]) if $\{X - F_\alpha \mid \alpha \in A\}$ is pairwise open.

Definition 2.2([5]). In a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$, the topology $\mathscr{P}_i$ is said to be regular with respect to $\mathscr{P}_j$, if for each $x \in X$ and each $(\mathscr{P}_i)$closed set A with $x \notin A$, there exist $U \in \mathscr{P}_i$ and $V \in \mathscr{P}_j$ such that $x \in U, A \subset V$ and $U \cap V = \emptyset$. X is said to be pairwise regular if $\mathscr{P}_i$ is regular with respect to $\mathscr{P}_j$ for both $i = 1$ and $i = 2$.

Definition 2.3([11]). Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces and $\mathscr{P}_i \times \mathscr{Q}_i$ be the product topology on $X \times Y$ of the topologies $\mathscr{P}_i$ and $\mathscr{Q}_i$ on X and Y respectively. Then the bitopological space $(X \times Y, \mathscr{P}_1 \times \mathscr{Q}_1, \mathscr{P}_2 \times \mathscr{Q}_2)$ is called the product bitopological space of the spaces $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$.

Definition 2.4([9]). A set $A \subset X$ is said to be (i, j)regularly open if $A = (\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}A)$.

A subset of X is said to be (i, j)regularly closed if its complement is (i, j)regularly open. In other words, a set $A \subset X$ is (i, j)regularly closed iff $A = (\mathscr{P}_i)\mathrm{cl}((\mathscr{P}_j)\mathrm{int}A)$.

Definition 2.5([9]). A bitopological space X is said to be pairwise semiregular iff for each $x \in X$ and each $(\mathscr{P}_i)$open set U with $x \in U$, there exists a $(\mathscr{P}_i)$open set V such that $x \in V \subset (\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}V) \subset U$.

Obviously, a pairwise regular space is pairwise semiregular.

Definition 2.6([9]). A bitopological space X is said to be pairwise almost regular if for each $x \in X$ and each (i, j)regularly closed set F with $x \notin F$, there exist a $(\mathscr{P}_i)$open set U and a $(\mathscr{P}_j)$open set V, $j \neq i$, $i, j \in \{1, 2\}$, such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Equivalently, X is pairwise almost regular iff for each $x \in X$ and each (i, j)regularly open set U with $x \in U$, there exists a $(\mathscr{P}_i)$open set V such that $x \in V \subset (\mathscr{P}_j)\mathrm{cl}V \subset U$.

We introduce the following definitions.

Definition 2.7. A bitopological space X is said to be nearly pairwise compact if for each pairwise open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}V) \mid V \in \mathscr{V} \cap \mathscr{P}_i,\ i \in \{1, 2\}\}$ covers X.

Obviously, a pairwise compact space is nearly pairwise compact. The following examples shows that, the notion of pairwise near compactness and near pairwise compactness are independent.

Example 2.1. Let b be a fixed real number. We define

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, R\}\bigcup\left\{\left(b-\frac{1}{n},\infty\right) \mid n\in N\right\}\cup\{[b,\infty)\},\\ \mathscr{P}_2 &= \{\emptyset, R\}\bigcup\left\{\left(-\infty, b-\frac{1}{n}\right) \mid n\in N\right\}\cup\{(-\infty,b),[b,\infty)\}\\ &\qquad\bigcup\left\{R-\left[b-\frac{1}{n},b\right) \mid n\in N\right\}.\end{aligned}$$

$(R,\mathscr{P}_1,\mathscr{P}_2)$ is pairwise nearly compact but it is not nearly pairwise compact.

Example 2.2(cf. [11], p. 142)**.** Let

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, R\}\bigcup\{(-\infty,n) \mid n\in Z\},\\ \mathscr{P}_2 &= \{\emptyset, R\}\bigcup\{(n,\infty) \mid n\in Z\}\end{aligned}$$

where Z is the set of integers. The bitopological space $(R,\mathscr{P}_1,\mathscr{P}_2)$ is not ij-nearly compact for any $i\in\{1,2\}$. Hence the space is not pairwise nearly compact. The space is pairwise compact and hence it is also nearly pairwise compact.

Example 2.3. Let b be a fixed real number. We define

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, R\}\bigcup\{(-\infty,b],(b,\infty)\},\\ \mathscr{P}_2 &= \{\emptyset, R\}\bigcup\{(b,\infty)\}\bigcup\left\{\left(b+\frac{1}{n},\infty\right) \mid n\in N\right\}.\end{aligned}$$

$(R,\mathscr{P}_1,\mathscr{P}_2)$ is nearly pairwise compact but it is not pairwise compact.

Definition 2.8. A bitopological space X is said to be almost pairwise compact if for each pairwise open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V}\subset\mathscr{U}$ such that $\{(\mathscr{P}_j)\mathrm{cl}V \mid V\in\mathscr{V}\cap\mathscr{P}_i,\ i\in\{1,2\}\}$ covers X.

It readily follows from definitions, a nearly pairwise compact space is an almost pairwise compact space.

Definition 2.9. A cover $\mathscr{C}$ of X is said to be a pairwise basic cover if there exist two bases $\mathscr{B}_1$ and $\mathscr{B}_2$ of the topologies $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively such that $\mathscr{C}\subset\mathscr{B}_1\cup\mathscr{B}_2$ and for each $i\in\{1,2\}$, $\mathscr{C}\cap\mathscr{B}_i\neq\emptyset$.

Definition 2.10. A collection $\mathscr{U}$ (resp. $\mathscr{F}$) of subsets of X is said to be pairwise regularly open (resp. pairwise regularly closed) if each member of $\mathscr{U}$ (resp. $\mathscr{F}$) is (i,j)regularly open (resp. (i,j)regularly closed) for some $i\in\{1,2\}$ and contains at least one (i,j)regularly open (resp. (i,j)regularly closed) set for each $i \in \{1,2\}$. $\mathscr{U}$ (resp. $\mathscr{F}$) is said to be a pairwise regularly open (resp. pairwise regularly closed) cover if it covers X.

Definition 2.11. A bifilter is a collection $\mathscr{F}$ of nonempty subsets of X with the following properties:

(a) $\mathscr{F} \subset \mathscr{P}_1 \cup \mathscr{P}_2$ and $\mathscr{F} \cap \mathscr{P}_i \neq \emptyset$ for each $i \in \{1,2\}$.
(b) If $E, F \in \mathscr{F}$ with $E, F \in \mathscr{P}_i$ for some $i \in \{1,2\}$ then $E \cap F \in \mathscr{F}$.
(c) If $G \in \mathscr{F}$ and $H \supset G$ with $G, H \in \mathscr{P}_i$ for some $i \in \{1,2\}$ then $H \in \mathscr{F}$.

Definition 2.12. A bifilter $\mathscr{F}$ on a bitopological space X is said to be maximal provided

(a) for any bifilter $\mathscr{G}$ on X, $\mathscr{G} \subset \mathscr{F}$,
(b) if $\mathscr{G}$ is a bifilter with $\mathscr{F} \subset \mathscr{G}$, then $\mathscr{F} = \mathscr{G}$.

Definition 2.13. A point $p \in X$ is said to be a bicluster point of a bifilter $\mathscr{F}$ if for each $F \in \mathscr{F}$, $p \in (\mathscr{P}_i)\mathrm{cl}F$ whenever F is $(\mathscr{P}_i)$open for some $i \in \{1,2\}$.

Definition 2.14. A $(\mathscr{P}_i)$open set containing a point $p \in X$ is said to be a $(\mathscr{P}_i)$open neighbourhood (abbreviated as $(\mathscr{P}_i)$open nbd) of p.

Definition 2.15. A point p is said to be a biconvergent point of a bifilter $\mathscr{F}$ if each $(\mathscr{P}_i)$open nbd of p is a member of $\mathscr{F}$.

Throughout the paper, N denotes the set of natural numbers and R, the set of real numbers. For a pairwise open (resp. closed) collection $\mathscr{U}$ (resp. $\mathscr{F}$) of subsets of a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$, we write $\mathscr{U}^i$ (resp. $\mathscr{F}^i$) to denote the collection of all $(\mathscr{P}_i)$open (resp. $(\mathscr{P}_i)$closed) sets in $\mathscr{U}$ (resp. $\mathscr{F}$). $(\mathscr{T})\mathrm{int}A$ (resp. $(\mathscr{T})\mathrm{cl}A$) denotes the interior (resp. closure) of a set A in a topological space $(X, \mathscr{T})$. Always $i, j \in \{1,2\}$ and whenever i, j appear together, $j \neq i$.

3. Results

We now establish the following theorems on nearly pairwise compact spaces.

Theorem 3.1. *In a bitopological space X, the following statements are equivalent:*

(a) *X is nearly pairwise compact.*
(b) *Each pairwise basic cover $\mathscr{U}$ of X possesses a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_i)int((\mathscr{P}_j)clV) \mid V \in \mathscr{V} \cap \mathscr{P}_i,\ i \in \{1,2\}\}$ covers X.*
(c) *Each pairwise regularly open cover of X has a finite subcover.*
(d) *Each pairwise regularly closed collection of subsets of X with finite intersection property has nonempty intersection.*
(e) *Each pairwise closed collection $\mathscr{F} = \{F_\alpha \mid \alpha \in B\}$ of subsets of X with the property that for any finite subcollection $\mathscr{E} \subset \mathscr{F}$, $\bigcap\{(\mathscr{P}_i)cl((\mathscr{P}_j)intF_\alpha) \mid F_\alpha \in \mathscr{E},\ i \in \{1,2\}\} \neq \emptyset$, has a nonempty intersection.*

Proof. $(a) \Rightarrow (b)$: Obvious.

$(b) \Rightarrow (c)$: Let $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ be a pairwise regularly open cover of X and let $\mathscr{B}_i$ be a base of the topology $\mathscr{P}_i$. For each $G_\alpha \in \mathscr{G}$ with $G_\alpha \in \mathscr{P}_i$, there exist $\mathscr{H}_{i\alpha} = \{H_\lambda \mid \lambda \in \Lambda_\alpha, H_\lambda \in \mathscr{B}_i\}$ such that $G_\alpha = \bigcup\{H_\lambda \mid H_\lambda \in \mathscr{H}_{i\alpha}\}$. Then $\mathscr{U} = \{H_\lambda \mid \lambda \in \Lambda_\alpha, \alpha \in A\}$ is a pairwise basic cover of X. So by (b), we obtain a finite subcollection $\mathscr{V} = \{H_{\lambda_k} \mid k = 1, 2, \ldots, m\}$ of $\mathscr{U}$ such that $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}H_{\lambda_k}) \mid H_{\lambda_k} \in \mathscr{V} \cap \mathscr{P}_i, k = 1, 2, \ldots, m\}$ covers X. For each $H_{\lambda_k} \in \mathscr{P}_i$, there exists a $G_{\alpha_k} \in \mathscr{P}_i$, $\alpha_k \in A$ such that $H_{\lambda_k} \subset G_{\alpha_k}$ which implies $(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}H_{\lambda_k}) \subset (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_{\alpha_k}) = G_{\alpha_k}$. Then $\{G_{\alpha_k} \mid k = 1, 2, \ldots, m\}$ is a finite subcover of $\mathscr{G}$.

$(c) \Rightarrow (d)$: We suppose that $\mathscr{F} = \{F_\alpha \mid \alpha \in I\}$ is a pairwise regularly closed collection of subsets of X with finite intersection property i.e. for each $n \in N$, $\bigcap\{F_{\alpha_k} \mid k = 1, 2, \ldots, n\} \neq \emptyset$. If possible, let $\bigcap\{F_\alpha \mid \alpha \in I\} = \emptyset$. Then $\{X - F_\alpha \mid \alpha \in I\}$ is a pairwise regularly open cover of X. So by (c), $\{X - F_\alpha \mid \alpha \in I\}$ has a finite subcover $\{X - F_{\alpha_k} \mid k = 1, 2, \ldots, m\}$ which in turn implies $\bigcap\{F_{\alpha_k} \mid k = 1, 2, \ldots, m\} = \emptyset$. This is a contradiction to our assumption. Thus we have $\bigcap\{F_\alpha \mid \alpha \in I\} \neq \emptyset$.

$(d) \Rightarrow (e)$: We suppose that $\mathscr{F} = \{F_\alpha \mid \alpha \in B\}$ is a pairwise closed collection of subsets of X and $X - F_\alpha \in \mathscr{P}_i$ such that for any finite subcollection $\mathscr{E}$ of $\mathscr{F}$, $\bigcap\{(\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}F_\alpha) \mid F_\alpha \in \mathscr{E}, i \in \{1, 2\}\} \neq \emptyset$. Thus $\{(\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}F_\alpha) \mid \alpha \in B\}$ is a pairwise regularly closed collection of subsets of X with finite intersection property. So by (d), we have $\bigcap\{(\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}F_\alpha) \mid \alpha \in B\} \neq \emptyset$. Since F_α is $(\mathscr{P}_i)$closed, $(\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}F_\alpha) \subset F_\alpha$. Thus it follows that $\bigcap\{F_\alpha \mid \alpha \in B\} \neq \emptyset$.

$(e) \Rightarrow (a)$: Suppose $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ is a pairwise open cover of X. If possible, suppose X is not nearly pairwise compact. So for any finite subcollection $\{U_{\alpha_k} \mid \alpha_k \in A, k = 1, 2, \ldots, m\}$ of $\mathscr{U}$, $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1, 2, \ldots, m; U_{\alpha_k} \in \mathscr{P}_i, i \in \{1, 2\}\}$ is not a cover of X. Thus $\bigcap\{X - (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1, 2, \ldots, m; U_{\alpha_k} \in \mathscr{P}_i, i \in \{1, 2\}\} \neq \emptyset$. Since $X - (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}U_{\alpha_k}) \subset (\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}(X - U_{\alpha_k}))$, $\bigcap\{(\mathscr{P}_i)\text{cl}((\mathscr{P}_j)\text{int}(X - U_{\alpha_k})) \mid \alpha_k \in A, k = 1, 2, \ldots, m; U_{\alpha_k} \in \mathscr{P}_i, i \in \{1, 2\}\} \neq \emptyset$. Thus $\{X - U_\alpha \mid \alpha \in A\}$ is a pairwise closed collection of subsets of X satisfying the properties of (e). Hence $\bigcap\{X - U_\alpha \mid \alpha \in A\} \neq \emptyset$ which in turn implies $\bigcup\{U_\alpha \mid \alpha \in A\} \neq X$, which is a contradiction. □

Theorem 3.2. *A pairwise semiregular space is nearly pairwise compact iff it is pairwise compact.*

Proof. Firstly, suppose X is pairwise semiregular and nearly pairwise compact. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ be a pairwise open cover of X. For each $x \in X$, there exists a $U_{\alpha(x)} \in \mathscr{U}$, $\alpha(x) \in A$ with $x \in U_{\alpha(x)}$. Suppose $U_{\alpha(x)} \in \mathscr{P}_i$. So by pairwise semiregularity, there exists a $(\mathscr{P}_i)$open set G_x such that $x \in G_x \subset (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_x) \subset U_{\alpha(x)}$. Here $\mathscr{G} = \{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_x) \mid x \in X\}$ is a pairwise regularly open cover of X. Using (c) of Theorem 3.1, we obtain a finite subcover $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_{x_k}) \mid k = 1, 2, \ldots, n\}$ of $\mathscr{G}$ which in turn implies $\{U_{\alpha(x_k)} \mid k = 1, 2, \ldots, n\}$ is a finite subcover of $\mathscr{U}$. The converse part is obvious. □

Theorem 3.3. *A pairwise almost regular space is nearly pairwise compact if it is almost pairwise compact.*

Proof. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ be a pairwise regularly open cover of a pairwise almost regular and almost pairwise compact space X. For each $x \in X$, we have a $U_{\alpha(x)} \in \mathscr{U}$, $\alpha(x) \in A$ such that $x \in U_{\alpha(x)}$. Suppose $U_{\alpha(x)} \in \mathscr{P}_i$. Hence using the notion of pairwise almost regularity, we obtain a $(\mathscr{P}_i)$open set G_x such that $x \in G_x \subset (\mathscr{P}_j)\mathrm{cl}G_x \subset U_{\alpha(x)}$. Obviously, $\mathscr{G} = \{G_x \mid x \in X\}$ is a pairwise open cover of X. So there exists a finite subcollection $\{G_{x_k} \mid k = 1, 2, \ldots, n\}$ of $\mathscr{G}$ such that $\{(\mathscr{P}_j)\mathrm{cl}G_{x_k} \mid k = 1, 2, \ldots, n\}$ covers X. Thus $\{U_{\alpha(x_k)} \mid k = 1, 2, \ldots, n\}$ is a finite subcover of $\mathscr{U}$ for X. Hence X is nearly pairwise compact by (c) of Theorem 3.1.□

Lemma 3.1. *Each $(\mathscr{P}_i)$open cover $\mathscr{U}$ of a (j,i)regularly closed subset F of a nearly pairwise compact space X has a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $\{(\mathscr{P}_i)int((\mathscr{P}_j)clA) \mid A \in \mathscr{V}\}$ covers F.*

Proof. The proof is straightforward and hence omitted. □

Theorem 3.4. *Every pairwise Hausdorff, nearly pairwise compact space is pairwise almost regular.*

Proof. Suppose X is a pairwise Hausdorff and nearly pairwise compact space. Let G be a (i,j)regularly open set and x be a point of X with $x \in G$. For each $y \in X - G$, we obtain a $(\mathscr{P}_i)$open set U_y and a $(\mathscr{P}_j)$open set V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Then $\mathscr{G} = \{V_y \mid y \in X - G\}$ is a $(\mathscr{P}_j)$open cover of the (i,j)regularly closed set $X - G$. So by Lemma 3.1, $\mathscr{G}$ has a finite subcollection $\mathscr{H} = \{V_{y_k} \mid k = 1, 2, \ldots, n\}$ with $X - G \subset \bigcup\{(\mathscr{P}_j)\mathrm{int}((\mathscr{P}_i)\mathrm{cl}V_{y_k}) \mid k = 1, 2, \ldots, n\}$. We write $U = \bigcap_{k=1}^{n} U_{y_k}$ and $V = \bigcup_{k=1}^{n}(\mathscr{P}_j)\mathrm{int}((\mathscr{P}_i)\mathrm{cl}V_{y_k})$. Here U is $(\mathscr{P}_i)$open with $x \in U$ and V is $(\mathscr{P}_j)$open with $X - G \subset V$ and $U \cap V = \emptyset$. Thus $(\mathscr{P}_j)\mathrm{cl}U \subset X - V$. Therefore it follows that $x \in U \subset (\mathscr{P}_j)\mathrm{cl}U \subset G$. □

Theorem 3.5. *If the topological space $(X, \mathscr{T})$ is nearly compact and the bitopological space $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ is nearly pairwise compact, then the product space $(X \times Y, \mathscr{T} \times \mathscr{Q}_1, \mathscr{T} \times \mathscr{Q}_2)$ is nearly pairwise compact.*

Proof. Let $\mathscr{U}$ be a pairwise basic cover of $X \times Y$. For each $U \in \mathscr{U}$, we have $U = G \times H$, $G \in \mathscr{T}$ and $H \in \mathscr{Q}_i$, $i \in \{1, 2\}$. For each $x \in X$, the space $\{x\} \times Y$ is nearly pairwise compact. Hence we get a finite number of elements $G_x^k \times H_x^k$, $k = 1, 2, \ldots, n$ of $\mathscr{U}$ such that $\{x\} \times Y \subset \bigcup_{k=1}^{n}(\mathscr{T} \times \mathscr{Q}_i)\mathrm{int}((\mathscr{T} \times \mathscr{Q}_j)\mathrm{cl}(G_x^k \times H_x^k))$ where we assume $H_x^k \in \mathscr{Q}_i$. We suppose that all the sets $G_x^k \times H_x^k$ intersects $\{x\} \times Y$. Then $x \in G_x$ where $G_x = \bigcap_{k=1}^{n} G_x^k \in \mathscr{T}$. The $(\mathscr{T})$open cover $\{G_x \mid x \in X\}$ of X has a finite subfamily $G_{x_1}, G_{x_2}, \ldots, G_{x_m}$ such that $X = \bigcup_{l=1}^{m}(\mathscr{T})\mathrm{int}((\mathscr{T})\mathrm{cl}G_{x_l})$. Hence the collection $\{(\mathscr{T} \times \mathscr{Q}_i)\mathrm{int}((\mathscr{T} \times \mathscr{Q}_j)\mathrm{cl}(G_{x_l}^k \times H_{x_l}^k)) \mid k = 1, 2, \ldots, n;\ l = 1, 2, \ldots, m\}$ covers $X \times Y$ and $\{G_{x_l}^k \times H_{x_l}^k \mid k = 1, 2, \ldots, n;\ l = 1, 2, \ldots, m\}$ is a finite subcollection of $\mathscr{U}$. □

But the product of two nearly pairwise compact space need not be nearly pairwise compact. For, we consider Example 2.2. The space $(R, \mathscr{P}_1, \mathscr{P}_2)$ is nearly pairwise compact, but the product space $(R \times R, \mathscr{P}_1 \times \mathscr{P}_1, \mathscr{P}_2 \times \mathscr{P}_2)$ is not nearly pairwise compact.

Lemma 3.2. *A bifilter $\mathscr{F}$ is maximal iff for some $i \in \{1,2\}$, each $(\mathscr{P}_i)$open set A intersecting every member of $\mathscr{F}^i$ belongs to $\mathscr{F}$.*

Proof. Firstly, suppose $\mathscr{F}$ is maximal. We write $\mathscr{G} = \{G \mid G \supset A \cap B$ for some $B \in \mathscr{F}^i$ and G is $(\mathscr{P}_i)$open$\} \bigcup \mathscr{F}^j$. Obviously, $\mathscr{G}$ is a bifilter with $\mathscr{G} \supset \mathscr{F}$ and $A \in \mathscr{G}$. Since $\mathscr{F}$ is a maximal bifilter, we have $\mathscr{G} = \mathscr{F}$.

Conversely, suppose the condition holds. If $\mathscr{F}$ is not maximal, there exists a bifilter $\mathscr{H}$ such that $\mathscr{H} \supset \mathscr{F}$. Let $H \in \mathscr{H}$ and H be $(\mathscr{P}_i)$open. Then by definition of a bifilter, H intersects every member of $\mathscr{H}^i$ and hence every member of $\mathscr{F}^i$. Thus $H \in \mathscr{F}$ and hence we have $\mathscr{H} = \mathscr{F}$. □

Lemma 3.3. *A bicluster point of a bifilter is a biconvergent point if it is a maximal bifilter.*

Proof. Suppose the maximal bifilter $\mathscr{F}$ has a bicluster point p. Then for each $F \in \mathscr{F}$, $p \in (\mathscr{P}_i)\mathrm{cl}F$ whenever F is $(\mathscr{P}_i)$open for some $i \in \{1,2\}$. So each $(\mathscr{P}_i)$open nbd V of p intersects every $F \in \mathscr{F}^i$. Thus by Lemma 3.2, $V \in \mathscr{F}$ which implies p is a biconvergent point of $\mathscr{F}$. □

Lemma 3.4. *Each pairwise open collection of subsets of X with finite intersection property is contained in a maximal bifilter.*

Proof. The proof is straightforward and hence omitted. □

Theorem 3.6. *Let X be pairwise almost regular and each bifilter $\mathscr{A}$ in X has the following property: For $A, B \in \mathscr{A}$ with $A \in \mathscr{P}_1$ and $B \in \mathscr{P}_2$, $A \cap B$ is nonempty $(\mathscr{P}_i)$open for each $i \in \{1,2\}$. Then the following statements are equivalent:*

- (a) *X is nearly pairwise compact.*
- (b) *Each bifilter in X has a bicluster point.*
- (c) *Each maximal bifilter in X has a biconvergent point.*

Proof. $(a) \Rightarrow (b)$: Let $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ be a bifilter. For each $\alpha \in A$, we write $F_\alpha = (\mathscr{P}_i)\mathrm{cl}G_\alpha$ if $G_\alpha \in \mathscr{P}_i$. Then $\mathscr{F} = \{F_\alpha \mid \alpha \in A\}$ is a pairwise closed collection of subsets of X with following property: For any finite subcollection $\mathscr{E} \subset \mathscr{F}$, $\bigcap\{(\mathscr{P}_i)\mathrm{cl}((\mathscr{P}_j)\mathrm{int}F) \mid F \in \mathscr{E}\} \neq \emptyset$. Hence by Theorem 3.1$(e)$, $\bigcap\{F_\alpha \mid \alpha \in A\} \neq \emptyset$. Thus there exists a $p \in X$ with $p \in F_\alpha$ for each $\alpha \in A$. So p is a bicluster point of $\mathscr{G}$.

$(b) \Rightarrow (c)$: A maximal bifilter is of course a bifilter. So by (b), each maximal bifilter has a bicluster point p. It then follows by Lemma 3.3, p is a biconvergent point of the maximal bifilter.

$(c) \Rightarrow (a)$: Let $\mathscr{U}$ be a pairwise regularly open cover of X. Suppose $\mathscr{U}$ has no finite subcollection covering X. Again for each $x \in X$, there exists a $U_x \in \mathscr{U}$ such that $x \in U_x$. Suppose U_x is (i,j)regularly open. Since X is pairwise almost regular, we obtain a $(\mathscr{P}_i)$open set G_x such that $x \in G_x \subset (\mathscr{P}_j)\mathrm{cl}G_x \subset U_x$. We note here that $\mathscr{G} = \{G_x \mid x \in X\}$ is a pairwise open cover of X. Also $\mathscr{H} = \{X - (\mathscr{P}_j)\text{cl}G_x \mid G_x \in \mathscr{G}\}$ is a pairwise open collection of subsets of X with finite intersection property. Now by Lemma 3.4, we obtain a maximal bifilter $\mathscr{E}$ which contains $\mathscr{H}$. So by (c), $\mathscr{E}$ has a biconvergent point p. A biconvergent point of a maximal bifilter is also a bicluster point. So if $E \in \mathscr{E}$ is $(\mathscr{P}_i)$open then $p \in (\mathscr{P}_i)\text{cl}E$ for each $E \in \mathscr{E}$. Hence $p \in (\mathscr{P}_j)\text{cl}(X - (\mathscr{P}_j)\text{cl}G_x)$ for each $G_x \in \mathscr{G}$. Now we show $p \notin G_x$ for any $G_x \in \mathscr{G}$. We need only to prove the case when $p \notin X - (\mathscr{P}_j)\text{cl}G_x$ but p is a $(\mathscr{P}_j)$limit point of $X - (\mathscr{P}_j)\text{cl}G_x$. If possible, let $p \in G_z$ for some $G_z \in \mathscr{G}$. For definiteness suppose, G_z is $(\mathscr{P}_i)$open. Now each $(\mathscr{P}_j)$open set A with $p \in A$ intersects each $E \in \mathscr{E}$ whenever E is $(\mathscr{P}_j)$open. Again G_z intersects each $E \in \mathscr{E}$ whenever E is $(\mathscr{P}_i)$open. Therefore by Lemma 3.2, $A, G_z \in \mathscr{E}$. So $A \cap G_z$ is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and $p \in A \cap G_z \subset G_z$. Since p is a $(\mathscr{P}_j)$limit point of $X - (\mathscr{P}_j)\text{cl}G_z$ we have $(A \cap G_z) \cap (X - (\mathscr{P}_j)\text{cl}G_z) \neq \emptyset$ which is not possible since $G_z \cap (X - (\mathscr{P}_j)\text{cl}G_z) = \emptyset$. Thus our anticipation $p \notin G_x$ for any $G_x \in \mathscr{G}$ is true. This contradicts the fact that $\mathscr{G}$ is a pairwise open cover of X. So $\mathscr{U}$ must have a finite subcover. Hence X is nearly pairwise compact. □

Remark 3.1. Theorem 3.6 also holds good if the expression 'X be pairwise almost regular' of the theorem is replaced by 'X be a bitopological space with each $(X, \mathscr{P}_i)$ being regular'.

We now give an example of a bitopological space which satisfies the conditions of Theorem 3.6.

Example 3.1. For any $a \in R$, we define

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, R, (-\infty, a), (-\infty, a], (a, \infty), R - \{a\}\}, \\ \mathscr{P}_2 &= \{\emptyset, R, (-\infty, a), (-\infty, a]\}. \end{aligned}$$

The bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise almost regular. The possible bifilters of this space are $\{(-\infty, a], R\}, \{(-\infty, a), (-\infty, a], R - \{a\}, R\}$. Clearly, they satisfy the conditions of Theorem 3.6.

It also follows, the bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$ is not pairwise regular.

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