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The Polynomial Numerical Index of $L_p(\mu)$

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Abstract. We show that for $1 < p < \infty, k, m \in \mathbb{N}$, $n^{(k)}(l_p) = \inf\{n^{(k)}(l_p^m) : m \in \mathbb{N}\}$ and that for any positive measure μ, $n^{(k)}(L_p(\mu)) \geq n^{(k)}(l_p)$. We also prove that for every $Q \in \mathcal{P}(^k l_p : l_p)$ $(1 < p < \infty)$, if $v(Q) = 0$, then $\|Q\| = 0$.

1. Introduction

Given a complex or real Banach space E we write B_E for the closed unit ball and S_E for the unit sphere of E. The dual space of E is denoted by E^*. For $k \in \mathbb{N}$, a mapping $P : E \to E$ is called a (continuous) k-homogeneous polynomial if there is a k-multilinear (continuous) mapping $A : E \times \cdots \times E \to E$ such that $P(x) = A(x, \ldots, x)$ for every $x \in E$. $\mathcal{P}(^k E : E)$ denotes the Banach space of all k-homogeneous continuous polynomials from E into itself with the norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We refer to [6] for background of polynomials on a Banach space. Let

$$\Pi(E) := \{(x, x^*) \ : \ x \in S_E, x^* \in S_{E^*}, x^*(x) = 1\}.$$

The *numerical radius* of P is defined [3] by

$$v(P) := \sup\{|x^*(Px)| : (x, x^*) \in \Pi(E)\}.$$

The polynomial numerical index of order k of E is defined [4] by

$$\begin{aligned} n^{(k)}(E) &:= \inf\{v(P) : P \in \mathcal{P}(^k E : E), \|P\| = 1\} \\ &= \sup\{M \geq 0 : \|P\| \leq M \ v(P) \text{ for all } P \in \mathcal{P}(^k E : E)\}. \end{aligned}$$

Of course, $n^{(1)}(E)$ is the classical numerical index of E. Note that $0 \leq n^{(k)}(E) \leq 1$, and $n^{(k)}(E) > 0$ if and only if $v(\cdot)$ is a norm on $\mathcal{P}(^k E : E)$ equivalent to the usual

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norm. It is obvious that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$.

The concept of the classical numerical index (in our terminology, the polynomial numerical index of order 1) was first suggested by G. Lumer [12]. In [4] the authors proved $n^{(k)}(C(K)) = 1$ when $C(K)$ is the complex spaces and some inequality $n^{(k)}(E) \leq n^{(k-1)}(E) \leq \frac{k^{(k+\frac{1}{k-1})}}{(k-1)^{k-1}}\, n^{(k)}(E)$ for every Banach space E. It was shown that $n^{(k)}(E^{**}) \leq n^{(k)}(E)$. The authors [10] found a lower bound for the polynomial numerical index of real lush spaces. They used this bound to compute the polynomial numerical index of order 2 of the real spaces c_0, ℓ_1 and ℓ_∞. In fact, they showed that for the real spaces $X = c_0, l_1, l_\infty$, $n^{(2)}(X) = 1/2$. They also presented an example of a real Banach space X whose polynomial numerical indices are positive while the ones of its bidual are zero. We refer to ([1–5, 7–12]) for some results about the polynomial numerical index. For general information and background on numerical ranges, we refer to [1–2].

In this paper, we show that for $1 < p < \infty, k, m \in \mathbb{N}$, $n^{(k)}(l_p) = \inf\{n^{(k)}(l_p^m) : m \in \mathbb{N}\}$ and that for any positive measure μ, $n^{(k)}(L_p(\mu)) \geq n^{(k)}(l_p)$. We also prove that for every $Q \in \mathcal{P}({}^k l_p : l_p)$ $(1 < p < \infty)$, if $v(Q) = 0$, then $\|Q\| = 0$.

2. Results

For $1 < p < \infty$ and $m \in \mathbb{N}$, l_p^m denotes $\mathbb{K}^m$ endowed with the usual p-norm, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We may consider l_p^m as a subspace of l_p. Let $\{e_n\}_{\mathbb{N}}$ be the canonical basis of l_p and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$. Note that in general if X is a Banach space and Y is a subspace of X there is no comparison between $n^{(k)}(X)$ and $n^{(k)}(Y)$ for $k \in \mathbb{N}$.

Theorem 2.1. *Let $1 < p < \infty$ and $k \in \mathbb{N}$ be fixed. Then $n^{(k)}(l_p) = \inf\{n^{(k)}(l_p^m) : m \in \mathbb{N}\}$ and the sequence $\{n^{(k)}(l_p^m)\}_{m\in\mathbb{N}}$ is decreasing.*

Proof. We proceed by steps. Let $m \in \mathbb{N}$. We define $P_{\{1,\cdots,m\}} : l_p \to l_p^m$ by $P_{\{1,\cdots,m\}}(\sum_{j=1}^{\infty} \lambda_j e_j) = \sum_{j=1}^{m} \lambda_j e_j$.

Step 1: The sequence $\{n^{(k)}(l_p^m)\}_{m\in\mathbb{N}}$ is decreasing.

Proof of Step 1. Let $Q \in S_{\mathcal{P}({}^k l_p^m : l_p^m)}$. We define $\tilde{Q} \in \mathcal{P}({}^k l_p^{m+1} : l_p^{m+1})$ by $\tilde{Q}(x) = (Q \circ P_{\{1,\cdots,m\}}(x), 0)$ for $x \in l_p^{m+1}$. It is obvious that $\tilde{Q} \in S_{\mathcal{P}}({}^k l_p^{m+1} : l_p^{m+1})$.

Claim A: $v(Q) = v(\tilde{Q})$

Let $(x, x^*) \in \Pi(l_p^m)$. Then $(\ (x, 0), (x^*, 0)\ ) \in \Pi(l_p^{m+1})$ and

$(*)\quad |x^*(Q(x))| = |(x^*, 0)(\tilde{Q}(\ (x, 0)\ ))| \leq v(\tilde{Q})$.

By taking supremum in the left side of $(*)$ over $(x, x^*) \in \Pi(l_p^m)$, we have $v(Q) \leq v(\tilde{Q})$. For the reverse inequality let $\epsilon > 0$. Then there exists $z_0 := \sum_{j=1}^{m+1} a_j e_j \in$

$S_{l_p^{m+1}}$ such that $(z_0, \sum_{j=1}^{m+1} sign(a_j)|a_j|^{p-1}e_j^*) \in \Pi(l_p^{m+1})$ and

$$\begin{aligned}
v(\tilde{Q}) - \epsilon &< |\sum_{j=1}^{m+1} sign(a_j)|a_j|^{p-1}e_j^*(\tilde{Q}(z_0))| \\
&= |\sum_{j=1}^{m} sign(a_j)|a_j|^{p-1}e_j^*(Q(\sum_{j=1}^{m} a_j e_j))| \\
&= C^{k+p-1}|\sum_{j=1}^{m} sign(a_j)|\frac{a_j}{C}|^{p-1} e_j^*(Q(\sum_{j=1}^{m} \frac{a_j}{C} e_j))| \\
&\quad (\text{where } C := (\sum_{j=1}^{m} |a_j|^p)^{\frac{1}{p}} \le 1) \\
&\le |\sum_{j=1}^{m} sign(a_j)|\frac{a_j}{C}|^{p-1} e_j^*(Q(\sum_{j=1}^{m} \frac{a_j}{C} e_j))| \\
&\le v(Q), \text{ because } (\sum_{j=1}^{m} \frac{a_j}{C} e_j, \sum_{j=1}^{m} sign(a_j)|\frac{a_j}{C}|^{p-1} e_j^*) \in \Pi(l_p^m),
\end{aligned}$$

which show $v(\tilde{Q}) \le v(Q)$. Thus $v(Q) = v(\tilde{Q})$.

It follows that

$$\begin{aligned}
n^{(k)}(l_p^m) &= \inf_{Q \in S_{\mathcal{P}(^k l_p^m : l_p^m)}} v(Q) \\
&= \inf_{Q \in S_{\mathcal{P}(^k l_p^m : l_p^m)}} v(\tilde{Q}) \\
&\ge \inf_{R \in S_{\mathcal{P}(^k l_p^{m+1} : l_p^{m+1})}} v(R) \\
&= n^{(k)}(l_p^{m+1}).
\end{aligned}$$

Step 2: $n^{(k)}(l_p) \le n^{(k)}(l_p^m)$ for every $m \in \mathbb{N}$

Proof of Step 2. Let $Q \in S_{\mathcal{P}(^k l_p^m : l_p^m)}$. We define $\tilde{Q} \in \mathcal{P}(^k l_p : l_p)$ by $\tilde{Q}(z) = (Q \circ P_{\{1,\cdots,m\}}(z), 0, 0, \cdots)$ for $z \in l_p$. It is obvious that $\tilde{Q} \in S_{\mathcal{P}(^k l_p : l_p)}$. By the same argument as in Step 1, we have $v(\tilde{Q}) \le v(Q)$. Thus it follows.

Step 3: $\lim_{m\to\infty} n^{(k)}(l_p^m) = n^{(k)}(l_p)$

Proof of Step 3. Let $Q \in S_{\mathcal{P}(^k l_p : l_p)}$. For each $m \in \mathbb{N}$, we define $Q_m \in \mathcal{P}(^k l_p^m : l_p^m)$ by $Q_m(x) = P_{\{1,\cdots,m\}} \circ Q(x, 0, 0, \cdots)$ for $x \in l_p^m$. It is obvious that $\|Q_m\| \le 1, \|Q_m\| \le \|Q_{m+1}\|$ and $v(Q_m) \le v(Q)$. For each $m \in \mathbb{N}$, we define $\tilde{Q}_m \in \mathcal{P}(^k l_p : l_p)$ by $\tilde{Q}_m(z) = (Q_m \circ P_{\{1,\cdots,m\}}(z), 0, 0, \cdots)$ for $z \in l_p$. By the argument in Step 1, $v(\tilde{Q}_m) = v(Q_m)$.

Claim B: $\lim_{m\to\infty} \|Q_m\| = 1$

Let $\epsilon > 0$. Choose $x_0 \in S_{l_p}$ such that $\|Q(x_0)\| > 1 - \epsilon$. By continuity of Q at x_0 it follows that

$$\begin{aligned}
&\|Q_m \circ P_{\{1,\cdots,m\}}(x_0) - Q(x_0)\| \\
=&\|P_{\{1,\cdots,m\}} \circ Q \circ P_{\{1,\cdots,m\}}(x_0) - P_{\{1,\cdots,m\}} \circ Q(x_0)\| + \|P_{\{1,\cdots,m\}} \circ Q(x_0) - Q(x_0)\| \\
\leq&\|Q \circ P_{\{1,\cdots,m\}}(x_0) - Q(x_0)\| + \|P_{\{1,\cdots,m\}} \circ Q(x_0) - Q(x_0)\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Choose $N_0 \in \mathbb{N}$ such that $\|Q_m \circ P_{\{1,\cdots,m\}}(x_0) - Q(x_0)\| < \epsilon$ for all $m \geq N_0$. Then for all $m \geq N_0$, $1 \geq \|Q_m\| \geq \|Q_m \circ P_{\{1,\cdots,m\}}(x_0)\| > 1 - 2\epsilon$, which shows Claim B.

Claim C: $\lim_{m\to\infty} v(Q_m) = v(Q)$

There exists $(y_0, y^*) \in \Pi(l_p)$ such that $|y^*(Q(y_0))| > v(Q) - \epsilon$. Let $y_0 := \sum_{j=1}^{\infty} b_j e_j$. Then $y^* = \sum_{j=1}^{\infty} sign(b_j)\ |b_j|^{p-1} e_j^*$. For $m \in \mathbb{N}$, we define $y_0^{(m)} := \sum_{j=1}^{m-1} b_j e_j + (\sum_{j=m}^{\infty} |b_j|^p)^{\frac{1}{p}} e_m$ and $y_m^* := \sum_{j=1}^{m-1} sign(b_j)\ |b_j|^{p-1} e_j^* + (\sum_{j=m}^{\infty} |b_j|^p)^{\frac{p-1}{p}} e_m^*$. It is obvious that $(y_0^{(m)}, y_m^*) \in \Pi(l_p)$ and $\lim_{m\to\infty} \|y_0 - y_0^{(m)}\| = 0 = \lim_{m\to\infty} \|y^* - y_m^*\|$. Note that $\lim_{m\to\infty} y_m^*(Q(y_0^{(m)})) = y^*(Q(y_0))$. Indeed,

$$\begin{aligned}
& |y_m^*(Q(y_0^{(m)})) - y^*(Q(y_0))| \\
\leq\ & |y_m^*(Q(y_0^{(m)})) - y^*(Q(y_0^{(m)}))| + |y^*(Q(y_0^{(m)})) - y^*(Q(y_0))| \\
\leq\ & \|y_m^* - y^*\|\ \|Q(y_0^{(m)})\| + \|Q(y_0^{(m)}) - Q(y_0)\| \\
\leq\ & \|y_m^* - y^*\| + \|Q(y_0^{(m)}) - Q(y_0)\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Choose $N_1 \in \mathbb{N}$ such that $|y_m^*(Q(y_0^{(m)}))| > v(Q) - \epsilon$ for all $m \geq N_1$. It is easy to show that for all $m \geq N_1$, $y_{N_1}^*(\tilde{Q}_m(y_0^{(N_1)})) = y_{N_1}^*(Q(y_0^{(N_1)}))$. It follows that for all $m \geq N_1$,

$$\begin{aligned}
v(Q) - \epsilon &< |y_{N_1}^*(Q(y_0^{(N_1)}))| \\
&= |y_{N_1}^*(\tilde{Q}_m(y_0^{(N_1)}))| \\
&\leq v(\tilde{Q}_m) = v(Q_m) \\
&\leq v(Q),
\end{aligned}$$

which show $\lim_{m\to\infty} v(Q_m) = v(Q)$. Thus we have

$$\begin{aligned}
&(**)\ \ v(Q) = \lim_{m\to\infty} v(Q_m) \\
=\ & \limsup_{m\to\infty} [v(\frac{Q_m}{\|Q_m\|})\ \|Q_m\|] \\
=\ & \limsup_{m\to\infty} v(\frac{Q_m}{\|Q_m\|})\ \lim_{m\to\infty} \|Q_m\| \\
=\ & \limsup_{m\to\infty} v(\frac{Q_m}{\|Q_m\|})\ \ \text{(by claim B)} \\
\geq\ & \limsup_{m\to\infty} n^{(k)}(l_p^m)
\end{aligned}$$

Taking the infimum in the left side of $(**)$ over $Q \in S_{\mathcal{P}({}^k l_p : l_p)}$, we have $n^{(k)}(l_p) \geq \limsup_{m\to\infty} n^{(k)}(l_p^m)$. By Step 2, we have $n^{(k)}(l_p) \leq \liminf_{m\to\infty} n^{(k)}(l_p^m)$. Thus $\lim_{m\to\infty} n^{(k)}(l_p^m) = n^{(k)}(l_p)$. Therefore, we complete the proof. $\square$

Theorem 2.2. *Let $1 < p < \infty$. Let $Q \in \mathcal{P}({}^k l_p : l_p)$. Then $v(Q) = 0$ if and only if $\|Q\| = 0$.*

Proof. It is enough to show that if $v(Q) = 0$, then $Q = 0$. We will show that $Q_m := P_{\{1,\cdots,m\}} \circ Q|\text{span }\{e_1,\cdots,e_m\} : l_p^m \to l_p^m$ is the zero polynomial for every $m \in \mathbb{N}$. Write

$$Q_m(\sum_{k=1}^m x_k e_k) = \sum_{k_1+\cdots+k_m=m, 0\leq k_1,\cdots,k_m\leq m} \frac{m!}{k_1!\cdots k_m!} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} A_m(e_{k_1},\cdots,e_{k_m}),$$

where A_m is the corresponding symmetric k-linear mapping to the k-homogeneous polynomial Q_m. Let $a_{k_1\cdots k_m} := A_m(e_{k_1},\cdots,e_{k_m}) \in l_p^m$.

Let $p_1 := 0$ and p_n be the n-th prime ($n \geq 2$). Let $0 \leq t \leq 1$ be fixed and $q \in \mathbb{R}$ with $1/p + 1/q = 1$. Put

$$y := \frac{t^{\sqrt{p_1}} e_1 + t^{\sqrt{p_2}} e_2 + \cdots + t^{\sqrt{p_m}} e_m}{(1 + t^{p\sqrt{p_2}} + \cdots + t^{p\sqrt{p_m}})^{1/p}}$$

and

$$y^* := \frac{t^{(p-1)\sqrt{p_1}} e_1^* + t^{(p-1)\sqrt{p_2}} e_2^* + \cdots + t^{(p-1)\sqrt{p_m}} e_m^*}{(1 + t^{p\sqrt{p_2}} + \cdots + t^{p\sqrt{p_m}})^{1/q}}.$$

Then $(y, y^*) \in \Pi(l_p^m)$.

Claim: $a_{k_1\cdots k_m} = 0$ for every $k_1,\cdots,k_m$

It follows that for every $0 \leq t \leq 1$,

$$\begin{aligned} 0 &= y^*(Q_m(y)) \\ &= \frac{1}{(1 + t^{p\sqrt{p_2}} + \cdots + t^{p\sqrt{p_m}})^{1/q+k/p}} \times \\ &\quad (t^{(p-1)\sqrt{p_1}} e_1^* + t^{(p-1)\sqrt{p_2}} e_2^* + \cdots + t^{(p-1)\sqrt{p_m}} e_m^*) \\ &\quad (Q_m(t^{\sqrt{p_1}} e_1 + t^{\sqrt{p_2}} e_2 + \cdots + t^{\sqrt{p_m}} e_m)), \end{aligned}$$

so

$$\begin{aligned} 0 &= (t^{(p-1)\sqrt{p_1}} e_1^* + t^{(p-1)\sqrt{p_2}} e_2^* + \cdots + t^{(p-1)\sqrt{p_m}} e_m^*) \\ &\quad (Q_m(t^{\sqrt{p_1}} e_1 + t^{\sqrt{p_2}} e_2 + \cdots + t^{\sqrt{p_m}} e_m)) \\ &= \sum_{k_1+\cdots+k_m=m, 0\leq k_1,\cdots,k_m\leq m} t^{\sqrt{p_{k_2}}+\cdots+\sqrt{p_{k_m}}} \frac{m!}{k_1!\cdots k_m!} e_1^*(a_{k_1\cdots k_m}) \\ &+ \sum_{2\leq j\leq m} [\sum_{k_1+\cdots+k_m=m, 0\leq k_1,\cdots,k_m\leq m} t^{\sqrt{p_{k_2}}+\cdots+\sqrt{p_{k_m}}+(p-1)\sqrt{p_{k_j}}} \frac{m!}{k_1!\cdots k_m!} e_j^*(a_{k_1\cdots k_m})]. \end{aligned}$$

Note that the elements of the set

$$\{ \sqrt{p_{k_2}} + \cdots + \sqrt{p_{k_m}}, \sqrt{p_{k_2}} + \cdots + \sqrt{p_{k_m}} + (p-1)\sqrt{p_{k_j}} : \\ k_1 + \cdots + k_m = m, 0 \leq k_1, \cdots, k_m \leq m, 2 \leq j \leq m\}$$

are different. Thus $e_j^*(a_{k_1\cdots k_m}) = 0$ for every $1 \leq j \leq m$, which show $a_{k_1\cdots k_m} = 0$ for every $k_1, \cdots, k_m$. Therefore, $Q_m = 0$. Let $x = \sum_{k=1}^{\infty} x_k e_k \in l_p$ be fixed. By continuity of Q at x, we have

$$Q(x) = \lim_{m\to\infty} Q_m(x) = 0.$$

□

Corollary 2.3. *Let $1 < p < \infty$. Then for every $k, m \in \mathbb{N}$, we have $n^{(k)}(l_p^m) > 0$.*

Proof. Assume that $n^{(k)}(l_p^m) = 0$ for some $k, m \in \mathbb{N}$. Since the unit sphere of the finite dimensional space $\mathcal{P}({}^k l_p^m : l_p^m)$ is compact, there exists some $Q \in \mathcal{P}({}^k l_p^m : l_p^m)$ such that $\|Q\| = 1$ and $v(Q) = 0$. Theorem 2.2 shows that $Q = 0$, which is impossible. □

Let (Ω, Σ) be a measurable space and μ a positive measure on Ω. We denote by $\mathcal{P}$ the collection of all partitions π of Ω into finitely many pairwise disjoint members of Σ with finite strictly positive measures. We order this collection by $\pi_1 \leq \pi_2$ whenever each member of π_1 is the union of members of π_2. So $\mathcal{P}$ is a directed set. For each $\pi = \{E_1, \cdots, E_m\} \in \mathcal{P}$, we associate the subspace V_π of $L_p(\mu)$ defined by $V_\pi = \{\sum_{i=1}^m a_i 1_{E_i} : a_i \in \mathbb{K}\}$. By P_π we denote the projection of $L_p(\mu)$ onto V_π defined by

$$P_\pi(f) = \sum_{i=1}^m [\frac{1}{\mu(E_i)} \int_{E_i} f(t)dt] 1_{E_i}$$

for all $f \in L_p(\mu)$. V denotes the union of all subspaces V_π of $L_p(\mu)$. We recall that V is a dense subspace of $L_p(\mu)$, thus, for each $f \in L_p(\mu)$, the sequence $\{P_\pi(f)\}_\pi$ converges to f in $L_p(\mu)$.

We recall the following well-known result.

Theorem 2.4 *For $1 < p < \infty$ and for every partition $\pi = \{E_1, \cdots, E_m\} \in \mathcal{P}$, the subspace V_π is isometrically isomorphic to l_p^m. Thus $n^{(k)}(V_\pi) = n^{(k)}(l_p^m)$ for every $k \in \mathbb{N}$.*

Theorem 2.5. *Let $1 < p < \infty$ and $k \in \mathbb{N}$. Then for any positive measure μ,*

$$n^{(k)}(L_p(\mu)) \geq n^{(k)}(l_p).$$

Proof. Let $Q \in S_{\mathcal{P}({}^k L_p(\mu) : L_p(\mu))}$. Let $\epsilon > 0$. Choose $x_0 \in S_{L_p(\mu)}$ such that $\|Q(x_0)\| > 1 - \epsilon$. By uniform continuity of Q on the closed unit ball of $L_p(\mu)$, there exists some $\delta > 0$ such that $x, y \in B_{L_p(\mu)}$ with $\|x - y\| < \delta$ implies that $\|Q(x) - Q(y)\| < \epsilon$. Choose $\pi_0 \in \mathcal{P}$ such that $\|x_0 - P_{\pi_0}(x_0)\| < \delta$. Since $\|P_{\pi_0}(x_0)\| \leq 1$, we have $\|Q(x_0) - Q \circ P_{\pi_0}(x_0)\| < \epsilon$. Thus $\|Q \circ P_{\pi_0}(x_0)\| > \|Q(x_0)\| - \epsilon > 1 - 2\epsilon$. Thus we can choose $\pi_1 = \{E_1, \cdots, E_m\} \in \mathcal{P}$ such that

$\pi_1 \geq \pi_0$ and $\|P_{\pi_1} \circ Q \circ P_{\pi_0}(x_0)\| > 1 - 2\epsilon$. We define $R \in \mathcal{P}({}^k V_{\pi_1} : V_{\pi_1})$ by $R(P_{\pi_1}(x)) = P_{\pi_1} \circ Q \circ P_{\pi_1}(x)$ for $x \in L_p(\mu)$. Obviously $\|R\| \leq 1$. It follows that

$$\begin{aligned}
(\sharp)\quad \|R\| &\geq \|R(\frac{P_{\pi_0}(x_0)}{\|P_{\pi_0}(x_0)\|})\| = \frac{\|R(P_{\pi_0}(x_0))\|}{\|P_{\pi_0}(x_0)\|^k} \\
&\geq \|R(P_{\pi_0}(x_0))\| \\
&\geq \|P_{\pi_1} \circ Q \circ P_{\pi_0}(x_0)\| \\
&> 1 - 2\epsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
(\sharp\sharp)\quad v(R) &\geq n^{(k)}(V_{\pi_1})\ \|R\| \\
&> n^{(k)}(V_{\pi_1})\ (1 - 2\epsilon)\ (\text{by } \sharp) \\
&= n^{(k)}(l_p^m)\ (1 - 2\epsilon)\ (\text{by Theorem 2.4}) \\
&\geq n^{(k)}(l_p)\ (1 - 2\epsilon)\ (\text{by Theorem 2.1}).
\end{aligned}$$

Since V_{π_1} is a finite dimensional space, there exists $(y_0, y^*) \in \Pi(V_{\pi_1})$ such that $v(R) = |y^*(R(y_0))|$. It follows that

$$\begin{aligned}
v(R) &= |y^*(R(y_0))| = |y^*(P_{\pi_1} \circ Q(y_0))| \\
&= |P_{\pi_1}^* \circ y^*(Q(y_0))| \\
&\leq v(Q),\ \text{because } (y_0, P_{\pi_1}^* \circ y^*) \in \Pi(V_{\pi_1}).
\end{aligned}$$

By ($\sharp\sharp$), we have ($\sharp\sharp\sharp$) $v(Q) \geq v(R) > n^{(k)}(l_p)\ (1 - 2\epsilon)$. By taking infimum in the left side of ($\sharp\sharp\sharp$) over $Q \in S_{\mathcal{P}({}^k L_p(\mu) : L_p(\mu))}$, we conclude the proof. □

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