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Some Characterizations of Parabolas

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Abstract. We study some properties of tangent lines of parabolas. As a result, we establish some characterizations of parabolas.

1. Introduction and Preliminaries

Next to straight lines and circles, one of the most simple and interesting curves in a plane is a parabola. A characterization of ellipse was studied by the present authors in terms of the curvature and the support function ([5]). As was described in [2], a circle is characterized by the fact that the chord joining any two points on it meets the circle at the same angle.

Hammer and Smith ([4]) gave a characterization for a circle in the Euclidean plane and it was generalized to the isoperimetrix of the Minkowski plane ([1]). For some geometric characterizations of ellipses and hyperbolas (respectively, of parabolas), see [5] (respectively, [9]). In this regard, it is interesting to consider what simple geometric properties characterize a parabola.

In this paper, we examine the parabola concerning the chord connecting two points on a parabola and discuss the converse problems of well known properties about the parabola.

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Consider a parabola P, which is given by, say, $y = f(x)$, where $f(x)$ is a quadratic polynomial. Then the following are well-known.

Proposition 1([8]). *A pair of tangent lines to P at $x = x_1$ and at $x = x_2$ meet at $x = (x_1 + x_2)/2$.*

Proposition 2([7], pp.132-134)**.** *For any chord AB on P with $A = (x_1, y_1), B = (x_2, y_2)$, the tangent line to P at $x = (x_1 + x_2)/2$ is parallel to the chord.*

Proposition 3([6], p.535)**.** *A pair of tangent lines to P through a point on the directrix of P intersect at right angle and the chord through the points of tangency always contains the focus of P.*

As a matter of fact, it is natural to ask if the converses of such properties hold. We mainly focus on such in this paper.

2. Main Results

In this section, we prove the following:

Theorem 4([8]). *A curve C of class C^3 given by $y = f(x)$ is a parabola if it satisfies the following condition.*
(C_1) For any two numbers x_1 and x_2, the pair of tangent lines to C at $x = x_1$ and at $x = x_2$ meet at $x = (x_1 + x_2)/2$.

In [8], it was shown that a curve C given by $y = f(x)$ is a parabola if it satisfies (C_1) and $f(x)$ is analytic.

Theorem 5. *A curve C of class C^2 given by $y = f(x)$ is a parabola if it satisfies the following condition.*
(C_2) For any chord AB on C with $A = (x_1, y_1), B = (x_2, y_2)$, the tangent line to C at $x = (x_1 + x_2)/2$ is parallel to the chord.

Theorem 6. *A convex curve C of class C^2 is a parabola if it satisfies the following condition.*
(C_3) There are a line L and a point F such that for any point p on L there are two tangent lines of C through p which are perpendicular to each other, and the chord connecting the points of tangency passes through F.

First, suppose that C satisfies (C_1). Then the tangent lines given by

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1), \\ y - f(x_2) &= f'(x_2)(x - x_2) \end{aligned} \tag{1}$$

have the point of intersection at $x = (x_1 + x_2)/2$. Hence we get

$$2\{f(x_1) - f(x_2)\} = (x_1 - x_2)\{f'(x_1) + f'(x_2)\}. \tag{2}$$

Differentiating (2) with respect to x_1, we obtain

$$f'(x_1) - f'(x_2) = (x_1 - x_2)f''(x_1). \tag{3}$$

Once more, we differentiate (3) with respect to x_1. Then we have

$$(x_1 - x_2)f'''(x_1) = 0, \tag{4}$$

which shows that $f(x)$ is a quadratic polynomial. This completes the proof of Theorem 4.

From the proof of Theorem 4, we see that the point $x = x_2$ might be fixed.

Second, suppose that C satisfies (C_2). Then we have

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(\frac{x_1 + x_2}{2}). \tag{5}$$

Differentiating (5) with respect to x_1 and x_2, respectively, we get

$$f'(x_1) = f'(\frac{x_1 + x_2}{2}) + \frac{1}{2}(x_1 - x_2)f''(\frac{x_1 + x_2}{2}) \tag{6}$$

and

$$-f'(x_2) = -f'(\frac{x_1 + x_2}{2}) + \frac{1}{2}(x_1 - x_2)f''(\frac{x_1 + x_2}{2}). \tag{7}$$

It follows from (6) and (7) that

$$\frac{f'(x_1) + f'(x_2)}{2} = f'(\frac{x_1 + x_2}{2}). \tag{8}$$

Differentiating (8) with respect to x_1 and x_2, respectively, we obtain

$$f''(x_1) = f''(\frac{x_1 + x_2}{2}) \tag{9}$$

and

$$f''(x_2) = f''(\frac{x_1 + x_2}{2}). \tag{10}$$

It follows from (9) and (10) that $f''(x)$ is a constant, which completes the proof of Theorem 5.

Finally, suppose that C satisfies (C_3). Then we may introduce a coordinate system (x, y) of R^2 such that x-axis is the line L, $F = (b, c)$ and C is given by $y = f(x)$ with $f(x) > 0$. We denote by $V = (a, p)$ the point of C where p is the minimum value of $y = f(x)$.

For any point $(t, 0)$ of L, we denote by $m(t)$ and $-\frac{1}{m(t)}$ $(m(t) > 0)$ the slopes of the tangent lines to C through $(t, 0)$. Then C is nothing but the envelope of the following 1-parameter family of lines:

$$\begin{aligned} y &= m(t)(x - t)(x \geq t), \\ y &= -\frac{1}{m(t)}(x - t)(x \leq t). \end{aligned} \tag{11}$$

When $x \geq a$, letting $F(x, y, t) = m(t)x - y - tm(t)$, the curve C is given by ([3], p.59)

$$
\begin{aligned}
F(x, y, t) &= m(t)x - y - tm(t) = 0, \\
\frac{\partial F(x, y, t)}{\partial t} &= m'(t)x - m(t) - tm'(t) = 0.
\end{aligned}
\tag{12}
$$

From (12), the curve $C = (x_1, y_1)$ is given by

$$
\begin{aligned}
x_1 =& t + \frac{m(t)}{m'(t)}, \\
y_1 =& \frac{m(t)^2}{m'(t)}.
\end{aligned}
\tag{13}
$$

When $x \leq a$, using a similar argument as the above, we see that the curve $C = (x_2, y_2)$ is given by

$$
\begin{aligned}
x_2 =& t - \frac{m(t)}{m'(t)}, \\
y_2 =& \frac{1}{m'(t)}.
\end{aligned}
\tag{14}
$$

Since the curve C is convex, $m(t) : (-\infty, \infty) \to (0, \infty)$ is a strictly increasing function which satisfies

$$
\lim_{t \to -\infty} (x_1, y_1) = \lim_{t \to \infty} (x_2, y_2) = V = (a, p). \tag{15}
$$

Let's put $A = (x_1, y_1), B = (x_2, y_2)$. Since the chord AB passes through $F = (b, c)$, (13) and (14) show that

$$
\frac{m(t)^2 - 1}{2m(t)}(b - x_1(t)) + y_1(t) = c. \tag{16}
$$

Since $\lim_{t \to -\infty} m(t) = 0$, it follows from (15) and (16) that $a = b$, hence we have $F = (a, c)$.

Substituting x_1, y_1 in (13) into (16), we get a differential equation:

$$
m'(t)\{(m^2 - 1)(t - a) + 2cm\} = m(m^2 + 1), \tag{17}
$$

which is equivalent to

$$
-m(m^2 + 1)dt + \{(m^2 - 1)(t - a) + 2cm\}dm = 0. \tag{18}
$$

Letting $M = -m(m^2 + 1)$ and $N = (m^2 - 1)(t - a) + 2cm$, we have

$$
\frac{1}{M}(N_t - M_m) = \frac{-4m}{m^2 + 1}. \tag{19}
$$

Hence an integrating factor μ of the equation (18) is given by

$$\mu = e^{\int \frac{-4m}{m^2+1}dm} = (m^2+1)^{-2}. \tag{20}$$

Multiplying both sides of (18) by μ in (20), we get

$$\frac{-m}{m^2+1}dt + \{\frac{m^2-1}{(m^2+1)^2}(t-a) + \frac{2cm}{(m^2+1)^2}\}dm = 0, \tag{21}$$

which is an exact differential equation. By integrating (21), we find

$$\frac{(t-a)m+c}{m^2+1} = d, d \in R, \tag{22}$$

or equivalently,

$$dm^2 - (t-a)m - (c-d) = 0. \tag{23}$$

Since $m(t) \to 0$ as $t \to -\infty$, (23) implies that $(a-t)m(t)(>0)$ converges to $c-d$ as $t \to -\infty$, hence we see that $c-d > 0$. Since $\lim_{t\to\infty} m(t) \to \infty$, (22) shows that $d > 0$. Because $m(t) > 0$, it follows from (23) that

$$m(t) = \frac{1}{2d}\{t-a+\sqrt{(t-a)^2+\alpha^2}\}, \quad \alpha^2 = 4d(c-d). \tag{24}$$

Together with (24), (13) and (14) yield, respectively, that

$$\begin{aligned} y_1 &= \frac{1}{4d}(x_1-a)^2 + \frac{\alpha^2}{4d}, \quad x_1 \geq a, \\ y_2 &= \frac{d}{\alpha^2}(x_2-a)^2 + d, \quad x_2 \leq a. \end{aligned} \tag{25}$$

Since $p = f(a)$, it follows from (25) that $d = p$ and $\alpha = 2p$. Thus the curve C is the parabola given by $y = \frac{1}{4p}(x-a)^2 + p$ with focus $F = (a, 2p)$. This completes the proof of Theorem 6.

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