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Characterizations of Several Modules Relative to the Class of $B(M,X)$

Yahya Talebi* and Mehrab Hosseinpour
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
e-mail : talebi@umz.ac.ir *and* m.hpour@umz.ac.ir

Abstract. Let M and X be right R-modules. We introduce several modules relative to the class of $B(M,X)$ and we investigate relation among these modules. In this note, we show if M is X-$\oplus$-supplemented such that $M = M_1 \oplus M_2$ implies M_1 and M_2 are relatively B-projective, then M is an X-H-supplemented module.

1. Introduction

Throughout this paper, R will be an associative ring with identity, and all modules are unitary right R-modules. A submodule K of M is denoted by $K \leq M$. The notation $N \leq_{\oplus} M$ means that N is a direct summand of M. A submodule K of M is called *essential*(or *large*) in M (denoted by $K \leq_e M$), if $K \cap L \neq 0$ for every nonzero submodule L of M, and a submodule K of M is called *small* in M (denoted by $K \ll M$), if $N + K \neq M$ for any proper submodule N of M. A module M is called *hollow* if every proper submodule of M is small in M. Let N be a submodule of M, a *supplement* of N in M is a submodule K of M minimal with respect to the property $M = N + K$, equivalently, $M = N + K$ and $N \cap K \ll K$. Following [14], M is called *supplemented* if every submodule of M has a supplement in M. M is called a *lifting* module or (D_1)-module if for every submodule A of M there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M_2$. Following [11], M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M and M is called H-*supplemented* if for every submodule A of M there is a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$. H-supplemented modules are $\oplus$-supplemented [11, A.2]. Suppose $N \subseteq K$ are submodules of M, N is said to be a *cosmall* submodule of K in M if $K/N \ll M/N$(denoted by $N \leq^{cs} K$). A submodule N of M is *coclosed* in M if it has no proper cosmall submodules in M(denoted by $N \leq^{cc} M$). N is called a *coclosure* of K in M, if $N \leq^{cs} K$ and $N \leq^{cc} M$.

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* Corresponding Author.
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Recall that a module M has the *Summand Intersection Property* (*SIP*) if the intersection of any two direct summands of M is again a direct summand (see [6]) and M has the *Summand Sum Property* (*SSP*) if the sum of any two direct summands of M is again a direct summand (see [5]). Let M be a module, a submodule N of M is called *fully invariant* if for every $h \in End_R(M)$, $h(N) \subseteq N$.
Supplemented and lifting modules are worthy of study in module theory since they are dual of complemented and extending modules, and there has been a great deal of work on lifting modules by many authors. Supplemented modules, lifting modules are also studied in [11] and [14].

Let M and X be modules. Lopez-Permouth, Oshiro and Tariq Rizvi in [10], defined the family

$$A(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in Hom(Y,M), f(Y) \leq_e A\}$$

They studied extending, quasi-continuous, or continuous modules relative to this class.
In [8], D. Keskin and A. Harmanci dualized the class $A(X,M)$ and defined the family

$$B(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in Hom(M,X/Y), Kerf/A \ll M/A\}$$

They considered the following conditions:

$B(M,X)$-(D_1): For every submodule $A \in B(M,X)$, there exists a direct summand $A^* \leq_{\oplus} M$ such that $A/A^* \ll M/A^*$.
$B(M,X)$-(D_2): For any $A \in B(M,X)$, if $B \leq_{\oplus} M$ and $M/A \cong B$ implies $A \leq_{\oplus} M$.
$B(M,X)$-(D_3): For any $A \in B(M,X)$ and $B \leq_{\oplus} M$, if $A \leq_{\oplus} M$ and $M = A + B$ then $A \cap B \leq_{\oplus} M$.
They call M is *X-lifting*, *X-quasi-discrete* and *X-discrete*, respectively, if M satisfies $B(M,X)$-(D_1), $B(M,X)$-(D_1) and $B(M,X)$-(D_3), $B(M,X)$-(D_1) and $B(M,X)$-(D_2).
Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of submodules of a module M with $X_\lambda \in B(M,X)$, $\Sigma_{\lambda\in\Lambda} X_\lambda$ is called an *X-local summand* of M, if $\Sigma_{\lambda\in\Lambda} X_\lambda$ is direct and $\Sigma \oplus_{\lambda \in F} X_\lambda \leq_{\oplus} M$ for every finite subset $F \subseteq \Lambda$.
Let X and M be R-modules. Following [8], an R-module N is called $B(M,X)$-*projective* if for any submodule A of M with $A \in B(M,X)$, any homomorphism $\phi : N \longrightarrow M/A$ can be lifted to a homomorphism $\psi : N \longrightarrow M$. Two R-modules M_1 and M_2 are called *relatively B-projective* if M_1 is $B(M_2,X)$-projective and M_2 is $B(M_1,X)$-projective.
Let A and P be submodules of M with $P \in B(M,X)$. P is called an *X-supplement* of A if $M = A + P$ and $A \cap P \ll P$. The module M is called *X-supplemented* if every submodule N of M with $N \in B(M,X)$ has a X-supplement in M. Let X be an R-module. A non-zero module M is *X-hollow*, if for any proper submodule K of M with $K \in B(M,X)$, $K \ll M$.

In this paper, we consider H-supplemented and $\oplus$-supplemented relative to this class.
Therefore we define *X-H-supplemented*, *X-⊕-supplemented*, and *X-FI-lifting* modules. M is called X-H-supplemented if for any $A \in B(M, X)$ there exists a direct summand D of M such that $M = A + Y$ if and only if $M = D + Y$. M is called *X-⊕-supplemented* if every $N \in B(M, X)$ has an X-supplement that is a direct summand of M. A module M is called *X-FI-lifting* if for every fully invariant submodule A with $A \in B(M, X)$ there exists a direct summand N of M such that $A/N \ll M/N$. It is easy to see that M is H-supplemented($\oplus$-supplemented) if and only if M is M-H-supplemented(M-$\oplus$-supplemented) if and only if M is X-H-supplemented(X-$\oplus$-supplemented) for every module X. Clearly X-hollow modules are X-$\oplus$-supplemented and X-$\oplus$-supplemented modules are X-supplemented.

In Section 2, we will give some properties of X-$\oplus$-supplemented and X-H-supplemented modules. We investigate general properties of this modules, relation of them with other modules. We give a condition for an X-$\oplus$-supplemented module to be X-H-supplemented (see Proposition 2.18).

In Section 3, we define and investigate a generalization of X-H-supplemented modules.

2. Main results

A module M is called *X-supplement bounded*, if it is X-supplemented and every proper X-supplement submodule of M is contained in a nontrivial fully invariant submodule belongs to the class $B(M, X)$.

Lemma 2.1. *Let M be an X-supplemented module, then for every submodule K of M, M/K is X-supplemented.*

Proof. Simple to check. □

Proposition 2.2. *Let M be an X-supplemented module such that $B(M, X)$ is closed under taking arbitrary intersection. Then M is X-supplement bounded if and only if every proper coclosed submodule K of M is cosmall in a fully invariant submodule Y of M with $Y \in B(M, X)$.*

Proof. Assume M is X-supplement bounded. Let $K \leq^{cc} M$ be proper. Let Y be the intersection of fully invariant submodules in $B(M, X)$ containing K. Then $Y \in B(M, X)$ is a fully invariant submodule of M. By Lemma 2.1, let L/K be the X-supplement of Y/K in M/K. Then $L + Y = M$ and $L/K \cap Y/K \ll M/K$. Suppose $L \neq M$, by [8, Lemma 2.2], $L \in B(M, X)$. Since L is an X-supplement submodule of M, then there exists a fully invariant submodule $S \in B(M, X)$ such that $S \neq M$ and $L \subseteq S$. So $L + Y \subseteq Y \neq M$, a contradiction. Therefore $L = M$ and hence $K \leq^{cs} Y$ in M. The converse is trivial. □

Proposition 2.3. *Let M be X-supplement bounded such that $B(M, X)$ is closed under taking arbitrary intersection. If every submodule of M has a coclosure, then*

M is X-H-supplemented if and only if M is X-FI-lifting.

Proof. The necessity is clear. For the sufficiency assume M is X-FI-lifting. Let $Y \leq M$ and $Y \neq M$. Since Y has a coclosure, there exists a submodule K of M such that $K \subseteq Y$, $Y/K \ll M/K$ and $K \leq^{cc} M$. Since M is X-supplement bounded there exists a fully invariant submodule $B \in B(M, X)$ with $K \leq B$ and $B/K \ll M/K$ by Proposition 2.2. Since M is X-FI-lifting, there exists a direct summand D of M such that $D \leq B$ and $B/D \ll M/D$. Let $M = Y + L$ for some $L \leq M$. Then $M/K = Y/K + (L + K)/K = (L + K)/K$ implies that $M = L + K$. Then $M = L + B$ and hence $M/D = (L + D)/D + B/D = (L + D)/D$. Thus $M = L + D$. Conversely assume that $M = L + D$. Then $M = L + B$. Now $M/K = (L + K)/K + B/K$ implies that $M = L + K$ and hence $M = L + Y$. □

By analogy with the proof of [13, Proposition 2.5], we have the following proposition.

Proposition 2.4. *The following are equivalent for a module M:*
(1) M is X-FI-lifting.
(2) Every fully invariant submodule N of M with $N \in B(M, X)$ has a supplement which is a direct summand.

Now we consider the X-$\oplus$-supplemented module;

Proposition 2.5. *Let M be a nonzero module and let U be a fully invariant submodule of M with $U \in B(M, X)$ such that $M = U \oplus V$. If M is X-$\oplus$-supplemented, then V is X-$\oplus$-supplemented.*

Proof. Suppose that M is X-$\oplus$-supplemented. Let $L \in B(M, X)$ be a submodule of M which contains U. There exist submodules N and N' of M such that $M = N \oplus N'$, $M = L + N$, and $L \cap N$ is small in N and $N \in B(M, X)$. By [8, Lemma 2.2], $L/U \in B(M/U, X)$ and it is clear that $(N + U)/U$ is a X-supplement of L/U in M/U and by [8, Lemma 3.5], $(N + U)/U \in B(M/U, X)$. Since U be a fully invariant submodule of M, $U = (U \cap N) \oplus (U \cap N')$. Thus, $(N + U) \cap (N' + U) \leq (N + U + N') \cap U + (N + U + U) \cap N'$. Hence, $(N + U) \cap (N' + U) \leq U + (N + U \cap N + U \cap N') \cap N'$. It follows that $(N+U)\cap(N'+U) \leq U$ and $((N+U)/U)\oplus((N'+U)/U) = M/U$. Then $(N+U)/U$ is a direct summand of M/U. Consequently, M/U is X-$\oplus$-supplemented. □

Theorem 2.6. *Any finite direct sum of X-$\oplus$-supplemented modules is X-$\oplus$-supplemented.*

Proof. Let $M = M_1 \oplus M_2$ where M_1 and M_2 are two X-$\oplus$-supplemented modules. Let $N \in B(M, X)$, we have $N + M_2 = M_2 \oplus [(N + M_2) \cap M_1]$ and $(N + M_2) \cap M_1$ is a submodule of M_1. Since $N \in B(M, X)$, $N + M_2 \in B(M, X)$ by [8, Lemma 3.5 and 2.2]. By [12, Lemma 3.1], $(N + M_2) \cap M_1 \in B(M_1, X)$. Since M_1 is X-$\oplus$-supplemented, there exists a direct summand K_1 of M_1 with $K_1 \in B(M_1, X)$ such that $[(N + M_2) \cap M_1] + K_1 = M_1$ and $(N + M_2) \cap K_1 \ll K_1$. By [8, Lemma 3.5 and 2.2] and [12, Lemma 3.1], $(N + K_1) \cap M_2$ is a submodule of M_2 such that

$(N + K_1) \cap M_2 \in B(M_2, X)$, thus there exists a direct summand K_2 of M_2 with $K_2 \in B(M_2, X)$ such that $[(N + K_1) \cap M_2] + K_2 = M_2$ and $(N + K_1) \cap K_2 \ll K_2$. Let $\pi : M \longrightarrow M_1$ be projection along M_2. Since $K_1 \in B(M_1, X)$, by [8, Lemma 2.2(4)], $K_1 \oplus M_2 = \pi^{-1}(K_1) \in B(M, X)$. Since $M = K_1 + M_1 + M_2$, by [12, Lemma 3.1], $K_1 = (K_1 \oplus M_2) \cap M_1 \in B(M, X)$. Applying the same argument, we have $K_2 \in B(M, X)$. Let $K = K_1 \oplus K_2$, then K is a direct summand of M and by [3, Lemma 3.4], $K \in B(M, X)$. Moreover $M_1 \leq N + M_2 + K_1$ and $M_2 \leq N + K_1 + K_2$. Hence $M = N + K_1 + K_2 = N + K$. Since $N \cap (K_1 + K_2) \leq (N + K_1) \cap K_2 + (N + K_2) \cap K_1$, thus $N \cap (K_1 + K_2) \leq (N + K_1) \cap K_2 + (N + M_2) \cap K_1$. As $(N + M_2) \cap K_1 \ll K_1$ and $(N + K_1) \cap K_2 \ll K_2$, $N \cap K \ll K$. So M is X-$\oplus$-supplemented. □

Let X and M be R-modules. We call a module M *completely X-$\oplus$-supplemented* if every direct summand N of M with $N \in B(M, X)$ is X-$\oplus$-supplemented.

Proposition 2.7. *Let M be an X-$\oplus$-supplemented module with $B(M, X)$-(D_3). Then M is completely X-$\oplus$-supplemented.*

Proof. Let $N \leq_{\oplus} M$ and $A \leq N$ such that $N \in B(M, X)$ and $A \in B(N, X)$. We show that A has an X-supplement in N that is a direct summand of N. We have $M = N \oplus N'$ for some submodule N' of M. Let $\pi : M \longrightarrow N$ be projection along N'. Since $A \in B(N, X)$, by [8, Lemma 2.2(4)], $A \oplus N' = \pi^{-1}(A) \in B(M, X)$. Since $M = A + N + N'$, by [12, Lemma 3.1], $A = (A \oplus N') \cap N \in B(M, X)$. Since M is X-$\oplus$-supplemented, there exists a direct summand B of M with $B \in B(M, X)$ such that $M = A + B$ and $A \cap B \ll B$. Then $N = A + (N \cap B)$. Again by Lemma [12, Lemma 3.1], $N \cap B \in B(M, X)$. Furthermore $N \cap B \leq_{\oplus} M$ because M has $B(M, X)$-(D_3). Then $A \cap (N \cap B) = A \cap B$ is small in $N \cap B$ and by [12, Lemma 3.1], $N \cap B \in B(N, X)$. □

Proposition 2.8. *Let M be an indecomposable module. Then M is X-hollow if and only if M is completely X-$\oplus$-supplemented.*

Proof. Let M be completely X-$\oplus$-supplemented. If $N \in B(M, X)$ is a proper submodule of M then there exists an X-supplement A of N such that A is direct summand of M. By hypothesis we have $A = M$. Thus $N = N \cap M = N \cap A \ll M$. Therefore M is X-hollow. Conversely, if M is X-hollow and $N \in B(M, X)$ then $N \ll M$. Since $M \in B(M, X)$, so M is an X-supplement of N in M. □

Let M be any module. M is called a (D_3)-module if whenever M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M. Clearly (D_3) is $B(M, X)$-(D_3).
In the set $B(M, X)$, if we take $X = M$, then $B(M, X)$ coincides with the set of all submodules of M. Therefore we obtain the following corollaries:

Corollary 2.9. *Any finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented.*

Proof. See [7, Theorem 1.4]. □

Corollary 2.10. *Let M be a $\oplus$-supplemented module with (D_3). Then M is com-*

pletely $\oplus$-supplemented.

Proof. See [7, Proposition 2.3]. □

Corollary 2.11. *Let M be an indecomposable module. Then M is hollow if and only if M is completely $\oplus$-supplemented.*

Proof. See [7, Lemma 2.14]. □

Example 2.12. (1) $\mathbb{Z}_{p^\infty}$ is a lifting $\mathbb{Z}$-module and so an X-lifting $\mathbb{Z}$-module for every $\mathbb{Z}$-module X.

(2) Clearly $B(M, 0) = M$ for any module M. Therefore every module M is 0-$\oplus$-supplemented (completely 0-$\oplus$-supplemented), this means that the $\mathbb{Z}$-module $\mathbb{Z}_\mathbb{Z}$ is completely 0-$\oplus$-supplemented. But by Proposition 2.8, for every nonzero module X, it is not completely X-$\oplus$-supplemented.

(3) Let X be simple projective module and M any module. Then for any $A \in B(M, X)$, A is direct summand of M. Therefore M is X-H-supplemented module.

(4) If M is a divisible $\mathbb{Z}$-module, then $B(M, \mathbb{Z}) = \emptyset$, since $Hom(M, \mathbb{Z}_n) = 0$.

Lemma 2.13. *Let M be an X-$\oplus$-supplemented module with (SIP). Then every X-local summand Y of M such that $Y \in B(M, X)$ is a direct summand.*

Proof. Let $Y = \Sigma_{i\in I} Y_i$ be an X-local summand of M. Since M is X-$\oplus$-supplemented, there exists a direct summand K of M such that $M = K + Y$ and $K \cap Y \ll K$ such that $K \in B(M, X)$. For any finite subset F of I, $Y = \oplus_{i\in F} Y_i$ is a direct summand of M, hence $Y \cap K$ is a direct summand of M since M has (SIP). Thus $Y \cap K = 0$. Therefore $M = K \oplus Y$. □

Proposition 2.14. *Let M have the (SSP) with (D_3). Then M has the (SIP).*

Proof. By [2, Lemma 19(2)] □

Lemma 2.15. *If every X-local summand of a module M is a direct summand, then M has an indecomposable decomposition.*

Proof. See [3, Lemma 3.2] □

Theorem 2.16. *Let M be an X-$\oplus$-supplemented module with (SSP), (D_3) and every X-local summand Y of M such that $Y \in B(M, X)$. Then M is a direct sum of X-hollow modules.*

Proof. By Lemma 2.13, Lemma 2.15, M is a direct sum of indecomposable modules and since M has (D_3), therefore M has $B(M, X)$-(D_3), so by Proposition 2.7, every direct summand of M is X-$\oplus$-supplemented. Therefore M is a direct sum of indecomposable X-$\oplus$-supplemented modules, which are X-hollow. □

Remark 2.17. Let M be an X-H-supplemented module such that for every direct summand A of M with $A \in B(M, X)$. Then M is X-$\oplus$-supplemented.

Proposition 2.18. *Assume that M is X-$\oplus$-supplemented such that whenever*

$M = M_1 \oplus M_2$ then M_1 and M_2 are relatively B-projective. Then M is an X-H-supplemented module.

Proof. Let $N \in B(M, X)$, since M is X-$\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \ll M_2$ such that $M_2 \in B(M, X)$. By hypothesis, M_1 is $B(M_2, X)$-projective, by [8, Proposition 2.5], we obtain $M = A \oplus M_2$ for some submodule A of M such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. Let $Y \leq M$ with $M = N + Y$. Then $M = A + (M_2 \cap N) + Y$. Since $M_2 \cap N$ is small in M_2 and so is small in M, $M = A + Y$. Hence $M = N + Y$ if and only if $M = A + Y$. Thus M is X-H-supplemented module. □

Lemma 2.19. *For a submodule U of M , the following are equivalent:*

(1) there is a direct summand Y of M with $Y \subseteq U$ and $U/Y \ll M/Y$;

(2) there is a direct summand $Y \subseteq M$ and a submodule L of M with $Y \subseteq U$, $U = Y + L$ and $L \ll M$;

Proof. See [4, 22.1]. □

Let X and M_2 be R-modules. Following [12], an R-module M_1 is called *$B(M_2, X)$-cojective* if for any submodule A of M_2 with $A \in B(M_2, X)$, any homomorphism $\phi : M_1 \longrightarrow M_2/A$, there exist decompositions $M_1 = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$ and homomorphisms $\phi_1 : M_1' \longrightarrow M_2'$, $\phi_2 : M_2'' \longrightarrow M_1''$ such that ϕ_2 is onto, $\pi\phi_1 = \phi |_{M_1'}$ and $\phi\phi_2 = \pi |_{M_2''}$, where $\pi : M_2 \longrightarrow M_2/A$ is the natural epimorphism. Two R-modules M_1 and M_2 are called *relatively B-cojective* if M_1 is $B(M_2, X)$-cojective and M_2 is $B(M_1, X)$-cojective.

Proposition 2.20. *Assume that M is X-$\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively B-cojective. Then M is X-lifting.*

Proof. Let $A \in B(M, X)$. Then A has an X-supplement M_2 which is a direct summand of M, $M = M_1 \oplus M_2$. Then by hypothesis, M_1 is $B(M_2, X)$-cojective, since $M = A + M_2$, by [12, Proposition 3.2], we have $M = A' \oplus M_1'' \oplus M_2' = A' + M_2$, $A' \leq A$, $M_1'' \leq M_1$, $M_2' \leq M_2$. Then $A = A' + (A \cap M_2)$. Thus, since $A \cap M_2 \ll M_2$, Now from Lemma 2.19, M is an X-lifting module. □

3. X-H-cofinitely supplemented

A submodule N of M is called *cofinite* in M if the factor module M/N is finitely generated. A module M is called H-cofinitely supplemented if for every cofinite submodule A of M, there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$. Clearly H-supplemented modules are H-cofinitely supplemented. On the other hand, every finitely generated H-cofinitely supplemented module is H-supplemented.

We call M is called X-H-cofinitely supplemented if for cofinite $A \in B(M, X)$ there exists a direct summand D of M such that $M = A + Y$ if and only if $M = D + Y$. $\mathbb{Z}$-module $\mathbb{Q}$ has no proper cofinite submodule, so it is X-H-cofinitely-supplemented. By definition every X-H-supplemented is X-H-cofinitely supplemented.

Now we have the following hierarchy;

X-lifting$\Longrightarrow$ X-H-supplemented$\Longrightarrow$ X-H-cofinitely-supplemented

The module M is called *duo* module, if every submodule of M is fully invariant. M is called *distributive* if $N \cap (L+K) = (N \cap L) + (N \cap K)$ and $N + (L \cap K) = (N+L) \cap (N+K)$ for every submodules N,K,L of M.

Example 3.1. Let p be any prime number. Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}_P)$. Let L be any cofinite submodule of M. Hence $\mathbb{Q}/(\mathbb{Q} \cap L)$ is finitely generated. Thus $\mathbb{Q} \leq L$. It follows that $L = \mathbb{Q} \oplus L \cap (\mathbb{Z}/\mathbb{Z}_P)$. Then $L = \mathbb{Q}$ or $L = M$. So, M is H-cofinitely supplemented, then M is X-H-cofinitely supplemented.

Now we consider the X-H-cofinitely supplemented module;

Theorem 3.2. *Let M be a module. The following are equivalent:*
(1) M is X-H-cofinitely supplemented module;
(2) For each cofinite submodule $Y \in B(M,X)$ there exists a direct summand D of M such that $(Y+D)/D \ll M/D$ and $(Y+D)/Y \ll M/Y$;
(3) For each cofinite submodule $Y \in B(M,X)$ there exists $L \leq M$ and a direct summand D of M such that $L/Y \ll M/Y$, $L/D \ll M/D$.

Proof. (1) $\Longrightarrow$ (2) Let $Y \leq M$ be cofinite. By assumption there exists a direct summand D of M such that $M = Y + L$ holds if and only if $M = D + L$. Let $(Y+D)/D + L/D = M/D$ for some submodule L of M containing D. So, $Y + L = M$ and hence $D + L = M$, if follows that $L = M$. Thus $(Y+D)/D \ll M/D$. The second part is the same.
(2) $\Longrightarrow$ (3) Let $Y \in B(M,X)$ be cofinite. Then there exists a direct summand D of M such that $(Y+D)/D \ll M/D$ and $(Y+D)/Y \ll M/Y$. Now take $L = Y + D$.
(3) $\Longrightarrow$ (1) Let $Y \in B(M,X)$ be cofinite. Then there exist a submodule L of M and a direct summand D of M such that both Y and D are cosmall submodules of L in M. It is easy to see that $M = A + D$ if and only if $M = A + Y$ for all $A \leq M$. Thus M is X-H-cofinitely supplemented. □

Theorem 3.3.
(1) Let M be an X-H-cofinitely supplemented module and L a submodule of M. If for every direct summand K of M, $(L+K)/L$ is a direct summand of M/L then M/L is X-H-cofinitely supplemented.
(2) Let M be an X-H-cofinitely supplemented module with the (SSP). Then every direct summand of M is X-H-cofinitely supplemented module.
(3) Let M be an X-H-cofinitely supplemented distributive module. Then M/N is X-H-cofinitely supplemented for every submodule N of M.

Proof. (1) Let $N/L \in B(M/L,X)$ be cofinite where N is a cofinite submodule of M and $L \subseteq N$, then by [8, Lemma 2.2], $N \in B(M,X)$. Since M is X-H-cofinitely supplemented, for every cofinite $N \in B(M,X)$, there exists a direct summand D of M such that $M = N + Y$ if and only if $M = D + Y$. By hypothesis,

$(D + L)/L$ is a direct summand of M/L. Then $M/L = N/L + A/L$ if and only if $M/L = (D + L)/L + A/L$ for every $A/L \leq M/L$ so M/L is X-H-cofinitely supplemented.
(2) Assume that M is X-H-cofinitely supplemented and M has the summand sum property. Let N be a direct summand of M. We show that N is X-H-cofinitely supplemented. Let $M = N \oplus K$ for some submodule K of M. Assume that A is a direct summand of M. Since M has the summand sum property, $A + K$ is a direct summand of M. Let $M = (A + K) \oplus B$ for some submodule B of M. Then $M/K = (A + K)/K \oplus (B + K)/K$. Hence M/K is X-H-cofinitely supplemented by (1) and so N is X-H-cofinitely supplemented.
(3) Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/N = [(D + N)/N] + [(D' + N)/N]$. Note that $N = N + (D \cap D') = (N + D) \cap (N + D')$ by distributive of M. So $M/N = [(D + N)/N] \oplus [(D' + N)/N]$. By (1), M/N is X-H-cofinitely supplemented. □

Theorem 3.4. *Let M be a duo module. Then M has the (SIP) and the (SSP).*

Proof. See [1, Theorem 3.5] □

As a result of Theorem 3.3 and Theorem 3.4, we can obtain the following corollary;

Corollary 3.5. *Let M be an X-H-cofinitely supplemented duo module. Then every direct summand of M is X-H-cofinitely supplemented module.*

Theorem 3.6. *Let $M = M_1 \oplus M_2$ be a duo module and for any $A \in B(M, X)$, $M = A + M_i$ $(i = 1, 2)$. If M_1 and M_2 are X-H-cofinitely supplemented modules, then M is X-H-cofinitely supplemented.*

Proof. Assume M_1 and M_2 are X-H-cofinitely supplemented modules. Let $L \in B(M, X)$ be cofinite. $L = (L \cap M_1) \oplus (L \cap M_2)$. Clearly, $L \cap M_1$ and $L \cap M_2$ are cofinite submodules of M_1 and M_2, then by [12, Lemma 3.1], $L \cap M_1 \in B(M_1, X)$ and $L \cap M_2 \in B(M_2, X)$. Since M_1, M_2 are X-H-cofinitely supplemented, there exists a direct summand A_1, A_2 of M_1, M_2 such that $M_1 = A_1 + Y$ if and only if $M_1 = (L \cap M_1) + Y$ for any submodule Y of M_1 that $L \cap M_1 \in B(M_1, X)$ and also $M_2 = A_2 + Y$ if and only if $M_2 = (L \cap M_2) + Y$ for any submodule Y of M_2 that $L \cap M_2 \in B(M_2, X)$. It is clear until that show that $M = (A_1 \oplus A_2) + Z$ if and only if $M = L + Z$ for any submodule Z of M. □

Corollary 3.7. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of duo modules and for any $A \in B(M, X)$, $M = A + M_i$ $(i = 1, ..., n)$. If every M_i is X-H-cofinitely supplemented modules, then M is X-H-cofinitely supplemented.*

Finally, we get the following results as corollaries of Theorem 3.3, Corollary 3.5 and Corollary 3.7.

Corollary 3.8([9, Theorem 2.1])**.** *(1) Let M be an H-cofinitely supplemented module and L a submodule of M. If for every direct summand K of M, $(L + K)/L$ is*

a direct summand of M/L then M/L is H-cofinitely supplemented.
(2) Let M be an H-cofinitely supplemented module with the (SSP). Then every direct summand of M is H-cofinitely supplemented module.
(3) Let M be an H-cofinitely supplemented distributive module. Then M/N is H-cofinitely supplemented for every submodule N of M.

Corollary 3.9[9, Corollary 2.3]**.** *Let M be an H-cofinitely supplemented duo module. Then every direct summand of M is H-cofinitely supplemented module.*

Corollary 3.10. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of duo modules. If every M_i is H-cofinitely supplemented modules, then M is H-cofinitely supplemented.*

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