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Module-theoretic Characterizations of Strongly t-linked Extensions

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Abstract. In this paper, we introduce and study the concept of "strongly t-linked extensions", which is a stronger version of t-linked extensions of integral domains. We show that for an extension of Prüfer v-multiplication domains, this concept is equivalent to that of "w-faithfully flat".

1. Introduction

Let R be an integral domain with quotient field K. Then for any nonzero (fractional) ideal I set $I^{-1} := \{x \in K \mid xI \subseteq R\}$ and an ideal J of R is called a *GV-ideal*, denoted by $J \in GV(R)$, if J is a finitely generated ideal of R with $J^{-1} = R$.

Let R be a subring of the integral domain T. Following [7], we say that T is *t-linked over R* if $J \in GV(R)$ implies $JT \in GV(T)$. As pointed out in [1], an extension $R \subseteq T$ of Krull domains is t-linked if and only if it satisfies Samuel's PDE (Pas d'éclatement) or NBU (No blowing up) condition, i.e., for a height one prime $P \in \text{Spec}(T)$, the set of prime ideals of T, we have $\text{ht}(P \cap R) \leq 1$. Anderson *et al.* in [1] showed that if T is t-linked over R, then the map $[I] \mapsto [(IT)_t]$ gives a homomorphism $Cl_t(R) \to Cl_t(T)$ of the t-class groups. Recall from [8] that an integral domain R is called *t-linkative* if each overring T of R is t-linked over R, equivalently, if every (nonzero) ideal of R is w-ideal ([13]). Examples of t-linkative domains are Prufer domains and domains with Krull dimension one ([7, Corollary 2.7]). In [13],

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module-theoretic characterizations of t-linked extensions and t-linkative domains are given. In [16], a stronger version of the PDE condition for an extension of Krull domains was introduced and studied. In this paper, we introduce and study the concept of "strongly t-linked extensions", which is a stronger version of t-linked extensions of integral domains. In fact, this is a continuous work on the project of studying some properties over Prüfer v-multiplication domains ([12, 14]).

We first introduce some definitions and notations. Let R be an integral domain with quotient field K. Let I be a nonzero fractional ideal I of R. Then $I_v := (I^{-1})^{-1}$, $I_t := \bigcup\{J_v | J \subseteq I$ is a nonzero finitely generated ideal$\}$, and $I_w := \{x \in K \mid Jx \subseteq I$ for some $J \in GV(R)\}$. We say that I is a *t-ideal* (resp., *w-ideal*) if $I = I_t$ (resp., $I = I_w$). A fractional ideal I of R is said to be *t-invertible* (resp., *w-invertible*) if $(II^{-1})_t = R$ (resp., $(II^{-1})_w = R$). It is known that a fractional ideal I is t-invertible if and only if I is w-invertible. We say that a fractional ideal I of R is of *w-finite type* if $I_w = J_w$ for some finitely generated ideal J of R. A maximal t-ideal (resp., w-ideal) is an ideal of R maximal among proper integral t-ideals (resp., w-ideals) of R. Let t-Max(R) (resp., w-Max(R)) be the set of maximal t-ideals (resp., w-ideals). Then it is easy to see that t-Max$(R) = w$-Max(R); if R is not a field, then t-Max$(R) \neq \emptyset$. An integral domain R is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal of R is t-invertible. It is well known that an integral domain R is a PvMD if and only if $R_{\mathfrak{p}}$ is a valuation domain for any prime t-ideal $\mathfrak{p}$ of R; if a domain R is a PvMD, then $t = w$; if T is t-linked over a PvMD R, then T is w-flat over R (The definition of w-flatness will be reviewed later). Let M be a module over the Prüfer domain R. Then it is well known that M is torsion-free if and only if M is flat. From this result it also follows that a finitely generated module over a valuation domain is torsion-free if and only if it is free, since a finitely generated module over a local ring is free or, equivalently, projective, if and only if it is flat.

Let M be a module over an integral domain R. Following [13] and [19], M is said to be *GV-torsion-free* (or *co-semi-divisorial*) if $\{x \in M \mid (ann_R(x))_w = R\} = 0$; equivalently, if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$. M is called *GV-torsion* (or *w-null*) if $\{x \in M \mid (ann_R(x))_w = R\} = M$. We call an R-module M *semi-divisorial* (or a *w-module*) if it is torsion-free and $M = W_R(M)$, where the *w-envelope of* M is defined as $W_R(M) = \bigcap_{P \in w\text{-Max}(R)} M_P$, where the intersection is taken within $K \otimes_R M$. In particular, the domain R itself is semi-divisorial as an R-module. Any R-linear map $u : M \to N$ between torsion-free R-modules induces a map $W_R(u) : W_R(M) \to W_R(N)$, i.e., W_R may be viewed as a covariant functor on torsion-free R-modules. Let M, N be semi-divisorial modules over R. Suppose that $f : M \to N$ is an R-homomorphism and $f_P : M_P \to N_P$ is an isomorphism for all $P \in w$-Max(R). Then it is easy to see that f is an isomorphism.

Let R be a PvMD. Then for any $\mathfrak{p} \in w$-Spec(R) the ring $R_{\mathfrak{p}}$ is a valuation domain, hence, an $R_{\mathfrak{p}}$-module is $R_{\mathfrak{p}}$-flat if and only if it is torsion-free. Since $R_{\mathfrak{p}}$ is a flat R-module, any $R_{\mathfrak{p}}$-module that is $R_{\mathfrak{p}}$-flat is R-flat. Hence any semi-divisorial R-module M is an intersection in $K \otimes_R M$ of flat R-modules $M_{\mathfrak{p}}$.

Any undefined terminology is standard, as in [9] or [10].

2. Main results

We begin this section by listing some characterizations of t-linked extensions of integral domains in the literature.

If R is an integral domain, we set $R\langle X\rangle := R[X]_{N_t}$, where $N_t := \{f \in R[X] \mid c(f)_t = R\}$, a multiplicative set in $R[X]$ ($c(f)$ is the ideal of R generated by the coefficients of $f \in R[X]$). $R\langle X\rangle$ is called the *t-Nagata ring of R.*

Theorem 2.1. *Let $R \subseteq T$ be an extension of domains. Then the following conditions are equivalent.*

(1) *T is t-linked over R.*

(2) *If I is a (finitely generated) ideal of R with $I_t = R$, then $(IT)_t = T$.*

(3) *If Q is a prime t-ideal of T with $Q \cap R \neq 0$, then $(Q \cap R)_t \subsetneq R$.*

(4) *If Q is a maximal t-ideal of T with $Q \cap R \neq 0$, then $(Q \cap R)_t \subsetneq R$.*

(5) *If I and J are t-invertible ideals of R with $I_t = J_t$, then $(IT)_t = (JT)_t$.*

(6) *If I is a t-invertible ideal of R, then $(IT)_t = (I_tT)_t$.*

(7) *$I_w \subseteq (IT)_w$, for any ideal I of R.*

(8) *$A \cap R$ is a w-ideal of R for any w-ideal A of T.*

(9) *$(IT)_w \cap R$ is a w-ideal of R for any ideal I of R.*

(10) *$(IT)_w \cap R$ is a w-ideal of R for any finitely generated ideal I of R.*

(11) *$P \cap R$ is a (prime) w-ideal of R for any prime w-ideal P of T.*

(12) *$T = T\langle X\rangle \cap qf(T)$, where X is an indeterminate over T.*

(13) *T is semi-divisorial as an R-module.*

(14) *Every GV-torsion-free T-module is a GV-torsion-free R-module.*

(15) *$M \otimes_R T$ is a GV-torsion T-module for any GV-torsion R-module M.*

Proof. The proof of [7, Proposition 2.1] shows that (1)-(3) are equivalent. For the equivalences of (2), (4), (5), and (6), See [1, Proposition 2.1]. It was shown in [17, Proposition 1.2] that (1) and (7)-(11) are equivalent. (1) $\Leftrightarrow$ (12). See the proof of [4, Lemma 3.2]. (1) $\Leftrightarrow$ (13) $\Leftrightarrow$ (14). See [13, Theorem 9.10]. (14) $\Leftrightarrow$ (15). See [23, Lemma 1.1(2)]. □

Corollary 2.2. *Let $R \subseteq T$ be a t-linked extension of domains and let M be an R-module. If $M \otimes_R T$ is a GV-torsion-free T-module, then M is a GV-torsion-free R-module.*

Proof. This follows from [23, Lemma 1.1(1)] and Theorem 2.1. □

Consider an inclusion of domains $i : R \hookrightarrow T$. Taking intersections with A, this inclusion induces a continuous map (for the Zariski topology)

$$^{a}i : \mathrm{Spec}(T) \to \mathrm{Spec}(R), Q \mapsto Q \cap R,$$

which does, in general, not restrict to a map w-Spec$(T) \to w$-Spec(R). If it does, i.e., if $Q \cap R \in w$-Spec(R), for all $Q \in w$-Spec(T), then we say that i is a *t-linked extension*.

It was shown in [17, Proposition 1.1] that for an extension $R \subseteq T$ of domains, if P is a prime ideal of T such that $P \cap R$ is a w-ideal of R, then $P_w \neq T_w$. As a corollary, for a prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ is a w-ideal if and only if $\mathfrak{p}_w \neq R$. It is easy to see that if a domain T is semi-divisorial over a domain R and I is a w-ideal of T. Then I is semi-divisorial over R (cf. [17, Remark 1]).

Let $R \subseteq T$ be an extension of rings. Suppose T is a flat R-module. Then it is known that if P is a prime ideal of T and write $\mathfrak{p} = P \cap R$, then T_P is a faithfully flat $R_{\mathfrak{p}}$-module. Note that if R is a valuation domain, then any nonzero (prime) ideal of R is w-ideal. Recall that for two local rings $(R, \mathfrak{m}_R)$ and $(T, \mathfrak{m}_T)$, a homomorphism $\psi : R \to T$ is called a *local homomorphism* if $\psi(\mathfrak{m}_R) \subseteq \mathfrak{m}_T$.

Theorem 2.3. *Let $i : R \hookrightarrow T$ be an extension of PvMDs. Then i is a t-linked extension if and only if the $R_{\mathfrak{p}}$-module T_P is (faithfully) flat, for every $P \in w$-Spec(T) and $\mathfrak{p} = P \cap R$.*

Proof. ($\Rightarrow$) If i is a t-linked extension, then $\mathfrak{p} := P \cap R \in w$-*Spec*$(R)$, for all $P \in w$-*Spec*(T). Thus $R_{\mathfrak{p}}$ is a valuation domain. Since T_P is torsion-free over $R_{\mathfrak{p}}$, it is flat over $R_{\mathfrak{p}}$.

($\Leftarrow$) If $P \in w$-Spec(T) and if T_P is flat over $R_{\mathfrak{p}}$ with $\mathfrak{p} = P \cap R$, then T_P is faithfully flat over $R_{\mathfrak{p}}$ since $R_{\mathfrak{p}} \to T_P$ is local. Therefore T_P is semi-divisorial over $R_{\mathfrak{p}}$ and $\mathfrak{p}_{\mathfrak{p}} T_P \neq T_P$, and so $(T_P)_{w_{\mathfrak{p}}} = T_P$ as an $R_{\mathfrak{p}}$-module (i.e., $w_{\mathfrak{p}}$ is the w-operation on $R_{\mathfrak{p}}$) and $\mathfrak{p}_{\mathfrak{p}} T_P \subseteq P_P$. Note that P_P is a prime w-ideal of T_P, since $R_{\mathfrak{p}}$ is a valuation domain. We will show that $\mathfrak{p}$ is a prime w-ideal of R. Suppose that $\mathfrak{p}_w = R$. Then $(\mathfrak{p}_{\mathfrak{p}})_{w_{\mathfrak{p}}} = R_{\mathfrak{p}}$. Thus we have $P_P = (P_P)_{w_{\mathfrak{p}}} \supseteq (\mathfrak{p}_{\mathfrak{p}} T_P)_{w_{\mathfrak{p}}} = ((\mathfrak{p}_{\mathfrak{p}})_{w_{\mathfrak{p}}} T_P)_{w_{\mathfrak{p}}} = (T_P)_{w_{\mathfrak{p}}} = T_P$ (the first equality follows from the remark just above, while the second equality follows from [20, Proposition 2.8]) as (torsion-free) $R_{\mathfrak{p}}$-modules, which is a contradiction. Therefore, $\mathfrak{p}_w \neq R$. Thus by [21, Proposition 1.1] $\mathfrak{p}$ is a prime w-ideal of R. □

The following result provides the first link between the notion of the t-linked extension and that of a semi-divisorial module.

Proposition 2.4([13, Corollary 9.11])**.** *Let $R \subseteq T$ be a t-linked extension of integral domains. If M is a semi-divisorial T-module, then M is also semi-divisorial as an R-module.*

If the map $^{a}i : w$-Spec$(T) \to w$-Spec(R) is surjective, i.e., if for every $P \in w$-Spec(R) there exists some $Q \in w$-Spec(T) with the property that $Q \cap R = P$, then we will say that i is a *strongly t-linked extension* or that T is *strongly t-linked* over

R. Thus it is clear to see that a t-linked extension $R \subseteq T$ of domains is a strongly t-linked extension if and only if the pair (R, T) satisfies "lying over" property for prime w-ideals of R and T.

Following [22], an ideal J of a commutative ring R is called a *Glaz-Vasconcelos ideal* or a *GV-ideal*, denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism $\alpha : R \to \mathsf{Hom}_R(J, R)$, defined by $\alpha(r)(a) = ra, \forall r \in R, \forall a \in J$, is an isomorphism. An R-module M is said to be *GV-torsion-free* if whenever $Jx = 0$, for some $J \in GV(R)$ and $x \in M$, then $x = 0$.

Now we extend this concept to any module. Let R be a commutative ring with identity and let M be an R-module. Define $\mathsf{r}(M) := \{x \in M \mid (ann_R(x))_w = R\}$. Then $\mathsf{r}(M)$ is a submodule of M. It is easy to see that M is GV-torsion-free if and only if $\mathsf{r}(M) = 0$ and that $M/\mathsf{r}(M)$ is GV-torsion-free. Define $M_w := \{x \in E(M) \mid Jx \subseteq M/\mathsf{r}(M) \text{ for some } J \in GV(R)\}$, where $E(M)$ denotes the *injective envelope* (or *injective hull*) of M. Then it is also easy to see that $W(M) = M_w$ for any torsion-free R-module M. An R-module M is said to be a *w-module* if $M_w = M$. Let M and N be any modules over any commutative ring R. Then we define the *w-tensor product* of M and N as follows: $M \hat{\otimes} N := (M \otimes_R N)_w$.

The following definitions and proposition are easily derived from [3, 1.4, 1.5 Proposition, 1.6 Proposition]: Let M be an R-module. Then it is clear that the functor $M\hat{\otimes}-$ is right exact. We call M a *w-flat* R-module if $M\hat{\otimes}-$ is exact. Then M is w-flat if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in w\text{-Max}(R)$. M is said to be *w-faithfully flat* if it satisfies one of the equivalent conditions of the following proposition.

Proposition 2.5. *Let M be an R-module. Then the following statements are equivalent:*

(1) *For all GV-torsion-free semi-divisorial R-modules A, B, and C, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $M\hat{\otimes}A \xrightarrow{1_M \hat{\otimes} f} M\hat{\otimes}B \xrightarrow{1_M \hat{\otimes} g} M\hat{\otimes}C$ is exact.*

(2) *M is w-flat and for all GV-torsion-free semi-divisorial R-module N we have $M\hat{\otimes}N = 0$ if and only if $N = 0$.*

(3) *M is w-flat and for all $\mathfrak{p} \in w\text{-Max}(R)$ we have $(M/\mathfrak{p}M)_w \neq 0$.*

(4) *For all $\mathfrak{p} \in w\text{-Max}(R)$ the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is faithfully flat.*

It is well known that for an extension $R \subseteq T$ of integral domains having the same quotient field, if T is faithfully flat over R, then $R = T$. The following result is the w-theoretic analogue of this result.

Corollary 2.6. *Let $R \subseteq T$ be an extension of integral domains having the same quotient field. If T is w-faithfully flat over R, then $R = T$.*

Proof. Let $\mathfrak{p} \in w\text{-Max}(R)$. Then $T_{R\backslash\mathfrak{p}}$ is $R_{\mathfrak{p}}$-faithfully flat by Proposition 2.5.

Since $T_{R\setminus\mathfrak{p}}$ and $R_{\mathfrak{p}}$ have the same quotient field, we have that $T_{R\setminus\mathfrak{p}} = R_{\mathfrak{p}}$. Hence $R = \bigcap_{\mathfrak{p}\in w-\text{Max}(R)} R_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in w-\text{Max}(R)} T_{R\setminus\mathfrak{p}} \supseteq T \supseteq R$, and thus $R = T$. □

To address the question of "what is a w-faithfully flat ideal of an integral domain R?", we need the following lemma.

Lemma 2.7([19, Theorem 2.6.22])**.** *Let $(R, \mathfrak{m})$ be a local ring and let I be a faithfully flat ideal of R. Then I is a principal ideal, which is also free as an R-module. More precisely, if $a \in I \setminus \mathfrak{m}I$, then $I = (a)$.*

A nonzero ideal I of R is called a *w-cancellation ideal* if $(IA)_w = (IB)_w$ for nonzero ideals A and B of R implies $A_w = B_w$. In [6, Corollary 2.4], it was shown that I is a w-cancellation ideal if and only if I is w-locally principal. The following result is the w-theoretic analogue of [19, Theorem 2.6.23].

Theorem 2.8. *Let R be a domain and let I be a nonzero ideal of R. Then I is w-faithfully flat if and only if I is w-locally principal.*

Proof. Suppose that I is w-faithfully flat and let $\mathfrak{m}$ be a maximal w-ideal of R. Then by Proposition 2.5, $I_{\mathfrak{m}}$ is a faithfully flat ideal of $R_{\mathfrak{m}}$. Hence $I_{\mathfrak{m}}$ is a principal ideal by Lemma 2.7.

Conversely, suppose that $I_{\mathfrak{m}}$ is a principal ideal of $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R. Then I is w-flat by the remark before Proposition 2.5. Clearly, $I_{\mathfrak{m}} \neq \mathfrak{m}I_{\mathfrak{m}}$, and hence $(I/\mathfrak{m}I)_w \neq 0$. Therefore by Proposition 2.5, I is w-faithfully flat. □

Now we give an example of a w-faithfully flat, but not w-invertible ideal.

Example 2.9. It is known that a nonzero ideal I of R is w-invertible if and only if I is of w-finite type and I is w-locally principal. Consider $R := \mathbb{Z} + X\mathbb{Q}[[X]]$. Then it is known that R is a two-dimensional Prüfer domain. Let I be an ideal of R generated by the set $\{\frac{1}{p}X\}$, where p ranges over the set of prime numbers in $\mathbb{Z}$. Then it is shown in [19, Example 8.6.25] that I is a faithfully flat ideal, which is not finitely generated. Since R is a Prüfer domain, R is t-linkative, i.e., every (nonzero) ideal of R is a w-ideal. Therefore by Theorem 2.8, I is a not w-invertible but w-locally principal ideal of R.

Lemma 2.10. *Let $R \subseteq T$ be an extension of domains. If T is a w-faithfully flat R-module, then $(IT)_w \cap R = I_w$ for any ideal I of R.*

Proof. Let $\mathfrak{p}$ be a maximal w-ideal of R. Then we have that $(IT \cap R)_{\mathfrak{p}} = I_{\mathfrak{p}}T_{\mathfrak{p}} \cap R_{\mathfrak{p}} = I_{\mathfrak{p}}$ (the second equality follows from the fact that $T_{\mathfrak{p}}$ is faithfully flat as an $R_{\mathfrak{p}}$-module). Thus $(IT)_w \cap R = (IT \cap R)_w = I_w$. □

Lemma 2.11. *Let $R \subseteq T$ be an extension of domains and let $\mathfrak{p} \in w$-*Spec(R)*. Then there is a $P \in w$-*Spec(T) *lying over $\mathfrak{p}$ if and only $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$.*

Proof. Suppose that there is a $P \in w$-Spec(T) lying over $\mathfrak{p}$. Then we have that $\mathfrak{p} \subseteq (\mathfrak{p}T)_w \cap R \subseteq P_w \cap R = P \cap R = \mathfrak{p}$. Therefore $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$.

Conversely, suppose that $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$. Set $S := R \setminus \mathfrak{p}$. Then $(\mathfrak{p}T)_w \cap S = \emptyset$. Thus there is a prime w-ideal P of T such that $(\mathfrak{p}T)_w \subseteq P$ and $P \cap S = \emptyset$. Hence $\mathfrak{p} \subseteq P \cap R$. Let $x \in P \cap R$. Since $P \cap S = \emptyset$, $x \notin S$, and so $x \in \mathfrak{p}$. Therefore $P \cap R = \mathfrak{p}$. □

Lemma 2.12([19, Theorem 5.2.17]). *Let $(R, \mathfrak{m}_R)$ and $(T, \mathfrak{m}_T)$ be local rings and let $f : R \to T$ be a homomorphism with $f^{-1}(\mathfrak{m}_T) = \mathfrak{m}_R$. Suppose T is a flat R-module.*

(1) *T is faithfully flat and f is monomorphic.*

(2) *The map ${}^a f : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ defined by ${}^a f(P) = f^{-1}(P)$ is a surjection.*

Let T be an R-algebra and M be a T-module. Then it is known that if T is flat over R and M is flat over T, then M is flat over R. Let $R \subseteq T$ be a t-linked extension. We say that (R, T) satisfies w-GD if (R, T) satisfies "going down" in the sense that for $\mathfrak{p}, \mathfrak{q} \in w\text{-Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and let $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$, there exists a $Q \subseteq P$ such that $Q \cap R = \mathfrak{q}$. We also define $w\text{-dim}(R) := \sup\{ht(\mathfrak{p}) \mid \mathfrak{p} \in w\text{-Max}(R)\}$.

Theorem 2.13. *Let $R \subseteq T$ be an extension of rings.*

(1) *If T is a w-flat R-module, then (R, T) satisfies w-GD.*

(2) *If T is a w-faithfully flat R-module, then $w\text{-dim}(R) \leq w\text{-dim}(T)$.*

Proof. (1) Let $\mathfrak{p}, \mathfrak{q} \in w\text{-Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and let $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$. Consider the extension $R_\mathfrak{p} \subseteq T_P$. Since $T_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for every $\in w\text{-Spec}(R)$ and T_P is a quotient ring of $R_\mathfrak{p}$, it follows from the above remark that T_P is flat over $R_\mathfrak{p}$. Thus by Lemma 2.12, there is a $Q \in \mathrm{Spec}(T)$ with $Q \subseteq P$ and $QT_P \cap R_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$. Note that $Q \in w\text{-Spec}(T)$ since $Q \subseteq P$. It follows that $Q \cap R = \mathfrak{q}$.

(2) Let $\mathfrak{p}$ be a maximal w-ideal of R and let $\mathfrak{p}_s \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}$ be a chain of prime (w-)ideals of R. By Lemma 2.10 and Lemma 2.11, there is a maximal w-ideal P of T such that $P \cap R = \mathfrak{p}$. By (1), (R, T) satisfies the w-GD. Hence there is a chain of prime (w-)ideals $P_s \subset \cdots \subset P_1 \subset P$ such that $P_i \cap R = \mathfrak{p}_i$. It follows that $ht(\mathfrak{p}) \leq ht(P)$. Therefore $w\text{-dim}(R) \leq w\text{-dim}(T)$. □

The following result is a connection between strong t-linkedness and w-faithful flatness among extensions of PvMDs.

Theorem 2.14. *Let $R \subseteq T$ be a t-linked extension of PvMDs. Then T is a strongly t-linked extension over R if and only if T is a w-faithfully flat R-module.*

Proof. Assume that T is a strongly t-linked extension over R and let $\mathfrak{p} \in w\text{-Spec}(R)$. Then $T_\mathfrak{p}$ is torsion-free over the valuation domain $R_\mathfrak{p}$, and hence it is flat over $R_\mathfrak{p}$. To prove that $T_\mathfrak{p}$ is faithfully flat, we have to show that $IT_\mathfrak{p} \neq T_\mathfrak{p}$ for every maximal ideal I of $R_\mathfrak{p}$. Note that $\mathfrak{p}_\mathfrak{p}$ is a unique maximal of $R_\mathfrak{p}$. We thus have to show that $\mathfrak{p}T_\mathfrak{p} \neq T_\mathfrak{p}$. By assumption, there is a prime ideal $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$. Then clearly $\mathfrak{p}T_\mathfrak{p} \subseteq PT_\mathfrak{p} \neq T_\mathfrak{p}$ (as $P \cap (R \setminus \mathfrak{p}) = \emptyset$).

Conversely, assume that for every $\mathfrak{p} \in w\text{-Spec}(R)$, the $R_{\mathfrak{p}}$-module $T_{\mathfrak{p}}$ is faithfully flat. Consider, for the moment, a fixed $\mathfrak{p} \in w\text{-Spec}(R)$ and the induced map $R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$, which is faithfully flat. Since $T_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, the extension $R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$ is t-linked, equivalently $T_{\mathfrak{p}}$ is semi-divisorial over $R_{\mathfrak{p}}$, and so $(T_{\mathfrak{p}})_w = T_{\mathfrak{p}}$. Again since $T_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, there exists $Q' \in \text{Max}(T_{\mathfrak{p}})$ with $Q' \cap R_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ ([2, I, 3.5, Proposition 9]). We will show that $Q' \in w\text{-Spec}(T_{\mathfrak{p}})$. Since $R_{\mathfrak{p}}$ is a valuation domain, $\mathfrak{p}_{\mathfrak{p}}$ is a w-prime ideal of $R_{\mathfrak{p}}$. Then by [17, Proposition 1.1], $(Q')_w \neq (T_{\mathfrak{p}})_w = T_{\mathfrak{p}}$. Thus by [21, Proposition 1.1], we have $Q' \in w\text{-Spec}(T_{\mathfrak{p}})$. Set $Q = Q' \cap T$. Then $Q \in w\text{-Spec}(T)$, since $T \hookrightarrow T_{\mathfrak{p}}$ is t-linked, with $Q \cap R = \mathfrak{p}$. □

The following result follows immediately from Proposition 2.5 and Theorem 2.14.

Corollary 2.15. *Let T be an extension of a PvMD R. Then the following assertions are equivalent:*

(1) *T is strongly t-linked over R.*

(2) *For any GV-torsion-free semi-divisorial R-module M, we have that $M\hat{\otimes}_R T = 0$ if and only if $M = 0$.*

In particular, it follows from Corollary 2.15 that the functor $T\hat{\otimes}_R -$ is left exact on GV-torsion-free semi-divisorial R-modules.

Example 2.16. We provide examples of strongly t-linked extensions of domains.

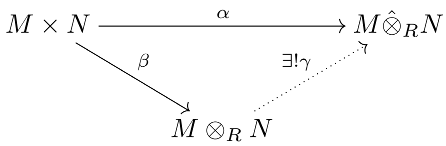
(1) Any faithfully flat extension $i : R \hookrightarrow T$ of PvMDs is strongly t-linked. Indeed, the fact that i is a t-linked extension follows from the flatness of i and it is trivial to see that for any $\mathfrak{p} \in w\text{-Spec}(R)$ the induced map $i_{\mathfrak{p}} : R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$ is faithfully flat.

(2) If $i : R \hookrightarrow T$ is an extension of PvMDs which makes T into a semi-divisorial R-lattice, then T is strongly t-linked over R. Indeed, since T is semi-divisorial over R, i is a t-linked extension. On the other hand, for every $\mathfrak{p} \in w\text{-Spec}(R)$, the $R_{\mathfrak{p}}$-module $T_{\mathfrak{p}}$ is free of finite rank, hence certainly faithfully flat.

(3) As in [18], we say that an element $u \in K$ is w-integral over R if $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of R. Set $R^w := \{x \in K \mid x \text{ is } w\text{-integral over } R\}$. It is known that R^w is an integrally closed overring of R (see [18, section 3]); R^w is called the *w-integral closure of* R. The w-integral closure R^w of an integral domain R is strongly t-linked over R. Indeed, this follows from [5, Lemma 1.2 and Corollary 1.4].

The w-tensor product $M\hat{\otimes}_R N$ of two R-modules M and N can be redefined as the image under the reflector W of their ordinary tensor product $M \otimes_R N$. If M and N are torsion-free, then $M\hat{\otimes}_R N = W_R(MN)$, where MN is the image of $M \otimes_R N$ in $K \otimes_R (M \otimes_R N) \cong (K \otimes_R M) \otimes_K (K \otimes_R N)$. The w-tensor product behaves in many ways as the ordinary tensor product of R-modules.

For (m, n) in $M \times N$ let $\alpha(m, n) = 1 \otimes m \otimes n \in K \otimes_R (M \times_R N)$. View α as a map to $M \hat{\otimes}_R N$. Let β be the map sending (m, n) to $m \otimes n$ in $M \hat{\otimes}_R N$. Because α is bilinear, there is a commutative diagram



An integral domain R is said to be *of w-finite character* if every nonzero nonunit of R belongs to at most finitely many maximal w-ideals of R. The proof of the following result is easy, and so we omit it.

Lemma 2.17. *Let S be a multiplicative set of a domain of w-finite character and M an R-module. If M is semi-divisorial over R, then M_S is semi-divisorial over R_S.*

The following proposition summarizes some basic properties of the w-tensor product.

Proposition 2.18. *Let L, M, M_i be torsion-free R-modules. Then:*

(1) *Given an R-homomorphism $f : M \to N$, there exists a unique R-homomorphism $W_R(f) : W_R(M) \to W_R(N)$ which on M restricts to f. For $g : L \to M$ we have $W_R(fg) = W_R(f)W_R(g)$.*

(2) *$M \hat{\otimes}_R N$ is semi-divisorial.*

(3) *$M \hat{\otimes}_R N$ has the universal mapping property in the following sense. If L is semi-divisorial and $\delta : M \times N \to L$ is R-bilinear, then there exists a unique R-homomorphism $\lambda : M \hat{\otimes}_R N \to L$ satisfying $\lambda\alpha = \delta$ (with α as above).*

(4) *If M and N are R-lattices, so is $M \hat{\otimes}_R N$.*

(5) *If M is R-flat, then it is semi-divisorial. If in addition R is of w-finite character and N is semi-divisorial, then the map γ above is an isomorphism.*

(6) *If B is an R-algebra and M is a B-module, then there is a B-module structure on $M \hat{\otimes}_R N$ which makes γ a B-module homomorphism.*

(7) *$R \hat{\otimes}_R M = W_R(M)$.*

(8) *$L \hat{\otimes}_R (M \hat{\otimes}_R N) \cong (L \hat{\otimes}_R M) \hat{\otimes}_R N$.*

(9) *$L \hat{\otimes}_R (\bigoplus_i M_i) \cong \bigoplus_i (L \hat{\otimes}_R M_i)$.*

(10) *$M \hat{\otimes}_R N \cong N \hat{\otimes}_R M$.*

(11) *Let S be a multiplicative subset of R, a domain of w-finite character. Then $(M \hat{\otimes}_R N)_S \cong M_S \hat{\otimes}_{R_S} N_S$.*

Proof. (2) is clear. (1) and (3) follow from the fact that W_R is a reflector functor. (4) follows from the well-known fact that if M and N are R-lattices, so is MN.

The first assertion of (5) is given in [11]. The second assertion of (5) follows from [15, Corollary 2 to Proposition 1]. (6) follows from (2) and (3). Assertions (7) to (10) are easy to prove.

To prove (11), we will establish that there are maps in both directions between $(M\hat{\otimes}_R N)_S$ and $M_S\hat{\otimes}_{R_S} N_S$ whose composites are clearly the identity maps. First note that each of the modules involved is semi-divisorial over R_S. For $(M\hat{\otimes}_R N)_S$ this is true by (2) and Lemma 2.17. For $M_S\hat{\otimes}_{R_S} N_S$ we need only invoke (2) with R replaced by R_S. The existence of the maps we want is now easily established using (3) and properties of the localizing functor $(\ )_S$. □

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