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Fekete-Szegö Problem for a Generalized Subclass of Analytic Functions

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Abstract. In this present work, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disk for which

$$\frac{\lambda\beta z^3(L(a,c)f(z))''' + (2\lambda\beta+\lambda-\beta)z^2(L(a,c)f(z))'' + z(L(a,c)f(z))'}{\lambda\beta z^2(L(a,c)f(z))'' + (\lambda-\beta)z(L(a,c)f(z))' + (1-\lambda+\beta)(L(a,c)f(z))} \quad (0 \leq \beta \leq \lambda \leq 1)$$

lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by Hadamard product (or convolution) are given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives are obtained.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. For functions $f, g \in \mathcal{A}$, given by

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$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or *convolution*) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z) \qquad (z \in \mathcal{U})\,. \tag{1.2}$$

Note that $f * g \in \mathcal{A}$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathcal{U})$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that $f(z) = g(w(z)),\ z \in \mathcal{U}$. In particular, if the function $g(z)$ is univalent in $\mathcal{U}$, then we have that $f(z) \prec g(z)$ $(z \in \mathcal{U})$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

Let

$$\varphi(a, c; z) := z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \qquad (c \neq 0, -1, -2, ...;\ z \in \mathcal{U}) \tag{1.3}$$

where $(\kappa)_n$ is the Pochhammer symbol (or the *shifted factorial*) in terms of the gamma function, given by

$$(\kappa)_n := \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} 1 & n = 0, \\ \kappa(\kappa+1)(\kappa+2)...(\kappa+n-1) & n \in \mathbb{N} := \{1, 2, ...\}. \end{cases} \tag{1.4}$$

Further, for $f \in \mathcal{A}$

$$L(a, c) f(z) = \varphi(a, c; z) * f(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} a_k z^k \tag{1.5}$$

where $L(a, c)$ is called Carlson-Shaffer operator [1] and the operator "$*$" stands for Hadamard product (or convolution product) of two power series as given by (1.2).

We notice that $L(a, a) f(z) = f(z)$, $L(2, 1) f(z) = z f'(z)$, $L(n + 1, 1) f(z) = D^n f(z)$, where $D^n f(z)$ is the Ruscheweyh derivative of $f(z)$.

Let $\phi(z)$ be an analytic function with positive real part on $\mathcal{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc $\mathcal{U}$ onto a region starlike with respect to 1 which is symmetric with to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{z f'(z)}{f(z)} \prec \phi(z) \qquad (z \in \mathcal{U})$$

and let $C(\phi)$ be the class of functions $f \in \mathcal{S}$ for which

$$1 + \frac{z f''(z)}{f'(z)} \prec \phi(z) \qquad (z \in \mathcal{U})$$

where "$\prec$" denotes the subordination between analytic functions. Above classes were defined and studied by Ma and Minda [4]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'\in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close to convex functions, see the recent papers by Srivastava *et. al.* [11], Deniz and Orhan [2], Orhan and Güneş [5], Orhan *et. al.* [7, 8], Orhan and Raducanu [6].

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk $\mathcal{U}$ onto a region in the half plane which is symmetric with respect to the real axis, $\phi(0)=1$, and $\phi'(0)>0$. A function $f\in\mathcal{A}$ is in the class $P_{\lambda,\beta}^{a,c}(\phi)$, $0\le\beta\le\lambda\le 1$ if

(1.6)
$$\frac{\lambda\beta z^3(L(a,c)f(z))'''+(2\lambda\beta+\lambda-\beta)z^2(L(a,c)f(z))''+z(L(a,c)f(z))'}{\lambda\beta z^2(L(a,c)f(z))''+(\lambda-\beta)z(L(a,c)f(z))'+(1-\lambda+\beta)(L(a,c)f(z))}\prec\phi(z).$$

If we write $D_{\lambda,\beta}L(a,c)f(z)=\lambda\beta z^2(L(a,c)f(z))''+(\lambda-\beta)z(L(a,c)f(z))'+(1-\lambda+\beta)(L(a,c)f(z))$ then $f\in P_{\lambda,\beta}^{a,c}(\phi)\Leftrightarrow D_{\lambda,\beta}L(a,c)f\in S^*(\phi)$.

Also, we have

$$D_{\lambda,\beta}L(a,c)f(z)=\Psi_{\lambda,\beta}^{a,c}(z)*f(z) \tag{1.7}$$

where

$$\Psi_{\lambda,\beta}^{a,c}(z)=z+\sum_{k=2}^{\infty}[1+(\lambda\beta k+\lambda-\beta)(k-1)]\frac{(a)_{k-1}}{(c)_{k-1}}z^k. \tag{1.8}$$

Lemma 1.1([4]). *If $p_1=1+c_1z+c_2z^2+...$ is analytic function with positive real part in $\mathcal{U}$, then*

$$\left|c_2-vc_1^2\right|\leqslant\begin{cases}-4v+2 & if \quad v\leqslant 0,\\ 2 & if \quad 0\leqslant v\leqslant 1,\\ 4v-2 & if \quad v\geqslant 1.\end{cases} \tag{1.9}$$

When $v<0$ or $v>1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0<v<1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v=0$, the equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z}\qquad(0\leqslant\gamma\leqslant 1),$$

or one of its rotations. If $v=1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v=0$. The above upper bound is sharp. When $0<v<1$, it can be improved as follows:

$$\left|c_2-vc_1^2\right|+v\left|c_1\right|^2\leqslant 2\qquad\left(0<v\leqslant\frac{1}{2}\right)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \leqslant 2 \qquad \left(\frac{1}{2} < v \leqslant 1\right).$$

2. Fekete-Szegö Problem

In this section, we will give some upper bounds for the Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$. In order to prove our main results we have to recall the following. Firstly, the following information will be used in the proof of the Theorem 2.1. By geometric interpretation there exists a function w satisfying the conditions of the Schwarz lemma such that

$$\frac{z\left(D_{\lambda,\beta}L(a,c)f(z)\right)'}{D_{\lambda,\beta}L(a,c)f(z)} = \phi\left(w(z)\right) \qquad (z \in \mathcal{U}). \tag{2.1}$$

Secondly, we introduce the following functions which will be used in the discussion of sharpness of our results.

Corresponding to the function $\Psi_{\lambda,\beta}^{a,c}(z)$ defined by (1.8), we also consider the function $\Psi_{\lambda,\beta}^{a,c}(z)^{(-1)}$ given by

$$\Psi_{\lambda,\beta}^{a,c}(z)^{(-1)} = z + \sum_{k=2}^{\infty} \frac{(c)_{k-1}}{(a)_{k-1}[1+(\lambda\beta k + \lambda - \beta)(k-1)]} z^k \tag{2.2}$$

where inverse is taken with respect to Hadamard product.

Using (1.7), (2.1) and logarithmic differentiation it can be deduce that $f \in P_{\lambda,\beta}^{a,c}(\phi)$ if and only if

$$f(z) = \Psi_{\lambda,\beta}^{a,c}(z)^{(-1)} * \left\{ z \exp\left(\int_0^z \frac{\phi(w(t)) - 1}{t} dt \right) \right\}$$

for some function $w(z)$ satisfying the conditions of the Schwarz Lemma.

Define the function G in $\mathcal{U}$ by

$$G(z) = \frac{1}{z}\left[\Psi_{\lambda,\beta}^{a,c}(z)^{(-1)} * \left\{ z \exp\left(\int_0^z \frac{\phi(\xi) - 1}{\xi} d\xi \right) \right\} \right]. \tag{2.3}$$

Also we consider the following extremal function
(2.4)

$$K(z,\theta,\tau) = \Psi_{\lambda,\beta}^{a,c}(z)^{(-1)} * z \exp\left(\int_0^z \left[\phi\left(\frac{e^{i\theta}\xi(\xi+\tau)}{1+\tau\xi} \right) - 1 \right] \frac{d\xi}{\xi} \right) \qquad (0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \tau \leqslant 1).$$

Note that $K(z,0,1) = zG(z)$ defined by (2.3) and $K(z,\theta,0)$ is an odd function.

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *If* $f(z)$ *given by* (1.1) *belongs to the class* $P^{a,c}_{\lambda,\beta}(\phi)$, *then*

$$(2.5)\quad \left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{c(c+1)B_2}{2a(a+1)A_2} - \frac{\mu c^2 B_1^2}{a^2 A_1^2} + \frac{c(c+1)B_1^2}{2a(a+1)A_2} & if \quad \mu \leqslant \sigma_1, \\ \frac{c(c+1)B_1}{2a(a+1)A_2} & if \quad \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ -\frac{c(c+1)B_2}{2a(a+1)A_2} + \frac{\mu c^2 B_1^2}{a^2 A_1^2} - \frac{c(c+1)B_1^2}{2a(a+1)A_2} & if \quad \mu \geqslant \sigma_2, \end{cases}$$

where

$$\sigma_1 \ : \ = \frac{a(c+1)A_1^2\left\{(B_2 - B_1) + B_1^2\right\}}{2c(a+1)A_2 B_1^2},$$
$$\sigma_2 \ : \ = \frac{a(c+1)A_1^2\left\{(B_2 + B_1) + B_1^2\right\}}{2c(a+1)A_2 B_1^2}$$

and

$$A_1 = (2\lambda\beta + \lambda - \beta + 1), \quad A_2 = (6\lambda\beta + 2\lambda - 2\beta + 1) \qquad (0 \le \beta \le \lambda \le 1).$$

Each of the estimates in (2.5) *is sharp for the function* $K(z, \theta, \tau)$ *given by* (2.4).

Proof. For $f(z) \in P^{a,c}_{\lambda,\beta}(\phi)$, let

$$(2.6)\quad \begin{aligned} p(z) &= \frac{\lambda\beta z^3 (L(a,c)f(z))''' + (2\lambda\beta + \lambda - \beta) z^2 (L(a,c)f(z))'' + z(L(a,c)f(z))'}{\lambda\beta z^2 (L(a,c)f(z))'' + (\lambda - \beta) z (L(a,c)f(z))' + (1 - \lambda + \beta)(L(a,c)f(z))} \\ &= 1 + b_1 z + b_2 z^2 + ... \end{aligned}$$

From (2.6), we obtain

$$\frac{a}{c} A_1 a_2 = b_1 \quad \text{and} \quad \frac{2a(a+1)}{c(c+1)} A_2 a_3 = A_1^2 \frac{a^2}{c^2} a_2^2 + b_2.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + ...$$

is analytic and has positive real part in $\mathcal{U}$. We also have

$$(2.7)\qquad p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right)$$

and thus, we get

$$b_1 = \frac{1}{2} B_1 c_1 \text{ and } b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2.$$

Hence, we have

$$a_3 - \mu a_2^2 = \frac{c(c+1)B_1}{4a(a+1)A_2}\left\{c_2 - vc_1^2\right\},$$

where

$$v := \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2ac(a+1)A_2\mu - a^2(c+1)A_1^2}{a^2(c+1)A_1^2}B_1\right).$$

If $\mu \leqslant \sigma_1$, then, according to Lemma 1.1, we get

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &= \frac{c(c+1)B_1}{4a(a+1)A_2}\left|c_2 - c_1^2\left[\frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2ac(a+1)A_2\mu - a^2(c+1)A_1^2}{a^2(c+1)A_1^2}B_1\right)\right]\right| \end{aligned} \tag{2.8}$$

and thus,

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{c(c+1)B_2}{2a(a+1)A_2} - \frac{\mu c^2 B_1^2}{a^2 A_1^2} + \frac{c(c+1)B_1^2}{2a(a+1)A_2},$$

which is the first assertion of (2.5).

Next, if $\mu \geqslant \sigma_2$, by applying Lemma 1.1, we get

$$\left|a_3 - \mu a_2^2\right| \leqslant -\frac{c(c+1)B_2}{2a(a+1)A_2} + \frac{\mu c^2 B_1^2}{a^2 A_1^2} - \frac{c(c+1)B_1^2}{2a(a+1)A_2}$$

which is the third assertion of (2.5).

If $\sigma_1 \leqslant \mu \leqslant \sigma_2$, by using again Lemma 1.1, we obtain

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{c(c+1)B_1}{2a(a+1)A_2}$$

which is the second part of the assertion (2.5).

We now obtain sharpness of the estimates in (2.5). If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds in (2.5) if and only if equality holds in (2.8). This happens if and only if $c_1 = 2$ and $c_2 = 2$. Thus $w(z) = z$. It follows that the extremal function is of the form $K(z,0,1)$ defined by (2.4) or one of its rotations.

If $\mu = \sigma_2$, the equality holds if and only if $|c_2| = 2$. In this case, we have

$$\frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2ac(a+1)A_2\mu - a^2(c+1)A_1^2}{a^2(c+1)A_1^2}B_1\right) = 0.$$

Therefore the extremal function f is $K(z,\theta,\tau)$ or one of its rotations.

Similarly, $\mu = \sigma_1$ is equivalent to

$$p_1(z) = \frac{1+\tau}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-\tau}{2}\left(\frac{1-z}{1+z}\right) \quad (0 < \tau < 1;\ z \in \mathfrak{U}).$$

Thus the extremal function is $K(z,0,\tau)$ or one of its rotations.

Finally if $\sigma_1 \leqslant \mu \leqslant \sigma_2$, then equality holds if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$h(z) = \frac{1+\tau z^2}{1-\tau z^2} \quad (0 \leqslant \tau \leqslant 1;\ z \in \mathcal{U}).$$

Therefore the extremal function f is $K(z, 0, 0)$ or one of its rotations. The proof of Theorem 2.1 is now completed. □

Remark 2.2. If $\sigma_1 \leqslant \mu \leqslant \sigma_2$, then, in view of Lemma 1.1, Theorem 2.1 can be improved. Let σ_3 given by

$$\sigma_3 := \frac{a(c+1)A_1^2\left\{B_1^2 + B_2\right\}}{2c(a+1)A_2B_1^2}.$$

If $\sigma_1 \leqslant \mu \leqslant \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{a(c+1)A_1^2}{2c(a+1)A_2B_1^2}\left[B_1 - B_2 + \frac{2ac(a+1)A_2\mu - a^2(c+1)A_1^2}{a^2(c+1)A_1^2}B_1^2\right]|a_2|^2 \leqslant \frac{c(c+1)B_1}{2a(a+1)A_2}.$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{a(c+1)A_1^2}{2c(a+1)A_2B_1^2}\left[B_1 + B_2 - \frac{2ac(a+1)A_2\mu - a^2(c+1)A_1^2}{a^2(c+1)A_1^2}B_1^2\right]|a_2|^2 \leqslant \frac{c(c+1)B_1}{2a(a+1)A_2}$$

where A_1 and A_2 are given by Theorem 2.1.

Proof. If $\sigma_1 \leqslant \mu \leqslant \sigma_3$, we have

$$\begin{aligned}
&\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 \\
= \;& \frac{c(c+1)B_1}{4a(a+1)A_2}\left|c_2 - vc_1^2\right| + (\mu - \sigma_1)\frac{c^2B_1^2}{4a^2A_1^2}|c_1|^2 \\
= \;& \frac{c(c+1)B_1}{4a(a+1)A_2}\left|c_2 - vc_1^2\right| + \left(\mu - \frac{a(c+1)A_1^2\left\{(B_2 - B_1) + B_1^2\right\}}{2c(a+1)A_2B_1^2}\right)\frac{c^2B_1^2}{4a^2A_1^2}|c_1|^2 \\
= \;& \frac{c(c+1)B_1}{2a(a+1)A_2}\left[\frac{1}{2}\left|c_2 - vc_1^2\right|\right. \\
& \left. + \frac{1}{2}\left(\frac{2\mu c(a+1)A_2B_1^2 - a(c+1)A_1^2[B_1^2 - B_1 + B_2]}{2a(c+1)A_2^2B_1}\right)|c_1|^2\right] \\
= \;& \frac{c(c+1)B_1}{2a(a+1)A_2}\left\{\frac{1}{2}\left[\left|c_2 - vc_1^2\right| + v|c_1|^2\right]\right\} \leqslant \frac{c(c+1)B_1}{2a(a+1)A_2}.
\end{aligned}$$

Similarly, if $\sigma_3 \leqslant \mu \leqslant \sigma_2$, we can write the following

$$\begin{aligned}
&\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2 \\
=\;& \frac{c(c+1)B_1}{4a(a+1)A_2}\left|c_2-vc_1^2\right|+(\sigma_2-\mu)\frac{c^2B_1^2}{4a^2A_1^2}\left|c_1\right|^2 \\
=\;& \frac{c(c+1)B_1}{4a(a+1)A_2}\left|c_2-vc_1^2\right|+\left(\frac{a(c+1)A_1^2[B_1^2+B_1+B_2]}{2c(a+1)A_2B_1^2}-\mu\right)\frac{c^2B_1^2}{4a^2A_1^2}\left|c_1\right|^2 \\
=\;& \frac{c(c+1)B_1}{2a(a+1)A_2}\left[\frac{1}{2}\left|c_2-vc_1^2\right|\right. \\
&\left.+\frac{1}{2}\left(\frac{a(c+1)A_1^2[B_1^2+B_1+B_2]-2\mu c(a+1)A_2B_1^2}{2a(c+1)A_2^2B_1}\right)\left|c_1\right|^2\right] \\
=\;& \frac{c(c+1)B_1}{2a(a+1)A_2}\left\{\frac{1}{2}\left[\left|c_2-vc_1^2\right|+(1-v)\left|c_1\right|^2\right]\right\}\leqslant\frac{c(c+1)B_1}{2a(a+1)A_2}.
\end{aligned}$$

Thus, the proof of Remark 2.2 is completed. □

3. Applications to functions defined by fractional derivatives

For fixed $g \in A$, let $P_{\lambda,\beta}^{a,c,g}(\phi)$ be class of functions $f \in A$ for which $(f * g) \in P_{\lambda,\beta}^{a,c}(\phi)$. In order to introduce the class $P_{\lambda,\beta}^{a,c,g}(\phi)$, we need the following:

Definition 3.1([9]). Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order γ is defined by

$$D_z^\gamma f(z)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^\gamma}d\zeta \qquad (0\leqslant\gamma<1)$$

where the multiplicity of $(z-\zeta)^\gamma$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta>0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [9] introduced the operator $\Omega^\gamma : A \to A$ defined by

$$\Omega^\gamma f(z)=\Gamma(2-\gamma)z^\gamma D_z^\gamma f(z) \qquad (\gamma\neq 2,3,4,...).$$

The class $P_{\lambda,\beta}^{a,c,\gamma}(\phi)$ consists of functions $f \in A$ for which $\Omega^\gamma f \in P_{\lambda,\beta}^{a,c}(\phi)$. Note that $P_{0,0}^{a,a}(\phi) \equiv S^*(\phi)$ and $P_{\lambda,\beta}^{a,c,\gamma}(\phi)$ is the special case of the class $P_{\lambda,\beta}^{a,c,g}(\phi)$ when

$$g(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}z^k.$$

Let

$$g(z)=z+\sum_{k=2}^{\infty}g_kz^k \qquad (g_k>0).$$

Since $L(a,c)f(z) \in P_{\lambda,\beta}^{a,c,g}(\phi)$ if and only if $L(a,c)f(z) * g(z) \in P_{\lambda,\beta}^{a,c}(\phi)$, we obtain the coefficient estimate for functions in the class $P_{\lambda,\beta}^{a,c,g}(\phi)$, from the corresponding estimate for functions in the class $P_{\lambda,\beta}^{a,c}(\phi)$.

Applying Theorem 2.1 for the function

$$\begin{aligned} L(a,c)f(z) * g(z) &= z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} a_k g_k z^k \\ &= z + \frac{a}{c} a_2 g_2 z^2 + ... \end{aligned}$$

We get the following Theorem 3.1 after an obvious change of the parameter μ :

Theorem 3.1. *Let* $g(z) = z + \sum\limits_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$, *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum\limits_{n=1}^{\infty} B_n z^n$. *If* $L(a,c)f(z)$ *defined by* (1.5) *belongs to the class* $P_{\lambda,\beta}^{a,c,g}(\phi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{g_3}\left[\frac{c(c+1)B_2}{2a(a+1)A_2} - \frac{\mu g_3 c^2 B_1^2}{g_2^2 a^2 A_1^2} + \frac{c(c+1)B_1^2}{2a(a+1)A_2}\right] & if \quad \mu \leqslant \sigma_1^*, \\ \frac{1}{g_3}\left[\frac{c(c+1)B_1}{2a(a+1)A_2}\right] & if \quad \sigma_1^* \leqslant \mu \leqslant \sigma_2^*, \\ \frac{1}{g_3}\left[-\frac{c(c+1)B_2}{2a(a+1)A_2} + \frac{\mu g_3 c^2 B_1^2}{g_2^2 a^2 A_1^2} - \frac{c(c+1)B_1^2}{2a(a+1)A_2}\right] & if \quad \mu \geqslant \sigma_2^*, \end{cases}$$

where

$$\begin{aligned} \sigma_1^* &:= \frac{g_2^2 a(c+1)A_1^2\left\{(B_2 - B_1) + B_1^2\right\}}{2g_3 c(a+1)A_2 B_1^2}, \\ \sigma_2^* &:= \frac{g_2^2 a(c+1)A_1^2\left\{(B_2 + B_1) + B_1^2\right\}}{2g_3 c(a+1)A_2 B_1^2}. \end{aligned}$$

The result is sharp.

Since

$$(\Omega^\gamma L(a,c)f)(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)(a)_{k-1}}{\Gamma(k+1-\gamma)(c)_{k-1}} a_k z^k,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}.$$

For g_2 and g_3 given by above equalities, Theorem 3.1 reduces to the following:

Theorem 3.2. *Let* $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. *If* $L(a,c)f(z)$ *defined by* (1.5) *belongs to the class* $P_{\lambda,\beta}^{a,c,g}(\phi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{c(c+1)B_2}{2a(a+1)A_2} - \frac{3(2-\gamma)\mu c^2 B_1^2}{2(3-\gamma)a^2 A_1^2} + \frac{c(c+1)B_1^2}{2a(a+1)A_2}\right] & if \quad \eta \leqslant \sigma_1^{**}, \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{c(c+1)B_1}{2a(a+1)A_2}\right] & if \quad \sigma_1^{**} \leqslant \eta \leqslant \sigma_2^{**}, \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{c(c+1)B_2}{2a(a+1)A_2} + \frac{3(2-\gamma)\mu c^2 B_1^2}{2(3-\gamma)a^2 A_1^2} - \frac{c(c+1)B_1^2}{2a(a+1)A_2}\right] & if \quad \eta \geqslant \sigma_2^{**}, \end{cases}$$

where

$$\sigma_1^{**} := \frac{(3-\gamma)a(c+1)A_1^2\left\{(B_2 - B_1) + B_1^2\right\}}{3(2-\gamma)c(a+1)A_2 B_1^2},$$
$$\sigma_2^{**} := \frac{(3-\gamma)a(c+1)A_1^2\left\{(B_2 + B_1) + B_1^2\right\}}{3(2-\gamma)c(a+1)A_2 B_1^2}.$$

The result is sharp.

Remark 3.3. When $a = c$, $\lambda = \beta = 0$, $B_1 = 8/\pi^2$ and $B_2 = 16/3\pi^2$, Theorem 3.2 reduces to a result of Srivastava and Mishra ([11], Theorem 8, p. 64) for a class of which $\Omega^\gamma f(z)$ is a parabolic starlike function (see [3], [10]).

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