Observer design with Gershgorin's disc

Chen Si, Yujia Zhai
Department of Electrical and Electronic Engineering, Xi'an Jiaotong-Liverpool University, Suahou, China

Abstract Observer design for system with unknown input was carried out. First, Kalman filter was considered to estimate system state with White noise. With the results of Kalman filter design, state observer, controller properties, including controllability and observability, and the Kalman filter structure and algorithm were also studied. Kalman filter algorithm was applied to Position and velocity measurement based on Kalman filter with white noise, and it was constructed and achieved by programming based on Matlab programming. Finally, observer for system with unknown input was constructed with the help of Gershgorin's disc theorem. With the designed observer, system states was constructed and applied to system with unknown input. By simulation results, estimation performance was verified. In this project, state feedback control theory, observer theory and relevant design procedure, as well as Kalman filter design were understood and used in practical application.

• Key Words : estimation; observer; linear system; linear minimum variance estimation; Kalman filter; Gershigorim's disc theorem; unknown input

1. Introduction

State feedback control provides better performance than unit feedback controller. Observer design has been studied to provide good performanceby way of state feedback [1]. Practically, when systems are affected by external disturbance or unknown inputs, more considerate approach is needed to design observer. To overcome the drawback because of unknown input, observer design method with unknown inputs was considered. In order to design state feedback controller, state observer is needed. In modern control system, observer design is a quite popular and fundamental concept. Since the state variable, which is with the function of describing the features of the dynamic behavior of the inner system cannot be measured directly and sometimes it has no actual physical meaning and just an abstract mathematically variable, the idea about state observer was put forward along with the desire of measuring it. The observer design has wide applications in field of flight control system, weather forecast, etc. One approach of constructing these observers is based on Kalman filter.However, lots of systems contain external disturbance or measurement noise, hence it make difficult to design observer. Hence, designing of observer for system with unknown inputs is very important for system with unknown input. Researches on observer design for system with unknown input have been reported by numerous researchers [2]. In this literature, we emphasized on designingan observer with unknown input for linear systems. The specific component for constructing anobserver is chosen to be Kalman filter. The design procedure mainly experienced two stages, firstly, constructing a normal Kalman filter model. Secondly, adding the unknown input into the model to construct an observer. In the following chapter, state

*교신저자: Zhai, Y.(Xi'an Jiaotong-Liverpool University)
접수일 2013년 10월 28일　수정일 2013년 11월 22일　게재확정일 2013년 11월 25일

estimation and observer theory will be briefly introduced. Then the controller properties will be presented, which can indicate the observer characters. Following that the detailed deduction of linear minimum variance estimation and its application into Kalman filter were proposed. Next five basic formulas for Kalman filter algorithm of discrete linear system was introduced, as well as the Kalman filter control system structure. A practical model for predicting the velocity and position was constructed by programming based on the algorithm of Kalman filter, firstly without unknown input and then adding it. Also, discussions about this project and the expected future work are illustrated. Finally, conclusions are followed. In the next chapter, observer design with unknown input was considered. In Chapter 3, two dimensional moving element was considered as an illustrative example. Finally, conclusions are followed in Chapter 4.

2. Observer for system with unknown input

2.1 System with unknown input

Observer design for linear and nonlinear system has been carried out by numerous researchers [2]. In general, for linear systems, the dynamic behavior, which is the "Inaccessible system state x", can be described by the continuous-time finite-dimensional linear system model, which is shown by equation (1) and (2).

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dw(t) & (1) \\ y(t) = Cx(t) & (2) \end{cases}$$

The system matrixes A, B and C are all known and in this model, they are time invariant.

Kalman filter, which is also known as linear quadratic estimation (LQE), is an algorithm that uses a series of measurements observed over time, containing noise (random variations) and other inaccuracies, and produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone. The basic thought involved in Kalman filter is to utilize the state space model of the signal and noise, plus the estimated value of previous moment and the observed value at present moment to obtain the optimal estimated value of the present moment. The estimation algorithm applied in the Kalman filter is linear minimum variance estimation. Kalman filter computes the mean and the covariance matrix recursively for a linear system [1]. Generally, Kalman filtering follows a fixed operation process, that is 'forecast－actual measurement－amendment'. And Kalman filter owns three fundamental properties, which are the average value of estimated state value equals to the actual one;the variance is minimum. In one cycle of the recursion of Kalman filtering algorithm, there are time updating (error calibration) and state updating (prediction of state variable). All the properties that appear in the algorithm of Kalman filter will be revealed by the five basic formulas below.

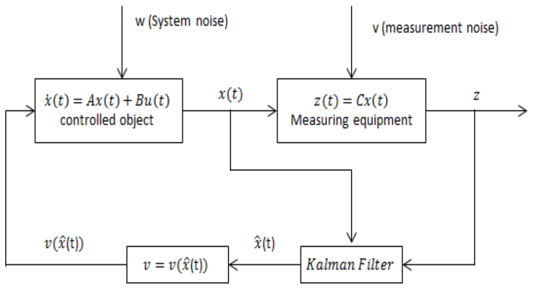
Firstly, a system of discrete control process should be introduced and can be expressed by a linear stochastic difference equation (3) and (4). For constructing the system, conversion process of physic system can be described as a discrete-time stochastic process was assumed [2].

$$\begin{cases} X(k) = AX(k-1) + BU(k) + W(k) & (3) \\ Z(k) = HX(k) + V(k) & (4) \end{cases}$$

where $X(k) \in R^n, Z(k) \in R^n$, $A \in R^{nXn}, BH$, and $X(k)$, $U(k)$, $Z(k)$ are system state, measured value, $W(k)$ and $V(k)$ are noise of process and measurement. In this study, $W(k)$ and $V(k)$ are assumed to be white Gaussian noise and are mutual independent. The covariance of them are set to be Q and R, respectively, in order to simplify the filtering process, Q and R are supposed to be constant.

$$W(k) \sim N(0, Q)$$
$$V(k) \sim N(0, R)$$

Fig. 1 displays the structure of the Kalman filter control system. Form the figure, it can be easily seen that Kalman filter utilizes system state and system output to produce the estimated value of system state.



[Fig. 1] Kalman filter control system structure

After introducing the basic structure of Kalman filter, the specific algorithm for it should also be known. There are five basic formulas involved, based on them, it is easy to achieve computer compiling.

Firstly, the two time update equations are illustrated by formulas (5) and (6).

$$X(k|k-1) = AX(k-1|k-1) + BU(k) \quad (5)$$

$X(k|k-1)$: predicting result according to state at k-1

$X(k-1|k-1)$: optimal result of state at k-1 Based on formula (5), the system outcome has already been updated, however, the covariance of $X(k|k-1)$ has not been renewed, and we represent it as P.

$$P(k|k-1) = AP(k-1|k-1)A' + Q \quad (6)$$

$P(k|k-1)$: covariance of $X(k|k-1)$

$P(k-1|k-1)$: covariance of $X(k-1|k-1)$

A': transposed matrix of matrix A

Then, itcomes the measurement update equations.

Since that the predicted value of present state has been obtained, combined with the measured value, the optimal value of present state can be achieved.

$$X(k|k) = X(k|k-1) + Kg(k)\big(Z(k) - HX(k|k-1)\big) \quad (7)$$

In the formula, Kg is named Kalman Gain.

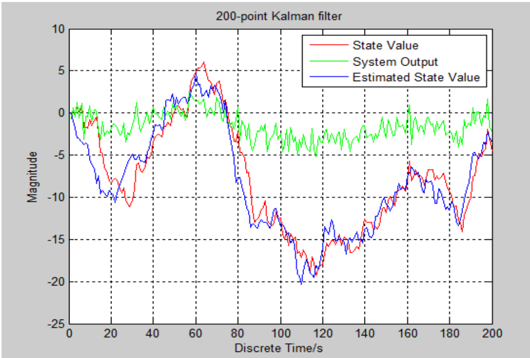
$$Kg(k) = \frac{P(k|k-1)H'}{HP(k|k-1)H' + R} \quad (8)$$

Although the optimal estimated value has been obtained, it is necessary to renew the covariance of $X(k|k)$for ensuring the process of Kalman filtering to continue predicting the optimal state value for next moment.

$$P(k|k) = (I - Kg(k)H)P(k|k-1) \quad (9)$$

The five equations above commendably illustrate the algorithm for the Kalman filtering.

Based on the five equations above, a simple Kalman filter can be built by programming. In this filter, 200 points has been selected. Fig.2 below displays the running outcome.



[Fig. 2] state value, estimated state value and output of 200-point Kalman filter

In Fig. 2, the green line stands for system output, while red and blue lines represent the actual state value and the estimated state value respectively. When comparing the red and blue lines, at the very beginning, approximate the first 20 points, difference between them is relative big. However, as the point number increases, these two lines resemble in the trace, which indicate that this Kalman filter is of sense.

2.2 Construction of Gerschgorin disc

For all systems, there are always some external and internal disturbances. For simplification, all these disturbances can be considered as an unknown input. Then another system was designed, which applies the output of the system as the feedback. And the dynamic model for the designed observer can be obtained, which is shown in below.

$$\dot{w} = Fw + Gy + Hu \qquad (10)$$

In the equation, $F = \begin{bmatrix} A & D_d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix}$, $G = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ and $H = \begin{bmatrix} B \\ 0 \end{bmatrix}$.

Since that in the observer parameters, A, D_d, C and B are all known, so to determine matrix F, just the value of L_1 and L_2 are needed to be selected.

$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ should be selected to ensure that matrix F can be arbitrarily assigned stable eigenvalues, which is equivalent to that all the eigenvalues should be located in the left-half plane.

In our model, matrix $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ should be of size 5×2. We assume that:

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{bmatrix}$$

Letters a to j all represents for real numbers.

Thus, matrix F can be expressed as:

$$F = \begin{bmatrix} A & D_d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} -a & 1 & -b & 0 & 1 \\ -c & 0 & -d & 0 & 1 \\ -e & 0 & -f & 1 & 1 \\ -g & 0 & -h & 0 & 1 \\ -i & 0 & -j & 0 & 0 \end{bmatrix}$$

Matrix F is of size 5×5, which is complicated for computing all the eigenvalues, another theorem called Gerschgorin disc theorem is introduced for convenient calculation of eigenvalues.

Definition 1 Assume a matrix $K \in R^{n \times n}$, for every row of the matrix:

$$S_i = \{r \in R: |r - k_{ii}| \le Q_i, Q_i = \sum_{j \ne i} |a_{ij}|\}$$

If the condition is satisfied, the matrix is said to hold stable eigenvalues.

According to the theorem, the following five constraints can be obtained:

$S_1: |r + a| \le 2 - b$

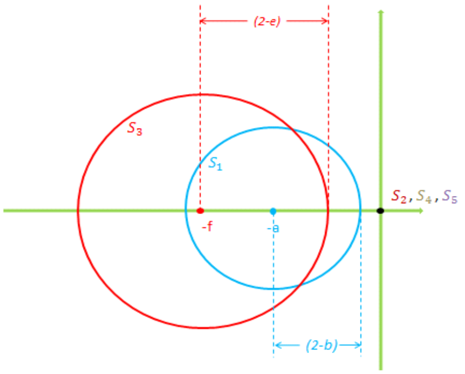
$S_2: |r| \le 1 - c - d$

$S_3: |r + f| \le 2 - e$

$S_4: |r| \le 1 - g - h$

$S_5: |r| \le -i - j$

The corresponding graph for satisfying these constraints can be drawn (Fig. 3).



[Fig. 3] Gerschgorin discs for stable system

To ensure the stability of the system, just need to ensure that all the disk area is located on the left half plane. Therefore, this set of constraints can be expressed as:

$$\begin{cases} -a < 0 \\ 0 < 2 - b < a \\ 1 - c - d = 0 \\ 1 - g - h = 0 \\ -i - j = 0 \\ -f < 0 \\ 0 < 2 - e < f \end{cases}$$

So based on these, the value of a to j can be selected and there are multiple options. Then matrix F can be determined.

One of the possible options is:

$$\begin{cases} a = 2 \\ b = 1 \\ c = 3 \\ d = -2 \\ e = 1 \\ f = 5 \\ g = 4 \\ h = -3 \\ i = 6 \\ j = -6 \end{cases}$$

Therefore, matrix $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ can be obtained:

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 5 \\ 4 & -3 \\ 6 & -6 \end{bmatrix}$$

In this model, since that as the object travels on the road, there are some unexpected disturbances affecting the motion. These disturbances can be wind, surface roughness of road and bad weather, such as snow and rain. Since these disturbances are difficult to determine, we consider all there unknown disturbances as an unknown input to the system for simplification.

So if the system is introduced with an unknown input, then linear system state equations can be described as:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_d u_d(t) & (11) \\ y(t) = Cx(t) & (12) \end{cases}$$

In this project, there are two types of unknown input considered. The first type is the square root of the system state and the second type is the output of another dynamic system.

3. Illustrative Examples

3.1 Output of dynamic system as unknown input

In dealing with the unknown input, we consider it as an output of a dynamic system, which can be expressed as [6]:

$$\begin{cases} u_d(t) = h & (13) \\ \dot{h} = 0 & (14) \end{cases}$$

The original assumption is that the unknown input is the linear combination of unit step and unit impulse function. The reason is that these two functions are commonly used in control systems. And the dynamic system above gives a satisfactory approximation of this assumption.

Based on the newly constructed system and the previous one which is without unknown input, another system can be developed by combing these two.

$$\begin{bmatrix} \dot{x} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} A & D_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix}$$

By assuming that $q = \begin{bmatrix} x \\ h \end{bmatrix}$, $F = \begin{bmatrix} A & D_d \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} C & 0 \end{bmatrix}$.

A new system can be established:

$$\begin{cases} \dot{q} = Fh + Gu & (15) \\ z = Qh & (16) \end{cases}$$

Taken this model into the practical constant model, the position and velocity on X and Y axis, which make up a matrix in the form of $\begin{bmatrix} position\ on\ X\ axis \\ velocity\ on\ X\ axis \\ position\ on\ Y\ axis \\ velocity\ on\ Y\ axis \end{bmatrix}$ will be extended to $\begin{bmatrix} position\ on\ X\ axis \\ velocity\ on\ X\ axis \\ position\ on\ Y\ axis \\ velocity\ on\ Y\ axis \\ unknown\ input \end{bmatrix}$.

From the new matrix, it seems that the unknown input has been added to the system; however, it is represented as an independent variable. As a result, the unknown input does not influence the motion. To modify the Matlab code for computing the newly designed system with unknown input, it can be noticed that there is no change in the motion of the object, both the position and the velocity.

This is a position and velocity measurement design and simulation is achieved based on Kalman Filter [3] [4]

The initial condition of the system is set as follow:

- Sampling time is one second
- Simulated period is 101 seconds
- Study objectives are position and velocity on X and Y axis, which make up a matrix in the form of $\begin{bmatrix} position\ on\ X\ axis \\ velocity\ on\ X\ axis \\ position\ on\ Y\ axis \\ velocity\ on\ Y\ axis \end{bmatrix}$

- Process and measurement noise are white noise
- No control quantity－no external input
- 2-dimensional constant velocity system

Value assignment at the beginning time is listed below:

The position on X-axis, velocity on X-axis, position on Y-axis and velocity on Y-axis at starting time (t=0) is set to be:

$x = [0 \quad 40 \quad 0 \quad 20]'$

And at t=0, the estimated values are assumed:

$x\_hat = [0 \quad 0 \quad 0 \quad 0]'$

The matrix parameters for the continuous - time state space model are:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Dn(t) \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Firstly, the controllability and observability should be examined to ensure that the system is of sense.

For the controllability:

$$S_c = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$S_c = 4 = full\ rank$

Therefore, this system is controllable.

For the observability:

$$S_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

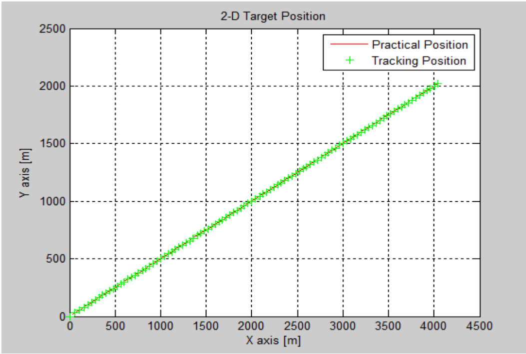
$S_o = 4 = full\ rank$

Therefore, this system is observable.

After checking out the controllability and the observability, the simulation process can be carried out.

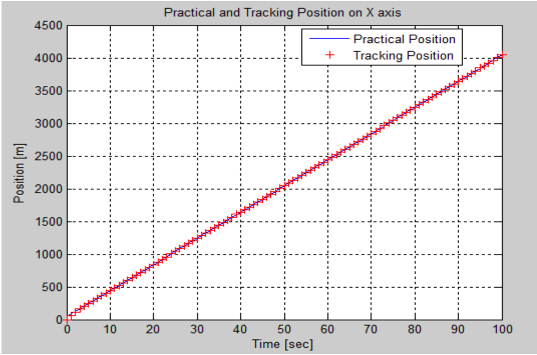
3.2 Simulation Results

The simulation process is achieved based on Matlab. The simulation results contain five figures, indicating the 2-dimensional practical and tracking position, the position error, the practical and tracking velocity of X-axis and the velocity error respectively [5].For Fig. 4, straight line represents the practical position, while the green '+'symbol indicates the predicted one and these two are nearly anastomotic, which proves that the Kalman filter model does a commendable prediction.



[Fig. 4] 2-D practical and tracing positions

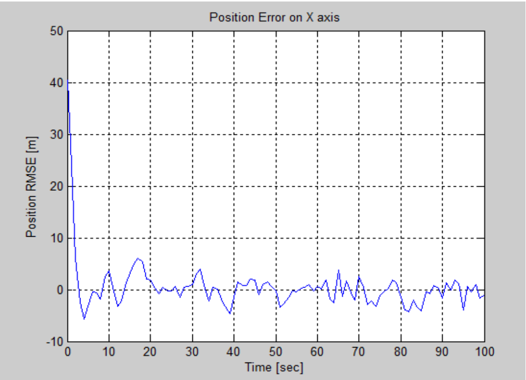
In order to simplify the analysis of the model, just X-axis has been studied to transfer 2-dimension problem into 1-dimension, since that both the situation of position and velocity on X-axis and Y-axis are similar.



[Fig. 5] Practical and tracking position on X- axis

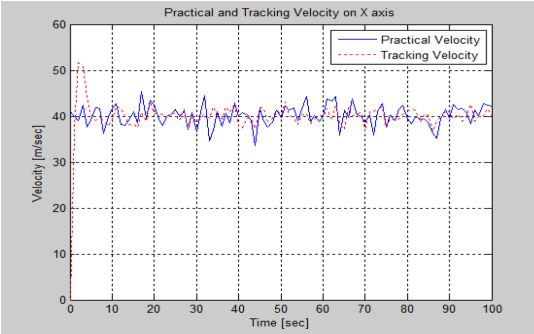
From Fig.5, for position on X-axis, practical and predicted ones are almost the same. Based on Fig.4 and 5, it can be deduced that practical and predicted position on Y-axis should also just have tiny difference.

Since that from the Fig. 4 and 5, the difference between the practical and predicted data is too tiny to identify, the error between these two are plotted directly for clear recognition, which is shown in Fig.6. At the beginning of the measurement, the difference is relative large, astime continues, the error is controlled in the range of approximate ±5m, which can be regarded as a reliable prediction. The gradually reduced difference between the real and the predicted ones is owing to the superimposed operation of the Kalman filter algorithm.

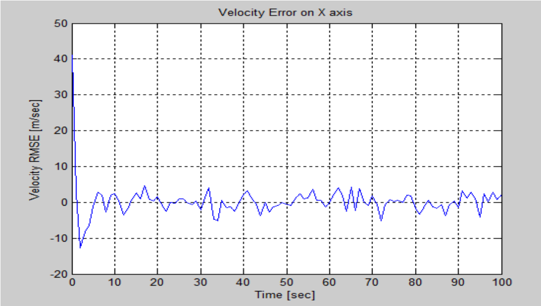


[Fig. 6] Position errors on X-axis

Apart from position, the situation about the velocity is also plotted, which are shown in Fig.7 and8 respectively, from Fig. 7, it can be seen that both the actual and the predicted velocity fluctuate around approximate 40m/sec. And the deviation is about 3m/sec at the steady state, which is acceptable. When noticing Fig. 6, at time equal zero, the tracking velocity is zero, which is caused by the initial condition setting rather than the system error.



[Fig. 7] Practical and tracking velocity on X-axis



[Fig. 8] Velocity errors on X-axis

4. Conclusions

State estimation for discrete-time linear system was proposed. State estimation could be achieved by multiple methods, such as Kalman filter and Luenbergerobserver. Kalman filter guarantees small account of calculation, low storage and high instantaneity. It can be applied for the discrete-time linear system and the algorithm for it is linear minimum variance estimation.Definition and deduction for linear minimum variance estimation was also proposed. It is an estimation method that could force the variance to be the minimum. It is linear and unbiased estimation. Under the condition that the joint probability density of the estimated state and the observed value are normally distributed, the value for minimum variance of estimated state is equivalent to the linear minimum variance estimation.Stability, controllability and observability for linear system are presented and applied to the practical model. For the design procedure, during the first stage, Kalman filter was applied to the prediction of position and velocity of an object by Matlab programming. It was proposed in terms of the track of practical and predicted position, the error between them, as well as the velocity. For the second stage, two types of unknown input are added, which are square root of system state and output of a dynamic system respectively. For square root of system state, it is of function of changing the uniform motion into accelerated one, while for output of a dynamic system, it represents the linear combination of two commonly used function, namely unit step and unit impulse function.

REFERENCES

[1] Charandabi, B.A.; Marquez, H.J., "Observer design for discrete-time linear systems with unknown disturbances," Decision and Control (CDC), 2012 IEEE 51st Annual Conference on, Vol. 10, No. 13, pp. 2563-2568, 2012.

[2] G.Welch and G.Bishop. An introduction to the Kalman filter. Technical Report 95-101, University of North Carolina, Department of Computer Science, 1995.

[3] Hiramatsu, T.; Ogawa, T.; Haseyama, M., "A Kalman filter-based approach for adaptive restoration of in-vehicle camera foggy images," Image Processing, 2008. ICIP 2008. 15th IEEE International Conference on, Vol. 12, No. 15, pp. 3160-3163, 2008.

[4] Jong-Yun Kim; Tae-Yong Kim, "Soccer Ball Tracking Using Dynamic Kalman Filter with Velocity Control," Computer Graphics, Imaging and Visualization, 2009. CGIV '09. Sixth International Conference on, Vol. 11, No. 14, pp. 367-374, 2009.

[5] Wei Wu; Wei Min, "The Mobile Robot GPS Position Based on Neural Network Adaptive Kalman Filter," Computational Intelligence and Natural Computing, 2009. CINC '09. International Conference, Vol. 1, pp. 26-29, 2009.

[6] Alexandridis, A.T.; Galanos, G. D., "Design of an optimal current regulator for weak AC/DC systems using Kalman filtering in the presence of unknown inputs," Generation, Transmission and Distribution, IEE Proceedings C, Vol. 136, No. 2, pp. 57-63, 1989.

저자소개

Si Chen

• 2013.Department of Electrical Engineering,, Xi'an Jiaotong-Liverpool University (Bachelor of Eng.)

Yujia Zhai

• 2001. Department of Electrical Engineering, Changchun University (Bachelor of Eng.)
• 2004. Department of Electrical Engineering and Electronics, University of Liverpool (UoL) (Master in Information and Intelligence Engineering,)
• 2009. Liverpool John Moores University (LJMU) (Ph.D. in Control Engineering,)
• E-Mail : yujia.zhai@xjtlu.edu.cn

<Research Interest> : Nonlinear Control and Robustness, Automotive Engine Modeling and Dynamics, Analysis Artificial Intelligence