Korean J. Math. **21** (2013), No. 1, pp. 63–73
http://dx.doi.org/10.11568/kjm.2013.21.1.63

SOME PROPERTIES OF EXTENDED ORDER ALGEBRAS

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Abstract. We study the properties of extended order algebras. In particular, we investigate the properties of adjoint, Galois pair in commutative extended-order algebras.

1. Introduction

Wille [8] introduced the structures on lattices which are important mathematical tools for data analysis and knowledge processing. MV-algebra was introduced by Chang [2] to provide algebraic models for many valued propositional logic. Recently, it is developed in many directions (BL-algebra, residuated algebra) [1,3,4,6,7]. Recently, Guido et al. [5] introduced extended order algebras as the generalization of residuated algebras.

In this paper, we study the properties of extended order algebras. In particular, we investigate the properties of adjoint, Galois pair in commutative extended-order algebras.

Received January 10, 2013. Revised February 27, 2013. Accepted March 5, 2013.

2010 Mathematics Subject Classification: 06A06, 06A15, 06B30, 54F05,68U35.

Key words and phrases: (Commutative, associative) weak extended-ordered algebras, adjoint pairs (triples), Galois pairs.

This work was supported by Research Institute of Natural Science of Gangneung-Wonju National University.

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2. Preliminaries

DEFINITION 2.1. [5] A triple $(L, \Rightarrow, \top)$ is called a *weak extended order algebra* (shortly, w-eo algebra) iff it satisfies the following properties:

(O1) $a \Rightarrow \top = \top$ (upper bounded condition),

(O2) $a \Rightarrow a = \top$ (reflexive condition),

(O3) if $a \Rightarrow b = \top$ and $b \Rightarrow a = \top$, then $a = b$,

(O4) if $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$, then $a \Rightarrow c = \top$.

A triple $(L, \Rightarrow, \top)$ is called a *right w-eo algebra* if it satisfies (O1), (O2), (O3) and

(O5) if $a \Rightarrow b = \top$, then $(c \Rightarrow a) \Rightarrow (c \Rightarrow b) = \top$.

A triple $(L, \Rightarrow, \top)$ is called a *left w-eo algebra* if it satisfies (O1), (O2), (O3) and

(O6) if $a \Rightarrow b = \top$, then $(b \Rightarrow c) \Rightarrow (a \Rightarrow c) = \top$.

A triple $(L, \Rightarrow, \top)$ is called an *eo algebra* if it is a right and left w-eo algebra.

A w-eo algebra is called a *right distributive w-eo algebra* if

(O7) $a \Rightarrow \bigwedge_i b_i = \bigwedge_i (a \Rightarrow b_i)$.

A w-eo algebra is called a *left distributive w-eo algebra* if

(O8) $\bigvee_i a_i \Rightarrow b = \bigwedge_i (a_i \Rightarrow b)$.

(1) A w-eo algebra has an *adjoint pair* $(\Rightarrow, \odot)$ if there exists a binary operation $\odot$ such that

$$a \odot b \leq c \text{ iff } b \leq a \Rightarrow c.$$

(2) A w-eo algebra has a *Galois pair* $(\Rightarrow, \rightarrow)$ if there exists a binary operation $\rightarrow$ such that

$$b \leq a \Rightarrow c \;\text{ iff } a \leq b \rightarrow c.$$

(3) A w-eo algebra has *symmetrical* if it has a Galois pair $(\Rightarrow, \rightarrow)$ and $(L, \rightarrow, \top)$ is a w-eo algebra.

(4) A w-eo algebra has an *adjoint triple* $(\Rightarrow, \odot, \rightarrow)$ if there exists binary operation $\odot$ and $\rightarrow$ such that $a \odot b \leq c$ iff $b \leq a \Rightarrow c$ iff $a \leq b \rightarrow c$.

(5) A w-eo algebra is called a *w-ceo algebra* if L is complete.

REMARK 2.2. We define that a w-eo algebra has a Galois pair (adjoint pair, adjoint triple) without the complete condition in [5].

THEOREM 2.3. (1) *If $(L, \Rightarrow, \top)$ is a right w-eo algebra, then it is a w-eo algebra.*

(2) *If $(L, \Rightarrow, \top)$ is a left w-eo algebra, then it is a w-eo algebra.*

Proof. (1) Let $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$. Since $(a \Rightarrow b) \Rightarrow (a \Rightarrow c) = \top$, then $\top \Rightarrow (a \Rightarrow c) = \top$ and $(a \Rightarrow c) \Rightarrow \top = \top$ from (O1). By (O3), $a \Rightarrow c = \top$. It satisfies (O4).

(2) Let $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$. Since $(b \Rightarrow c) \Rightarrow (a \Rightarrow c) = \top$, then $\top \Rightarrow (a \Rightarrow c) = \top$ and $(a \Rightarrow c) \Rightarrow \top = \top$ from (O1). By (O3), $a \Rightarrow c = \top$. It satisfies (O4). □

THEOREM 2.4. [5] *Let $(L, \Rightarrow, \top)$ be a right-distributive w-ceo algebra and $\odot$ be defined by*

$$a \odot x = \bigwedge\{y \in L \mid x \le a \Rightarrow y\}.$$

Then $\odot$ and $\Rightarrow$ form an adjoint pair,i.e.

$$x \odot y \le z \text{ iff } y \le x \Rightarrow z.$$

THEOREM 2.5. [5] *Let $(L, \Rightarrow, \top)$ be a left-distributive w-ceo algebra and $\rightarrow$ be defined by*

$$g_a(y) = y \rightarrow a = \bigvee\{x \in L \mid y \le x \Rightarrow a\}.$$

Then $(\Rightarrow, \rightarrow)$ forms a Galois pair,i.e.

$$y \le x \Rightarrow a \text{ iff } x \le y \rightarrow a.$$

3. Some properties of extended order algebras

We prove the properties of w-eo algebra having an adjoint pair without the complete condition in [5].

THEOREM 3.1. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having an adjoint pair $(\Rightarrow, \odot)$ and $\le$ be defined by*

$$a \Rightarrow b = \top \text{ iff } a \le b$$

For each $a, b, c, a_i, b_i \in L$, the following properties hold.

(1) *$a \odot b \le a$ and $a \odot (a \Rightarrow b) \le b \le a \Rightarrow a \odot b$;*

(2) *$a \odot \top = a$;*

(3) *$a \odot \bot = \bot \odot a = \bot$;*

(4) *If $b \le c$, then $a \odot b \le a \odot c$.*

(5) *$(L, \Rightarrow, \top)$ be a right w-eo algebra.*

(6) *If $(L, \Rightarrow, \top)$ is a complete lattice, $(L, \Rightarrow, \top)$ is a right distributive w-eo algebra and $a \odot (\bigvee_{i\in\Gamma} b_i) = \bigvee_{i\in\Gamma}(a \odot b_i)$.*

(7) *$(L, \Rightarrow, \top)$ is a left w-eo algebra iff $a \odot c \le b \odot c$ for $a \le b$.*

(8) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra, then* $(L, \Rightarrow, \top)$ *be a right w-eo algebra.*

(9) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra and* $a \odot (b \odot c) = (a \odot b) \odot c$, *then*

$$(a \Rightarrow b) \Rightarrow \Big((c \Rightarrow a) \Rightarrow (c \Rightarrow b)\Big) = \top.$$

(10) *If* $(L, \Rightarrow, \top)$ *is a complete and left w-eo algebra, then* $(L, \Rightarrow, \top)$ *is a left distributive w-eo algebra and* $(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b)$.

(11) *If* $\top \Rightarrow a = a$, *then* $\top \odot a = a$, $a = \bigwedge_b (b \Rightarrow (b \odot a))$ *and* $a = \bigvee_b (b \odot (b \Rightarrow a))$.

(12) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra and* $\top \Rightarrow a = a$, *then* $a \odot b \leq b$ *and* $a \Rightarrow (b \Rightarrow a) = \top$.

Proof. (1) Since $b \Rightarrow \top = \top$, then $b \leq \top = (a \Rightarrow a)$. By an adjoint pair, $a \odot b \leq a$. Since $a \Rightarrow b \leq a \Rightarrow b$ and $a \odot b \leq a \odot b$, we have $a \odot (a \Rightarrow b) \leq b \leq a \Rightarrow a \odot b$.

(2) Since $\top = a \Rightarrow a$, $a \odot \top \leq a$. Since $\top \leq a \Rightarrow a \odot \top$, $a \leq a \odot \top$. So, $a \odot \top = a$.

(3) Since $a \leq (\bot \Rightarrow \bot) = \top$, we have $\bot \odot a = \bot$. Since $\bot \leq a \Rightarrow \bot$, we have $a \odot \bot = \bot$.

(4) If $b \leq c$, then $b \leq c \leq a \Rightarrow a \odot c$. So, $a \odot b \leq a \odot c$.

(5) Let $b \leq c$ be given. Since $a \odot (a \Rightarrow b) \leq b \leq c$, then $a \Rightarrow b \leq a \Rightarrow c$.

(6) By (5), $\bigwedge_i (a \Rightarrow b_i) \geq a \Rightarrow \bigwedge_i b_i$. Since $a \odot \bigwedge_i (a \Rightarrow b_i) \leq a \odot (a \Rightarrow b_i) \leq b_i$, then $a \odot \bigwedge_i (a \Rightarrow b_i) \leq \bigwedge_i b_i$ iff $\bigwedge_i (a \Rightarrow b_i) \leq a \Rightarrow \bigwedge_i b_i$. Hence $\bigwedge_i (a \Rightarrow b_i) = a \Rightarrow \bigwedge_i b_i$.

Since $a \odot b_i \leq a \odot \bigvee_i b_i$, we have $\bigvee_{i\in\Gamma}(a \odot b_i) \leq a \odot (\bigvee_{i\in\Gamma} b_i)$.

Since $b_i \leq a \Rightarrow \bigvee_{i\in\Gamma}(a \odot b_i)$, then $\bigvee_{i\in\Gamma} b_i \leq a \Rightarrow \bigvee_{i\in\Gamma}(a \odot b_i)$. Thus, $a \odot (\bigvee_{i\in\Gamma} b_i) \leq \bigvee_{i\in\Gamma}(a \odot b_i)$.

(7) ($\Rightarrow$) Let $a \leq b$ be given. Since $c \leq b \Rightarrow b \odot c \leq a \Rightarrow b \odot c$, then $a \odot c \leq b \odot c$.

($\Leftarrow$) Let $a \leq b$ be given. Since $a \odot (b \Rightarrow c) \leq b \odot (b \Rightarrow c) \leq c$, we have $b \Rightarrow c \leq a \Rightarrow c$. Thus $(b \Rightarrow c) \Rightarrow (a \Rightarrow c) = \top$.

(8) It follows from (5) and Theorem 2.3(2).

(9)

$$\begin{aligned}
&c \odot ((c \Rightarrow a) \odot (a \Rightarrow b)) = (c \odot (c \Rightarrow a)) \odot (c \Rightarrow b)\\
&= (c \odot (c \Rightarrow a)) \odot (a \Rightarrow b) \leq a \odot (a \Rightarrow b) \leq b\\
&\text{iff } (c \Rightarrow a) \odot (a \Rightarrow b) \leq c \Rightarrow b \ \text{ iff } a \Rightarrow b \leq (c \Rightarrow a) \Rightarrow (c \Rightarrow b).
\end{aligned}$$

(10) By (7), $(\bigvee_{i\in\Gamma} a_i) \odot b \geq \bigvee_{i\in\Gamma}(a_i \odot b)$. We have $(\bigvee_{i\in\Gamma} a_i) \odot b \leq \bigvee_{i\in\Gamma}(a_i \odot b)$ from

$$\bigvee_{i\in\Gamma} a_i \Rightarrow \bigvee_{i\in\Gamma}(a_i \odot b) = \bigwedge_{i\in\Gamma}(a_i \Rightarrow \bigvee_{i\in\Gamma}(a_i \odot b)) \geq b.$$

(11) Since $\top \Rightarrow a = a$, then $\top \odot a \leq a$. Since $a \leq \top \Rightarrow \top \odot a = \top \odot a$, then $a \leq \top \odot a$. Since $a \leq \bigwedge_b(b \Rightarrow (b \odot a)) \leq \top \Rightarrow (\top \odot a) = a$, then $a = \bigwedge_b(b \Rightarrow (b \odot a))$. Since $a = \top \odot (\top \Rightarrow a) \leq \bigvee_b(b \odot (b \Rightarrow a)) \leq a$, then $\bigvee_b(b \odot (b \Rightarrow a)) = a$.

(12) By (11), $a \odot b \leq \top \odot b = b$. Moreover, $a \Rightarrow (b \Rightarrow a) = \top$ iff $a \leq b \Rightarrow a$ iff $b \odot a \leq a$. □

THEOREM 3.2. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having a Galois pair $(\Rightarrow, \rightarrow)$. For each $a, b, c \in L$, the following properties hold.*

(1) $a \leq (a \rightarrow b) \Rightarrow b$ and $a \leq (a \Rightarrow b) \rightarrow b$.

(2) $\top \rightarrow a = a$. Furthermore, if $(L, \Rightarrow, \rightarrow, \top)$ is a symmetrical w-eo algebra, then $\top \Rightarrow a = a$.

(3) If $\top \Rightarrow a = a$, then $a \rightarrow b = \top$ iff $a \leq b$.

(4) $(L, \Rightarrow, \top)$ is a left w-eo algebra. If $\top \Rightarrow a = a$, $(L, \rightarrow, \top)$ is a left w-eo algebra.

(5) $(L, \Rightarrow, \top)$ is a right w-eo algebra iff $(L, \rightarrow, \top)$ is a right w-eo algebra.

(6) $a = \bigwedge_b((a \Rightarrow b) \rightarrow b)$. If $\top \Rightarrow a = a$, then $a = \bigwedge_b((a \rightarrow b) \Rightarrow b)$.

(7) If $(L, \Rightarrow, \top)$ is a complete lattice, $\bigvee_i a_i \Rightarrow b = \bigwedge_i(a_i \Rightarrow b)$ and $\bigvee_i a_i \rightarrow b = \bigwedge_i(a_i \rightarrow b)$.

Proof. (1) Since $a \rightarrow b \leq a \rightarrow b$, we have $a \leq (a \rightarrow b) \Rightarrow b$. Since $a \Rightarrow b \leq a \Rightarrow b$, we have $a \leq (a \Rightarrow b) \rightarrow b$.

(2) Since $\top \leq a \Rightarrow a$, $a \leq \top \rightarrow a$. Since $\top \rightarrow a \leq \top \rightarrow a$, $\top \leq (\top \rightarrow a) \Rightarrow a$. Thus $(\top \rightarrow a) \leq a$. So, $a = \top \rightarrow a$.

Let $(L, \Rightarrow, \rightarrow, \top)$ be a symmetrical w-eo algebra. Then $(L, \rightarrow, \top)$ be a w-eo algebra. Similarly, $\top \Rightarrow a = a$.

(3) Let $a \leq b$ and $b = \top \Rightarrow b$ be given. Then $\top \leq a \rightarrow b$. Let $a \rightarrow b = \top$. Then $a \leq \top \Rightarrow b = b$.

(4) Let $a \Rightarrow b = \top$ be given. Since $a \leq b \leq (b \Rightarrow c) \rightarrow c$, then $b \Rightarrow c \leq a \Rightarrow c$. Let $a \rightarrow b = \top$ be given. By (2) and (3), $a \leq b$. Since $a \leq b \leq (b \rightarrow c) \Rightarrow c$, then $b \rightarrow c \leq a \rightarrow c$.

(O1) $a \rightarrow \top = \top$ because $\top \leq a \rightarrow \top$ iff $a \leq \top \rightarrow \top = \top$.

(O2) $a \rightarrow a = \top$ because $\top \leq a \rightarrow a$ iff $a \leq \top \Rightarrow a = a$.

(O3) Let $a \to b = \top$ and $b \to a = \top$ be given. Since $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$ and $\top \leq b \to a$ iff $b \leq \top \Rightarrow a = a$, then $a = b$.

(O4) Let $a \to b = \top$ and $b \to c = \top$ be given. Since $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$ and $\top \leq b \to c$ iff $b \leq \top \Rightarrow c = c$, then $a \leq c = \top \Rightarrow c$ iff $a \to c = \top$.

(O6) Let $a \to b = \top$. Then $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$. So, $a \leq b \leq (b \to c) \Rightarrow c$ implies $b \to c \leq (a \to c) = (\top \Rightarrow (a \to c))$. Thus $(b \to c) \to (a \to c) = \top$.

(5) Let $(L, \Rightarrow, \top)$ be a right w-eo algebra. If $a \to b = \top$, then $a \Rightarrow b = \top$. Thus $((c \to a) \Rightarrow a) \Rightarrow ((c \to a) \Rightarrow b) = \top$. So, $c \leq (c \to a) \Rightarrow a \leq (c \to a) \Rightarrow b$. Then $c \to a \leq c \to b$ iff $(c \to a) \to (c \to b) = \top$.

(6) Since $a \leq (a \Rightarrow b) \to b$ and $(a \Rightarrow a) \to a = \top \to a = a$ from (2), we have $a \leq \bigwedge_b((a \Rightarrow b) \to b) \leq (a \Rightarrow a) \to a = a$. If $\top \Rightarrow a = a$, $(L, \to, \top)$ is a left w-eo algebra. By Theorem 2.3(2), the result hods.

(7) $\bigvee_i a_i \Rightarrow b \leq \bigwedge_i(a_i \Rightarrow b)$. Since $a_i \leq (a_i \Rightarrow b) \to b \leq \bigwedge_i(a_i \Rightarrow b) \to b$, we have $\bigvee_i a_i \leq \bigwedge_i(a_i \Rightarrow b) \to b$. Hence $\bigwedge_i(a_i \Rightarrow b) \leq \bigvee_i a_i \Rightarrow b$. □

THEOREM 3.3. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having adjoint triple with $\odot$ and $\to$. Then the following properties hold.*

(1) *$(L, \Rightarrow, \top)$ is an eo algebra;*

(2) *If $\top \Rightarrow a = a$, then $(L, \to, \top)$ is an eo algebra;*

(3) *If $a \leq b$, then $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$.*

(4) *If $\top \Rightarrow a = a$, then $\top \odot a = a$.*

Proof. (1) Let $a \leq b$ be given. Since $a \leq b \leq (b \Rightarrow c) \to c$, then $b \Rightarrow c \leq a \Rightarrow c$. Since $c \odot (c \Rightarrow a) \leq a \leq b$, then $c \Rightarrow a \leq c \Rightarrow b$.

(2) (O1) $a \to \top = \top$ because $\top \leq a \to \top$ iff $a \leq \top \Rightarrow \top = \top$.

(O2) $a \to a = \top$ because $\top \leq a \to a$ iff $a \leq \top \Rightarrow a = a$.

(O3) Let $a \to b = \top$ and $b \to a = \top$ be given. Since $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$ and $\top \leq b \to a$ iff $b \leq \top \Rightarrow a = a$, then $a = b$.

(O4) Let $a \to b = \top$ and $b \to c = \top$ be given. Since $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$ and $\top \leq b \to c$ iff $b \leq \top \Rightarrow c = c$, then $a \leq c = \top \Rightarrow c$ iff $a \to c = \top$.

(O5) Let $a \to b = \top$. Then $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$. So, $(c \to a) \odot c \leq a \leq b$ implies $c \to a \leq (c \to b) = (\top \Rightarrow (c \to b))$. Thus $(c \to a) \to (c \to b) = \top$.

(O6) Let $a \to b = \top$. Then $\top \leq a \to b$ iff $a \leq \top \Rightarrow b = b$. So, $a \leq b \leq (b \to c) \Rightarrow c$ implies $b \to c \leq (a \to c) = (\top \Rightarrow (a \to c))$. Thus $(b \to c) \to (a \to c) = \top$.

(3) If $a \leq b$, then $a \leq b \leq c \to b \odot c$ implies $a \odot c \leq b \odot c$ and $a \leq b \leq c \Rightarrow c \odot b$ implies $c \odot a \leq c \odot b$.

(4) Since $\top \Rightarrow a = a$, then $\top \odot a \leq a$. Since $\top \leq a \to \top \odot a$, then $a \leq \top \odot a$. Thus, $\top \odot a = a$. □

Definition 3.4 (5). *A w-eo algebra* $(L, \Rightarrow, \top)$ *is commutative iff it satisfies*

$$a \Rightarrow (b \Rightarrow c) = \top \text{ iff } b \Rightarrow (a \Rightarrow c) = \top$$

Theorem 3.5. *Let* $(L, \Rightarrow, \top)$ *be a w-eo algebra having adjoint triple with* $\odot$ *and* $\to$. *Then the following statements are equivalent:*

(1) $(L, \Rightarrow, \top)$ *is commutative;*
(2) $(L, \odot, \top)$ *is commutative;*
(3) $(L, \to, \top)$ *is commutative eo algebra with* $\Rightarrow = \to$.
(4) $a \to b \leq a \Rightarrow b$ *for all* $a, b \in L$.
(5) $a \Rightarrow b \leq a \to b$ *for all* $a, b \in L$.
(6) $a \leq (a \Rightarrow b) \Rightarrow b$ *for all* $a, b \in L$.
(7) $a \leq (a \to b) \to b$ *for all* $a, b \in L$.
(8) $(a \Rightarrow b) \odot a \leq b$ *for all* $a, b \in L$.
(9) $a \odot (a \to b) \leq b$ *for all* $a, b \in L$.
(10) $b \leq a \Rightarrow b \odot a$ *for all* $a, b \in L$.
(11) $a \leq b \to b \odot a$ *for all* $a, b \in L$.

Proof. (1)⇔ (2) Since L is commutative, then $a \leq b \Rightarrow c$ iff $b \leq a \Rightarrow c$. Thus,

$$\begin{aligned} b \odot a \leq b \odot a \quad &\text{iff} \quad a \leq b \Rightarrow (b \odot a) \quad (\text{ by adjoint triple}) \\ &\text{iff} \quad b \leq a \Rightarrow (b \odot a) \quad (\text{ by commutative}) \\ &\text{iff } a \odot b \leq b \odot a. \end{aligned}$$

Similarly, $a \odot b \geq b \odot a$.

Conversely, $a \leq b \Rightarrow c$ iff $b \odot a \leq c$ iff $a \odot b \leq c$ iff $b \leq a \Rightarrow c$. Hence L is commutative.

(1)⇔ (3) We have $a \to b \leq a \Rightarrow b$ from:

$$\begin{aligned} a \to b \leq a \to b \quad &\text{iff } a \leq (a \to b) \Rightarrow b \ (\text{ by adjoint triple}) \\ &\text{iff } a \to b \leq a \Rightarrow b \ (\text{ by commutative}). \end{aligned}$$

We have $a \Rightarrow b \leq a \to b$ from:

$$a \Rightarrow b \leq a \Rightarrow b \quad \text{iff } a \leq (a \Rightarrow b) \Rightarrow b \text{ (by commutative)}$$
$$\text{iff } a \Rightarrow b \leq a \rightarrow b \text{ (by adjoint triple).}$$

Since $\top = a \Rightarrow a$ and it is commutative, we have $a \leq \top \Rightarrow a$. On the other hand, since $\top \Rightarrow a \leq \top \Rightarrow a$, by commutativity, $\top \leq (\top \Rightarrow a) \Rightarrow a$. So, $\top \Rightarrow a \leq a$. Thus $\top \Rightarrow a = a$. By Theorem 3.3 (2), $(L, \rightarrow, \top)$ is a commutative eo algebra.

Conversely, it is trival.

(1)$\Leftrightarrow$ (4) By the above proof, $a \rightarrow b \leq a \Rightarrow b$.

Conversely, let $a \Rightarrow (b \Rightarrow c) = \top$. By a adjoint triple, $a \leq b \Rightarrow c$ iff $b \leq a \rightarrow c$. So, $b \leq a \rightarrow c \leq a \Rightarrow c$;i.e. $b \Rightarrow (a \Rightarrow c) = \top$. Other case is similarly proved.

(1)$\Leftrightarrow$ (4)$\Leftrightarrow$ (7) $\Leftrightarrow$ (9) are follows from:

$$a \rightarrow b \leq a \Rightarrow b \text{ iff } a \leq (a \rightarrow b) \rightarrow b \text{ iff } a \odot (a \rightarrow b) \leq b.$$

(1)$\Leftrightarrow$(5)$\Leftrightarrow$ (6) $\Leftrightarrow$ (8) are follows from:

$$a \Rightarrow b \leq a \rightarrow b \text{ iff } a \leq (a \Rightarrow b) \Rightarrow b \text{ iff } (a \Rightarrow b) \odot a \leq b.$$

(2)$\Leftrightarrow$ (10) $\Leftrightarrow$ (11) are follows from:

$$b \leq a \Rightarrow b \odot a \text{ iff } a \leq b \rightarrow b \odot a.$$

□

EXAMPLE 3.6. *(1) Let* $([0,1], \Rightarrow)$ *be a unit interval defined as*

$$a \Rightarrow b = \begin{cases} 1, & \text{if } \ a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

We easily show that $([0,1], \Rightarrow, 1)$ *is an eo algebra having* $\bigvee_i x_i \Rightarrow y = \bigwedge_i (x_i \Rightarrow y)$. *Put* $f_b(x) = x \Rightarrow b$. *Define* $g_b(y) = y \rightarrow b = \bigvee\{x \in L \mid y \leq f_b(x)\}$. *Then* $y \leq f_b(x)$ *iff* $x \leq g_b(y)$. *We obtain:*

$$a \rightarrow b = \begin{cases} 1, & \text{if } \ a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

Since $1 \Rightarrow b = b$, $([0,1], \rightarrow, 1)$ *is an eo algebra. Furthermore,* $([0,1], \rightarrow, 1)$ *is commutative with* $a \Rightarrow b = a \rightarrow b$ *from:*

$$a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = \begin{cases} 1, & \text{if } \ a \leq c, \quad b \leq c \\ c, & \text{otherwise.} \end{cases}$$

(2) Let $([0,1], \Rightarrow)$ *be a unit interval defined as*

$$a \Rightarrow b = \begin{cases} 1, & \text{if} \quad a \leq b, \\ \frac{b}{2}, & \text{otherwise.} \end{cases}$$

We easily show that $([0,1], \Rightarrow, 1)$ *is an eo algebra having* $\bigvee_i x_i \Rightarrow y = \bigwedge_i (x_i \Rightarrow y)$*. Put* $f_b(x) = x \Rightarrow b$*. Define* $g_b(y) = y \to b = \bigvee\{x \in L \mid y \leq f_b(x)\}$*. Then* $y \leq f_b(x)$ *iff* $x \leq g_b(y)$*. We obtain:*

$$a \to b = \begin{cases} 1, & \text{if} \quad 2a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

Since $1 \Rightarrow b = \frac{b}{2}$ *and* $a \to a = a \neq 1$*,* $([0,1], \to, 1)$ *is not an eo algebra.*

For $a \neq 1$*,* $\frac{a}{4} = a \Rightarrow (1 \Rightarrow a) \neq 1 \Rightarrow (a \Rightarrow a) = 1$*. Thus* $([0,1], \Rightarrow, 1)$ *is not commutative.*

Example 3.7. *Let* $K = \{(x,y) \in R^2 \mid x > 0\}$ *be a set and we define an operation* $\otimes : K \times K \to K$ *as follows:*

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Then $(K, \otimes)$ *is a group with* $e = (1,0)$*,* $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$*.*

We have a positive cone $P = \{(a,b) \in R^2 \mid a = 1, b \geq 0 \text{ , or } a > 1\}$ *because* $P \cap P^{-1} = \{(1,0)\}$*,* $P \odot P \subset P$*,* $(a,b)^{-1} \odot P \odot (a,b) = P$ *and* $P \cup P^{-1} = K$*. For* $(x_1,y_1), (x_2,y_2) \in K$*, we define*

$$\begin{aligned} &(x_1, y_1) \leq (x_2, y_2) \\ &\Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P, \ (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then $(K, \leq \otimes)$ *is a lattice-group. (ref. [1])*

We define the structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1,0))$ *where* $\bot = (\frac{1}{2}, 1)$ *is the least element and* $\top = (1,0)$ *is the greatest element from the following statements:*

$$\begin{aligned}
(x_1, y_1) \odot (x_2, y_2) &= (x_1, y_1) \otimes (x_2, y_2) \vee (\frac{1}{2}, 1) \\
&= (x_1x_2, x_1y_2 + y_1) \vee (\frac{1}{2}, 1), \\
(x_1, y_1) \Rightarrow (x_2, y_2) &= ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1, 0) \\
&= (\frac{x_2}{x_1}, \frac{y_2 - y_1}{x_1}) \wedge (1, 0), \\
(x_1, y_1) \rightarrow (x_2, y_2) &= ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1, 0) \\
&= (\frac{x_2}{x_1}, -\frac{x_2y_1}{x_1} + y_2) \wedge (1, 0).
\end{aligned}$$

The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ *is a w-eo algebra having an adjoint triple as follows:*

$$\begin{aligned}
(x_1, y_1) \odot (x_2, y_2) \leq (x_3, y_3) \quad &\text{iff} \quad (x_2, y_2) \leq (x_1, y_1) \Rightarrow (x_3, y_3) \\
&\text{iff} \quad (x_1, y_1) \leq (x_2, y_2) \rightarrow (x_3, y_3)
\end{aligned}$$

Since $\bigvee_{n\in N}(\frac{2}{3}, n)$ *and* $\bigwedge_{n\in N}(\frac{2}{3}, -n)$ *does not exists,* $(L, \odot, \Rightarrow, \rightarrow)$ *is not a w-ceo algebra. A w-eo algebra* $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ *is not commutative because*

$$\begin{aligned}
(\frac{4}{5}, 1) \Rightarrow ((\frac{5}{6}, 3) \Rightarrow (\frac{2}{3}, -1)) &= (\frac{4}{5}, 1) \Rightarrow (\frac{4}{5}, -\frac{24}{5}) = (1, -\frac{29}{4}), \\
(\frac{5}{6}, 3) \Rightarrow ((\frac{4}{5}, 1) \Rightarrow (\frac{2}{3}, -1)) &= (\frac{5}{6}, 3) \Rightarrow (\frac{5}{6}, -\frac{5}{2}) = (1, -\frac{33}{5}), \\
(\frac{4}{5}, 1) \Rightarrow ((\frac{5}{6}, 3) \Rightarrow (\frac{2}{3}, -1)) &\neq (\frac{5}{6}, 3) \Rightarrow ((\frac{4}{5}, 1) \Rightarrow (\frac{2}{3}, -1)).
\end{aligned}$$

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