Korean J. Math. **21** (2013), No. 1, pp. 55–62
http://dx.doi.org/10.11568/kjm.2013.21.1.55

THE PRIMITIVE BASES OF THE SIGNED CYCLIC GRAPHS

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Abstract. The base $l(S)$ of a signed digraph S is the maximum number k such that for any vertices u, v of S, there is a pair of walks of length k from u to v with different signs. A graph can be regarded as a digraph if we consider its edges as two-sided arcs. A signed cyclic graph $\widetilde{C_n}$ is a signed digraph obtained from the cycle C_n by giving signs to all arcs. In this paper, we compute the base of a signed cyclic graph $\widetilde{C_n}$ when $\widetilde{C_n}$ is neither symmetric nor anti-symmetric. Combining with previous results, the base of all signed cyclic graphs are obtained.

1. Introduction

A *sign pattern matrix* A of order n is the $n \times n$ matrix with entries 1, 0 and -1. When we compute the entries of the powers of A, we use the operation rule that continues to hold the sign of the usual addition and multiplication, that is for any $a \in \{1, 0, -1\}$

$$1+1 = 1;\ (-1)+(-1) = -1;\ 1+0 = 0+1 = 1;\ (-1)+0 = 0+(-1) = -1$$

$$0 \cdot a = a \cdot 0 = 0;\ 1 \cdot 1 = (-1) \cdot (-1) = 1; 1 \cdot (-1) = (-1) \cdot 1 = -1.$$

Received January 2, 2013. Revised February 13, 2013. Accepted February 15, 2013.

2010 Mathematics Subject Classification: 15B35, 05C20, 05C22.

Key words and phrases: base, sign pattern matrix, directed cycle.

This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

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In this case we contact the ambiguous situations $1+(-1)$ and $(-1)+1$, which we will use the notation "$\sharp$" as in [3]. Define the addition and multiplication involving the symbol $\sharp$ as follows:

$$(-1)+1 = 1+(-1) = \sharp; \quad a+\sharp = \sharp+a = \sharp \text{ for any } a \in \{1,\ -1,\ \sharp,\ 0\}$$
$$0\cdot\sharp = \sharp\cdot 0 = 0; \quad b\cdot\sharp = \sharp\cdot b = \sharp \text{ for any } b \in \{1,\ -1,\ \sharp\}.$$

A *generalized sign pattern matrix* A of order n is the $n\times n$ matrix with entries 1, 0, -1 and the ambiguous sign $\sharp$. A least positive integer l such that there is a positive integer p satisfying $A^l = A^{l+p}$ is called the *base* of A, and denoted by $l(A)$. And the least such positive integer p is called to be the *period* of A, and denoted by $p(A)$. A generalized sign pattern matrix A is called *powerful* if there appears no $\sharp$ entry in any power of A. And A is *non-powerful* if it is not powerful. If a sign pattern matrix A is non-powerful and there is a number l such that every entry of A^l is $\sharp$, then the least such integer l is the base of A.

In [3], Li, Hall and Stuart showed that if the sign pattern matrix A is powerful, then $l(A) = l(|A|)$ where $|A|$ is the matrix by assigning each non-zero entry of A to 1. If A is non-powerful, then the $\sharp$ entry appears and we have a different situation. We introduce a graph theoretic method to study the powers of a sign pattern matrix.

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1. The *sign* of the walk W in S, denoted by $\mathrm{sgn}(W)$, is defined to be the product of signs of all arcs in W. If two walks W_1 and W_2 have the same initial points, the same terminal points, the same lengths and different signs, then we say W_1 and W_2 a *pair of SSSD walks.* A signed digraph S is *powerful* if S contains no pair of SSSD walks. So every non-powerful primitive signed digraph contains a pair of SSSD walks. Let $A = A(S) = (a_{ij})$ be the adjacency matrix of a signed digraph S, that is $\mathrm{sgn}(i,j) = \alpha$ if and only if $a_{ij} = \alpha$ where $\alpha = 1$, or -1 for an arc (i,j) of S. In this case A is a sign pattern matrix which satisfies that $(i,j)-$ entry of $A^k = 0$, if and only if there is no walk of length k from i to j. Moreover $(i,j)-$ entry of A^k is 1 (or -1), if and only if all walks of length k from i to j are of sign 1 (or, -1). Also $(i,j)-$ entry of A^k is $\sharp$ if and only if there is a pair of SSSD walks of length k from i to j. Thus we see from the above relations between matrices and graphs that each power of a signed digraph S contains no pair of SSSD walks if and only if the adjacency sign pattern matrix $A(S)$ is powerful. A signed digraph S is also said to be powerful or non-powerful if its adjacency sign pattern matrix is powerful or non-powerful respectively. There is an important characterization for powerful irreducible sign pattern matrices given in [2] which will be the starting point of our study on the bases of non-powerful irreducible sign pattern matrices. Let S be a strongly connected signed digraph and h be the index of imprimitivity of S (i.e., h is the greatest common divisor of the lengths of all the cycles of S). Then S is powerful if and only if S satisfies the following two conditions:

(A1) All cycles in S with lengths even multiples of h (if any) are positive.

(A2) All cycles in S with lengths odd multiples of h have the same sign.

From now on we assume that S is a primitive non-powerful signed digraph of order n. For each pair of vertices v_i, v_j of S, we define the *local base* $l_S(v_i, v_j)$ from v_i to v_j to be the smallest integer l such that for each $k \geq l$, there is a pair of SSSD walks of length k in S from v_i to v_j. The *base* $l(S)$ of S is defined to be $\max\{l_S(v_i, v_j)|v_i, v_j \in V(S)\}$. It follows directly from the definitions that $l(S) = l(A)$ where A is the adjacency matrix of S. You et al. [7] found upper bounds for the bases of primitive nonpowerful sign pattern matrices and completely characterized extremal cases. Gao, Huang and Shao [1], Shao and Gao [6] and Li and Liu [4] studied the base and local base of the primitive non-powerful signed symmetric digraphs with loops. Liang, Liu and Lai [5] gave the bounds on the $k-$th multiple generalized base index for a class of non-powerful generalized sign pattern matrices. They also characterized the extremal graphs for the (generalized) base for primitive anti-symmetric sign pattern matrices.

Let us assume that $\widetilde{C_n}$ is a signed digraph of order n which is the cyclic graph C_n on n vertices by assigning signs to each arc such that it becomes a signed digraph. Liang, Liu and Lai [5] proved that the base of anti-symmetric signed cyclic graph $\widetilde{C_n}$ on n vertices is $2n-1$. In this paper we find the base of $\widetilde{C_n}$ when $\widetilde{C_n}$ is neither symmetric nor anti-symmetric.

Let Q be the canonical cycle in C_n. We then can summarize the main contributions of the present paper as follows:

(C1) If the cycle Q and its inverse cycle $-Q$ have the same sign, then the base of $\widetilde{C_n}$ is $n+1$.

(C2) If the cycle Q and its inverse cycle $-Q$ have distinct sign, then the base of $\widetilde{C_n}$ is n.

Consequently the base of all signed cyclic graphs are obtained.

2. Main theorem

In this section we assume that n is an odd positive integer, $C_n = (V, E)$ where $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and $E = \{\{v_i, v_j\} | j \equiv i+1 \pmod n\}$. Thus C_n is a cyclic graph of odd order. If $A = \{(v_i, v_j) | \{v_i, v_j\} \in E\}$ and $f : A \to \{\pm 1\}$, then $\widetilde{C_n} = (V, A, f)$ is a signed digraph. If $a = (v, w) \in A$, then $a^{-1} = (w, v)$ is the inverse of a and $e = \{v, w\}$ is the underlying edge of a. If $W = w_0 w_1 \cdots w_k$ where $w_0, w_1, \cdots, w_k \in V$ is a walk of length k in C_n, then $-W = w_k w_{k-1} \cdots w_1 w_0$ is the inverse of W. If $W_1 = v_0 v_1 \cdots v_n$ and $W_2 = v_n v_{n+1} \cdots v_m$ are two walks in a graph, then we use $W_1 + W_2$ to be the walk $v_0 v_1 \cdots v_m$. We also use the notation $kW = W + W + \cdots + W$ (k-times) for a circuit W.

The *sign* $f(W)$ of W is $f(w_0 w_1) f(w_1 w_2) \cdots f(w_{k-1} w_k)$. If $e = \{v, w\}$, then the *sign* of e is $f(vw)f(wv)$. Note that $\widetilde{C_n}$ is symmetric when the sign of every edge is 1, and anti-symmetric when the sign of every edge is -1. If $W = w_0 w_1 \cdots w_k$ is a cycle of length k, then $W' = w_i w_{i+1} \cdots w_k w_0 w_1 \cdots w_i$ is a *rotation* of W for $0 \leq i \leq k$.

LEMMA 1. *If $W = w_0 w_1 \cdots w_k$ is a walk of length k in an odd cycle C_n with $w_0 = w_k$, then for each $e \in E$, the number of i such that $\{w_i, w_{i+1}\} = e$ is congruent to k modulo 2.*

Proof. Since $C_n - e$ is isomorphic to the path P_n, which is bipartite, there are $V_0, V_1 \subset V$ such that $V_0 \bigcup V_1 = V$ and $V_0 \bigcap V_1 = \phi$ and every edge except e joins a vertex of V_0 and a vertex of V_1. We may assume that the two vertices incident to e belong to V_0. For each $i = 0, 1, \cdots, k-1$, the membership of w_i and w_{i+1} among V_0 and V_1 is changed if and only if $\{w_i, w_{i+1}\} \neq e$. Since $w_0 = w_k$, the number of i such that $\{w_i, w_{i+1}\} \neq e$ is even. So the number of i such that $\{w_i, w_{i+1}\} = e$ is congruent to k modulo 2.. □

LEMMA 2. *Every even cycle in an odd cycle C_n is a 2-cycle.*

Proof. Let $Z = w_0 w_1 \cdots w_k$ be an even cycle. If $e = \{w_0, w_1\}$, then by Lemma 1 the number of i such that $\{w_i, w_{i+1}\} = e$ is even. Hence there is a t such that $t \geq 1$ and $\{w_t, w_{t+1}\} = e$. Since Z is a cycle, $w_i \neq w_j$ for $i \neq j$ except $i = 0, j = k$ or $i = k, j = 0$. Hence $t = 1$ and $w_2 = w_0$. Hence we have $k = 2$ and $Z = w_0 w_1 w_0$. □

Let Q be the canonical n-cycle $v_0v_1\cdots v_{n-1}v_0$ in C_n. Then $-Q = v_0v_{n-1}v_{n-2}\cdots v_0$.

LEMMA 3. *Let C_n be a cyclic graph of odd order. Then there are exactly two odd cycles Q and $-Q$ up to a rotation in C_n.*

Proof. If $Z = w_0w_1\cdots w_k$ is an odd cycle, then by Lemma 1, for each edge e, the number of i such that $\{w_i, w_{i+1}\} = e$ is odd. Since Z doesn't visit the same vertex twice, except $w_0 = w_k$, for all edge e of C_n, there is exactly one i such that $\{w_i, w_{i+1}\} = e$. Thus $k = n$ and $\{w_0, w_1, \cdots, w_{n-1}\} = V$. We may assume that $w_0 = v_0$. Since v_1 and v_{n-1} are only two vertices adjacent to v_0, w_1 are v_1 or v_{n-1}. If $w_1 = v_1$, then w_2 is v_0 or v_2. Since Z is a cycle, $w_2 \neq v_0$. Hence $w_2 = v_2$. Similarly we have $w_i = v_i$ for any $3 \leq i \leq n-1$ and $w_n = v_0$. Therefore $Z = Q$. If $w_1 = v_{n-1}$, then by the same method we have $Z = -Q$. □

PROPOSITION 1. *If a signed odd cyclic graph $\widetilde{C_n}$ is symmetric, then $\widetilde{C_n}$ is powerful.*

Proof. If Z is an even cycle, then by Lemma 2 Z is a 2-cycle. Hence $Z = w_0w_1w_0$ for some $w_0, w_1 \in V$. Thus $f(Z) = f(w_0w_1)f(w_1w_0)$ is the same with the sign of edge $\{v_0, v_1\}$. Since $\widetilde{C_n}$ is symmetric, $f(Z) = 1$. So there is no even cycle of sign -1. By Lemma 3 the odd cycles of C_n are Q and $-Q$ up to translation. Since $\widetilde{C_n}$ is symmetric, we have $f(-Q) = f(w_0w_{n-1})f(w_{n-1}w_{n-2})\cdots f(w_1w_0) = f(w_0w_1)f(w_1w_2)\cdots f(w_{n-1}w_0) = f(Q)$. Thus all odd cycles in $\widetilde{C_n}$ have the same signs. Hence every even cycle in $\widetilde{C_n}$ has sign 1 and every odd cycles, Q and $-Q$, have the same signs. By the characterization of powerful signed digraph provided in introduction, $\widetilde{C_n}$ is powerful. □

It is known [3] that the base of a primitive powerful signed digraph S is equal to the exponent of S. Hence we have the following Corollary.

COROLLARY 1. *If a signed odd cyclic graph $\widetilde{C_n}$ is symmetric, then the base of $\widetilde{C_n}$ is $n-1$.*

The following Proposition is due to Liang, Liu and Lai [5].

PROPOSITION 2. *If a signed odd cyclic graph $\widetilde{C_n}$ is anti-symmetric, then $l(\widetilde{C_n}) = 2n-1$.*

LEMMA 4. *There is only one walk of length $n-1$ from v_0 to v_{n-1} in an odd cycle C_n.*

Proof. If $W = w_0w_1 \cdots w_k$ is a walk of length $n-1$ from v_0 to v_{n-1} in C_n, then since $|E| = n$, there is $e \in E$ such that $\{w_i, w_{i+1}\} \neq e$ for all $i = 0, 1, \cdots, n-2$. If $e \neq \{w_0, w_{n-1}\}$, then since $C_n - e$ is bipartite, there is no walk of even length from v_0 to v_{n-1}. This contradicts to the fact that W is a walk of even length $n-1$ from v_0 to v_{n-1}. Thus $e = \{v_{n-1}, v_0\}$. Since the distance from v_0 to v_{n-1} in $C_n - \{v_0, v_{n-1}\}$ is $n-1$, we have $W = v_0v_1 \cdots v_{n-1}$ □

Lemma 5. *There are exactly two walks Q and $-Q$ of length n from v_0 to v_0 in an odd cycle C_n.*

Proof. If $W = w_0w_1 \cdots w_n$ is a walk of length n from v_0 to v_0 in C_n, then w_{n-1} is v_{n-1} or v_1. If $w_{n-1} = v_{n-1}$, then by Lemma 4 $w_0w_1 \cdots w_{n-1} = v_0v_1 \cdots v_{n-1}$. Hence $W = v_0v_1 \cdots v_{n-1} = Q$. By the same method, if $w_{n-1} = v_1$, then we have $W = -Q$. □

Proposition 3. *Assume that an odd cycle $\widetilde{C_n}$ is neither symmetric nor anti-symmetric. Then $l(\widetilde{C_n}) = n+1$ if $f(Q) = f(-Q)$, and $l(\widetilde{C_n}) = n$ if $f(Q) = -f(-Q)$.*

Proof. Let $v, w \in V$. We may assume that $v = v_0$ and $w = v_t$ for $0 \leq t \leq n-1$. Let $\alpha = n+1$ if $f(Q) = f(-Q)$, and $\alpha = n$ if $f(Q) = -f(-Q)$. Let $e_i = \{v_i, v_{i+1}\}$ for all $i = 0, 1, \cdots, n-2$ and $e_{n-1} = \{v_{n-1}, v_0\}$. Since $\widetilde{C_n}$ is neither symmetric nor anti-symmetric, there is s such that $0 \leq s \leq n-2$ and $f(v_sv_{s+1})f(v_{s+1}v_s) = -f(v_{n-1}v_0)f(v_0v_{n-1})$. Let $Z = v_0v_{n-1}v_0$, $Z_1 = v_sv_{s+1}v_s$ and $Z_2 = v_{s+1}v_sv_{s+1}$. Therefore $f(Z) = -f(Z_1) = -f(Z_2)$. Since n is odd, $\alpha \equiv t(\text{mod}2)$ or $\alpha \equiv n - t(\text{mod}2)$. We may assume that $\alpha \equiv t(\text{mod}2)$.

If $t \geq 1$ and $0 \leq s \leq t$, then since $\alpha - t - 2$ is even and $\alpha - t - 2 \geq n - (n-1) - 2 = -1$, $\alpha - t - 2 = 2k$ for all $k \geq 0$. Let $W_1 = v_0v_1 \cdots v_s$ and $W_2 = v_sv_{s+1} \cdots v_t$. Then $(k+1)Z + W_1 + W_2$ and $kZ + W_1 + Z_1 + W_2$ are SSSD walks of length α from v_0 to v_t.

If $t \geq 1$ and $t \leq s \leq n-2$, then since $n - t - 1 = (n-s-1) + (s-t)$, $s - t \leq \frac{n-t-1}{2}$ or $n - s - 1 \leq \frac{n-t-1}{2}$. Let $X_1 = v_0v_1 \cdots v_s$, $X_2 = v_tv_{t+1} \cdots v_s$ and $X_3 = v_0v_{n-1}v_{n-2} \cdots v_{s+1}$. If $s - t \leq \frac{n-t-1}{2}$, since $\alpha - 2s + t - 2$ is even and $\alpha - 2S + t - 2 \geq n - 2s + (2s + 1 - n) - 2 = -1$, $\alpha - 2s + t - 2 = 2k$ for some $k \geq 0$. Then $(k+1)Z + X_1 + X_2 - X_2$ and $kZ + X_1 + X_2 + Z_1 - X_2$ are SSSD walks of length α from v_0 to v_t. If $n - s - 1 \leq \frac{n-t-1}{2}$, by the similar method with $\alpha = 2k + 2(n-s) + t$, we can show that $(k+1)Z + X_3 - X_3 + X_1$ and $kZ + X_3 + Z_2 - X_3 + X_1$ are SSSD walks of length α from v_0 to v_t.

If $t = 0$ and $f(Q) = -f(-Q)$, then Q and $-Q$ are SSSD walks of length n from v_0 to v_t. So $l(\widetilde{C_n}) \leq n = \alpha$. If $t = 0$ and $f(Q) = f(-Q)$, then $s \leq \frac{n-1}{2}$ or $n - s - 1 \leq \frac{n-1}{2}$. Let $Y_1 = v_0v_1 \cdots v_s$ and $Y_2 = v_0v_{n-1}v_{n-2} \cdots v_{s+1}$. Since $\alpha = n + 1$ is even, $\alpha - 2s - 2$ is even. If $s \leq \frac{n-1}{2}$, then since $\alpha - 2s - 2 \geq n + 1 - (n - 1) - 2 = 0$, we have $\alpha - 2s - 2 = 2k$ for some $k \geq 0$. Hence $(k + 1)Z + Y_1 - Y_1$ and $kZ + Y_1 + Z_1 - Y_1$ are SSSD walks of length $n+1$ from v_0 to v_0. Similarly $\alpha - 2n - 2s = 2l$ for some $l \geq 0$. If $n - s - 1 \leq \frac{n-1}{2}$, then $(l+1)Z + Y_2 - Y_2$ and $lZ + Y_2 + Z_2 - Y_2$ are SSSD walks of length $n + 1$ from v_0 to v_0. So $l(\widetilde{C_n}) \leq n + 1 = \alpha$.

If $f(Q) = -f(-Q)$, then by Lemma 4 $l(\widetilde{C_n}) \geq n$. So $l(\widetilde{C_n}) = n = \alpha$. If $f(Q) = f(-Q)$, then by Lemma 5 Q and $-Q$ are only 2 walks of length n from v_0 to v_0. Since $f(Q) = f(-Q)$, there is no walk of length n from v_0 to v_0 with sign $-f(Q)$. Thus $l(\widetilde{C_n}) \leq n+1$. As a consequence we have $l(\widetilde{C_n}) = n + 1 = \alpha$. □

From Propositions 1, 2 and 3 we conclude the following.

Theorem 1. *Let $\widetilde{C_n}$ be a signed odd cyclic graph of order n. Then*

$$l(\widetilde{C_n}) = \begin{cases} n-1, & \text{if } \widetilde{C_n} \text{ is symmetric;} \\ 2n-1, & \text{if } \widetilde{C_n} \text{ is anti-symmetric;} \\ n+1, & \text{if } \widetilde{C_n} \text{ is neither anti-symmetric nor symmetric,} \\ & \text{and} f(Q) = f(-Q); \\ n, & \text{if } \widetilde{C_n} \text{ is neither anti-symmetric nor symmetric,} \\ & \text{and} f(Q) \neq f(-Q). \end{cases}$$

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