

# An Individual Risk Model and Its Uncertainty Distribution

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## ABSTRACT

In insurance statistics, the sum of homogeneous insurance is often needed. The sum is called individual risk model which is a fundamental model in risk analysis for insurance. This paper first presents an individual risk model based on the uncertainty theory. Then its uncertainty distribution is provided. Finally, its arithmetic is shown by a numerical example.

Keywords: Uncertainty Theory, Individual Risk Model, Uncertainty Distribution

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## 1. INTRODUCTION

It is well known that there are many types of uncertainty in real life. A. N. Kolmogoroff (1933) presented the probability theory dealing with a type of uncertainty that an event will happen (called randomness). Besides, Zadeh (1965) initialized the fuzzy set theory via membership function that aimed to study the uncertainty called fuzziness. The probability theory and the fuzzy set theory have been widely accepted to study uncertainty and have also been applied to many aspects of our life.

However, there are still many other uncertainty that can not be described by the probability theory and the fuzzy set theory. In order to rationally deal with the degree of belief, an uncertainty theory was introduced by Liu (2007). The uncertainty theory is a branch of axiomatic mathematics based on normality, duality, subadditivity and product axioms. And it was refined by Liu (2010a). Nowadays, the uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. Uncertain statistics is an important branch for uncertainty theory. It was introduced by Liu (2010a). Chen and Ralescu (2012) stated that the questionnaire survey may successfully acquire the expert's experimental data. Wang *et al.*, (2012) recast the Delphi

method as a process to determine the uncertainty distributions. Liu (2010a) suggested the principle of least squares method to propose an uncertainty distribution. Furthermore, the uncertainty theory has been applied to uncertain programming (Liu, 2009a; Gao, 2011, 2012; Meng and Zhang, 2013; Peng and Yao, 2011; Rong, 2011; Sheng and Yao, 2012; Zhang and Chen, 2012; Zhang and Meng, 2013), uncertain risk analysis (Liu, 2010b), uncertain logic (Chen *et al.*, 2012; Li and Liu, 2009; Zhang, 2011; Zhang and Peng, n.d.) uncertain process (Yao and Li, 2012), and others (Dai and Chen, 2012; Chen and Ralescu, 2011; Gao, 2009; Zhang *et al.*, 2013; Zhu, 2010).

In order to study dynamic systems in uncertainty theory, Liu (2009b) introduced uncertain process which is essentially a sequence of uncertain variables indexed by time. There is an important type of uncertain process called renewal process, which is an uncertain process in which events occur continuously and independently of one another in uncertain times. Yao and Li (2012) studied the uncertain alternating renewal process. Based on the uncertain renewal process, Yao and Ralescu (2013) investigated the age replacement policy in uncertain environment. Liu (2013) presented an uncertain insurance model by assuming that the claim is an uncertain renewal reward process, and proved a formula for calcula-

ting ruin index. Different from it, this paper aims to study insurance by uncertain statistics. We will build a flow shop scheduling model in the uncertain environment based on the uncertainty theory. It is well known that in insurance statistics the sum of homogeneous insurance is often needed, and the sum is called individual risk model, which is a fundamental model in risk analysis for insurance. These problems are studied by many scholars (Kass *et al.*, 2001; Peng, 2009; Wen and Wang, 2006).

The paper is organized as follows. Section 2 recalls several concepts of uncertain variables. Section 3 introduces an individual risk model and its uncertainty distribution. Section 4 gives a numerical example. Finally, a brief summary is included.

## 2. PRELIMINARIES

In this section, we introduce some foundational concepts and properties of uncertainty theory, which will be used throughout this paper.

Let  $\Gamma$  be a nonempty set, and  $L$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda \in L$  is called an event.

**Definition 1** (Liu, 2007). The set function  $M$  is called an uncertain measure if it satisfies the following three axioms:

- Axiom 1** (Normality).  $M\{\Gamma\} = 1$ ;
- Axiom 2** (Self-Duality).  $M\{\Lambda\} + M\{\Lambda^c\} = 1$  for any event  $\Lambda$ ;
- Axiom 3** (Countable Subadditivity). For every countable sequence of events  $\{\Lambda_i\}$ , we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Liu (2009b) presented the product measure axiom of uncertainty theory in 2009 as follows:

- Axiom 4** (Product Measure Axiom). Let  $\Gamma_k$  be nonempty sets on which  $M_k$  are uncertain measures,  $k = 1, 2, \dots, n$ , respectively. Then the product uncertain measure  $M$  is an uncertain measure on the product  $\sigma$ -algebra  $L_1 \times L_2 \times \dots \times L_n$  satisfying

$$M\left\{\prod_{k=1}^n \Lambda_k\right\} = \min_{1 \leq k \leq n} M_k\{\Lambda_k\}$$

That is, for each event  $\Lambda \in L$ , we have

$$M\{\Lambda\} = \begin{cases} \sup_{\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k\{\Lambda_k\} & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k\{\Lambda_k\} > 0.5 \\ 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k\{\Lambda_k\} & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda} \min_{1 \leq k \leq n} M_k\{\Lambda_k\} > 0.5 \\ 0.5 & \text{otherwise.} \end{cases}$$

**Definition 2** (Liu, 2010a). An uncertain variable is a measurable function  $\xi$  from an uncertainty space  $(\Gamma, L, M)$  to the set of real numbers, i.e., for any Borel set  $B$  of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

**Definition 3** (Liu, 2010a). The uncertainty distribution  $\Phi$  of an uncertain variable  $\xi$  is defined by

$$\Phi(x) = M\{\xi \leq x\}$$

for any real number  $x$ .

**Definition 4** (Liu, 2010a). An uncertain variable  $\xi$  is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ (x-a)/(b-a), & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases}$$

denoted by  $L(a, b)$  where  $a$  and  $b$  are real numbers with  $a < b$ .

**Definition 5** (Liu, 2010a). An uncertain variable  $\xi$  is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ (x-a)/2(b-a), & \text{if } a \leq x \leq b \\ (x+c-2b)/2(c-b), & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c \end{cases}$$

denoted by  $Z(a, b, c)$  where  $a, b, c$  are real numbers with  $a < b < c$ .

**Definition 6** (Liu, 2010a). An uncertain variable  $\xi$  is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

denoted by  $N(e, \sigma)$  where  $e$  and  $\sigma$  are real numbers with  $\sigma > 0$ .

**Definition 7** (Liu, 2010a). An uncertain variable  $\xi$  is called lognormal if  $\ln \xi$  is a normal uncertainty variable  $N(e, \sigma)$ . In other words, a lognormal uncertain variable has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

denoted by  $LOGN(e, \sigma)$  where  $e$  and  $\sigma$  are real num-

bers with  $\sigma > 0$ .

**Definition 8** (Liu, 2010a). The uncertain variables  $X_1, X_2, \dots, X_m$  are said to be independent if

$$M\left\{\prod_{i=1}^m X_i \in B_i\right\} = \min_{1 \leq i \leq m} M\{X_i \in B_i\}$$

for any Borel sets  $B_1, B_2, \dots, B_m$  of real numbers.

**Definition 9** (Liu, 2010a). An uncertainty distribution  $\Phi$  is said to be regular if its inverse function  $\Phi^{-1}(\alpha)$  exists and is unique for each  $\alpha \in (0, 1)$ .

**Theorem 1** (Liu, 2010a). Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f$  is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \dots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)).$$

**Theorem 2** (Liu, 2010a). Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f$  is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \dots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \dots, x_n)} \min_{1 \leq i \leq n} \Phi_i(x_i).$$

### 3. AN INDIVIDUAL RISK MODEL AND ITS UNCERTAINTY DISTRIBUTION

In this section, we will present an individual risk model based on the uncertainty theory. It is a fundamental model in risk analysis for insurance. Assume that in a stated time (such as one year's time), the insurer has done business with  $n$  customers, and every customer bought  $N$  policies. It is well known that in an insurance company, if customers have some prescriptive losses, then the insurer should shell out the payment. It is evident that the payment is an uncertain variable. Thus, we have the the following definition:

**Definition 10.** In a stated time, the total indemnity for the insurer is given by

$$S = \sum_{j=1}^N \sum_{i=1}^{n_j} \xi_{ij}, \quad (1)$$

which is called individual risk model, where  $\xi_{ij}$  are the payment for the  $j$ th insurance contract of  $i$ th insured, and  $n_j$  customers subscribe the  $j$ th insurance contract.

The quantity of the insurance contracts is supplied by the insurer.

Commonly, it is hard to obtain the accurate distribution of  $S$ . In order to study the distribution of  $S$  and obtain its properties, some hypotheses should be given.

- 1) The uncertainty distribution  $\varphi_{ij}$  of every uncertain variable  $\xi_{ij}$  are as follows:

$$\varphi_{ij}(x) = M\{\xi_{ij} \leq x\} = \begin{cases} 0, & \text{if } x < 0 \\ p(x), & \text{if } 0 \leq x < b \\ 1, & \text{if } x \geq b, \end{cases}$$

where  $0$  is the minimum payment,  $b$  is the maximum payment and  $p(x)$  is a strictly increasing function.

- 2) Uncertain variables  $\xi_{ij}$  are independent.
- 3) Every policy settles a claim once at most.
- 4) The homogeneous insurances have characteristic of the same insurance marks. From the thought of uncertainty theory, this kind of homogeneity shows that the payment variables  $\xi_{ij}$  of the same kind insurance policy are iid uncertain variables. This means

$$\varphi_{1j}(x) = \varphi_{2j}(x) = \varphi_{3j}(x) = \dots = \varphi_{n_jj}(x).$$

Based on the above hypothesis, we give the distribution of  $S$  in the following.

**Theorem 3.** In the individual risk model with the limit of  $N = 1$ , then  $S = \sum_{i=1}^n \xi_i$  is an uncertain variable with the uncertainty distribution

$$\Phi(x) = \varphi\left(\frac{x}{n}\right),$$

and its inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = n\varphi^{-1}(\alpha),$$

where  $\xi_1, \xi_2, \dots, \xi_n$  are independent uncertain variables with the common uncertainty distribution  $\varphi(x)$ .

**Proof.** Note that  $\xi_1, \xi_2, \dots, \xi_n$  are independent uncertain variables with the common uncertainty distribution  $\varphi(\alpha)$ . By Theorem 2, we have

$$\Phi(x) = \sup_{x_1 + x_2 + \dots + x_n = x} \min_{1 \leq i \leq n} \varphi\{x_i\}.$$

It is evident that  $n \times \frac{x}{n} = x$  and

$$\varphi\left(\frac{x}{n}\right) \wedge \varphi\left(\frac{x}{n}\right) \wedge \dots \wedge \varphi\left(\frac{x}{n}\right) = \varphi\left(\frac{x}{n}\right).$$

Counterevidence. Suppose  $\Phi(x) \neq \varphi\left(\frac{x}{n}\right)$ . If  $\Phi(x) > \varphi\left(\frac{x}{n}\right)$ , then there exists  $x_1, x_2, \dots, x_n$  such that  $x_1 + x_2 + \dots + x_n = x$  and  $\varphi(x_i) > \varphi\left(\frac{x}{n}\right)$ ,  $i=1, 2, \dots, n$ . Since  $\varphi(x)$  is a increasing function,  $x_2 > \frac{x}{n}$ , this is inconsistent with  $x_1 + x_2 + \dots + x_n = x$ . Thus

$$\Phi(x) = \varphi\left(\frac{x}{n}\right).$$

Since  $S = \sum_{i=1}^n \xi_i$  and it increase on each  $\xi_i$ ,  $i=1, 2, \dots, n$ , by Theorem 1 we have

$$\Phi^{-1}(\alpha) = \varphi^{-1}(\alpha) + \varphi^{-1}(\alpha) + \dots + \varphi^{-1}(\alpha) = n\varphi^{-1}(\alpha).$$

The theorem is proved.

**Theorem 4.** In the above individual risk model,  $S = \sum_{j=1}^N \sum_{i=1}^{n_j} \xi_{ij}$  is an uncertain variable with the uncertainty distribution

$$\Psi(x) = \sup_{x_1+x_2+\dots+x_N=x} \min_{1 \leq j \leq N} \Phi_j\left(\frac{x_j}{n_j}\right),$$

and its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = \sum_{j=1}^N n_j \Phi_j^{-1}(\alpha),$$

where  $\Phi_j(x_j) = M\{\xi_{ij} < x_j\}$ ,  $i \in \{1, 2, \dots, n_j\}$ ,  $0 < j \leq N$ .

**Proof.** Let

$$\xi_j = \sum_{i=1}^{n_j} \xi_{ij}.$$

Note that  $\xi_{1j}, \xi_{2j}, \dots, \xi_{n_j j}$  are independent uncertain variables with the common uncertainty distribution  $\varphi(x)$ . Since  $\Phi_j(x) = M\{\xi_{ij} < x_j\}$ ,  $i \in \{1, 2, \dots, n_j\}$ ,  $0 < j \leq N$ , by Theorem 3, we have  $\Phi_j(x_j) = \varphi_j\left(\frac{x_j}{n_j}\right)$ . Since  $S = \sum_{j=1}^N \sum_{i=1}^{n_j} \xi_{ij} = \sum_{j=1}^N \xi_j$ , by Theorem 2, we have

$$\psi(x) = \sup_{x_1+x_2+\dots+x_N=x} \min_{0 < j \leq N} \Phi_j\left(\frac{x_j}{n_j}\right).$$

By Theorem 3 and Theorem 1, inverse uncertainty distribution of  $S = \sum_{j=1}^N \sum_{i=1}^{n_j} \xi_{ij} = \sum_{j=1}^N \xi_j$  is

$$\psi^{-1}(\alpha) = \sum_{j=1}^N n_j \Phi_j^{-1}(\alpha).$$

The theorem is proved.

#### 4. A NUMERICAL EXAMPLE

Assume that the insurers have sold four kinds of policies to a fellowship with 1,000 members, and the distributions of these policies are as follows,

$$\varphi_{i1}(x) = M\{\xi_{i1} \leq x\} = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x < 2 \\ 1, & \text{if } x \geq 2, \end{cases}$$

$$\varphi_{i2}(x) = M\{\xi_{i2} \leq x\} = \begin{cases} 0, & \text{if } x < 0 \\ x/2, & \text{if } 0 \leq x < 1 \\ (x+1)/4, & \text{if } 1 \leq x \leq 3 \\ 1, & \text{if } x \geq 3, \end{cases}$$

$$\varphi_{i3}(x) = M\{\xi_{i3} \leq x\} = \begin{cases} 0, & \text{if } x < 0 \\ (1 + \exp(\pi(2-x)))^{-1}, & \text{if } 0 \leq x < 4 \\ 1, & \text{if } x \geq 4, \end{cases}$$

$$\varphi_{i4}(x) = M\{\xi_{i4} \leq x\} = \begin{cases} 0, & \text{if } x < 0 \\ (1 + \exp(\frac{\pi(1-\ln x)}{2}))^{-1}, & \text{if } 0 \leq x < 5 \\ 1, & \text{if } x \geq 5. \end{cases}$$

Then by the distribution of it we can get the uncertainty distribution for the total payments lacked in the coming year as follow:

$$\Psi(x) = \sup_{x_1+x_2+x_3+x_4=x} \varphi_{i1}\left(\frac{x_1}{1000}\right) \wedge \varphi_{i2}\left(\frac{x_2}{1000}\right) \wedge \varphi_{i3}\left(\frac{x_3}{1000}\right) \wedge \varphi_{i4}\left(\frac{x_4}{1000}\right) = \begin{cases} 0, & \text{if } x < 0 \\ (1 + \exp(\frac{\pi(1-\ln x)}{2}))^{-1}, & \text{if } 0 \leq x < 14000 \\ 1, & \text{if } x \geq 14000. \end{cases}$$

#### 5. CONCLUSION

This paper provided an individual risk model via uncertainty theory and gave its uncertainty distribution. The model and its arithmetic were shown by a numerical example.

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