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A Note on the Median Control Chart

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Abstract

This study reviews several well-known control charts for the location parameter with a discussion of the relationship between the maintenance of the control chart and a series of hypotheses testing. As a by-product, we then propose a new median control chart with the sign test statistic. We also modify the nonparametric control charts to easily understand the relation. Then we illustrate the construction of several median control charts with the industrial data and compare the efficiency among several control charts. Finally, we discuss some interesting features for the median control charts as concluding remarks.

Keywords: Median control chart, sign test, signed-rank test.

1. Introduction

For the detection of any shift of location in the production process, $\bar{X}$-chart has been widely used under the normality assumption for the process distribution. As an alternative control chart for the detection of location shift, a median control chart can also be applied when the distribution of the process may not be normal. Many median control charts have been proposed using the limit distribution of a sample median with various algorithms for the estimation of variance according to asymptotic normality (Nelson, 1982; Janacek and Meikle, 1997; Park, 2008, 2009). The non-availability of exact distribution of any sample median is the main reason for the difficulty of the construction of the median control charts. In order to detour this difficulty, the Hodges-Lehmann estimates with related interval estimates for median have been used to construct another type of median control charts (Farnum and Stanton, 1986; Alloway and Raghavachari, 1991).

We note that the maintenance of any type of the control chart for location shift would be equivalent to a series of hypothesis tests for the mean or median. If a point from a sample plots out of the pre-assigned control limits, then that point leads to a rejection of the null hypothesis for the two-sided test. We explain this development with a $\bar{X}$-chart to detail the relation of the control charts with the hypothesis test. For this, let μ and μ_0 be the mean of the process distribution and the target value of the process. Then μ_0 is the central line of the $\bar{X}$-chart. For testing

$$H_0 : \mu = \mu_0 \quad \text{against} \quad H_1 : \mu \neq \mu_0$$

based on a sample with sample size n, one may reject $H_0 : \mu = \mu_0$ under the significance level 0.0026 if $|\bar{X} - \mu_0| > 3\sqrt{\sigma^2/n}$, where σ^2 is the variance of the process distribution. In addition, the process is out of control when $|\bar{X} - \mu_0| > 3\sqrt{\sigma^2/n}$ if one adopts $3 - \sigma$ control limits. This shows the equivalence between the control chart and a series of hypothesis tests.

Section 2 reviews several median control charts with a discussion on the relationship of the maintenance of the control chart and a series of hypotheses testing. As a by-product, we propose a new

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median control chart based on a sign test. Section 3 illustrates the construction of several median control charts with industrial data and compares the efficiency of the control charts by obtaining the empirical probabilities of detection of out of the control state by varying the value of the location parameter through a simulation study. Finally, we discuss some interesting features of the median control charts in the concluding remarks of Section 4.

2. Median Control Charts

In this section, we review and detail some existing median control charts with a discussion on the relationship of the maintenance of the control chart and a series of hypotheses testing. We also propose a new median control chart as a by-product. For this, let F be the unknown continuous process distribution with median θ and $X_1, \ldots, X_n$, a sample from the production process. Then based on the sample, we determine that the process is in control or not using the established median control charts listed in Section 2.1.

2.1. Median control charts based on the estimated limiting variance

Many median control charts have been proposed based on various algorithms and approaches. For the simplicity of our discussion, let θ_0 be the target value for median θ in the production process. Then θ_0 can be used as the central line(CL) for a median control chart. In order to decide that the process is in control or not with $\hat{\theta}$, an estimate of θ from $X_1, \ldots, X_n$, we determine the lower and upper control limits denoted by LCL and UCL. First, we need the distribution of $\hat{\theta}$; subsequently, since the exact distribution of $\hat{\theta}$ is unknown, it is customary to apply the asymptotic normality for $\hat{\theta}$ using the large sample approximation theory. It would be inevitable to consistently estimate the limiting variance, $1/(4f^2(\theta))$ of $\sqrt{n}(\hat{\theta} - \theta)$, where f is the unknown density of F. The variety of the median control charts comes from a group of various consistent estimates of $1/(4f^2(\theta))$ (Nelson, 1982; Janacek and Meikle, 1997; Khoo, 2005). Park (2008, 2009) proposed several median control charts through an estimation of the variance of $\hat{\theta}$ with various bootstrap methods. In addition, Park also considered the moment estimate approach proposed by Maritz and Jarrett (1978) for the estimation of the variance of $\hat{\theta}$. Then with any consistent estimate $AV(\hat{\theta})$ of $1/(4f^2(\theta))$, a typical median control chart with $3 - \sigma$ control limits can be constructed by

(2.1.1) $\quad \text{CL} = \theta_0$

(2.1.2) $\quad \text{LCL} = \theta_0 - 3\sqrt{\text{AV}(\hat{\theta})/n} \quad \text{and} \quad \text{UCL} = \theta_0 + 3\sqrt{\text{AV}(\hat{\theta})/n},$

where n denotes the sample or subgroup size.

We note that for testing

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0 \tag{2.1}$$

with $\text{AV}(\hat{\theta})$, one may reject $H_0 : \theta = \theta_0$ in favor of $H_1 : \theta \neq \theta_0$ under the approximate significance level 0.0026 when the process is determined to be out of control with $3 - \sigma$ control limits and vice versa. In addition, one may consider the construction of median control charts using nonparametric test statistics for the one-sample problem shown in Section 2.2.

2.2. Median control charts from the Hodges-Lehmann estimates

For testing (2.1) with $X_1, \ldots, X_n$, one may consider applying the signed rank statistic W^+, which is defined as

$$W^+ = \sum_{i=1}^{n} I(X_i > \theta_0) R_i^+,$$

where $I(\cdot)$ is the indicator function and R^+ is the rank of $|X_i - \theta_0|$ among $|X_1 - \theta_0|, \ldots, |X_n - \theta_0|$. Let $w^+(\alpha, n)$ be the upper $\alpha \times 100\%$th percentile point of the null distribution of W^+ such that $\Pr\{W^+ \geq w^+(\alpha, n)|H_0\} = \alpha$. Then one may reject $H_0 : \theta = \theta_0$ in favor of $H_1 : \theta \neq \theta_0$ for $W^+ \geq w^+(\alpha/2, n)$ or $W^+ \leq n(n+1)/2 - w^+(\alpha/2, n)$ for any given significance level α. In order to discuss the Hodges-Lehmann estimate (Randles and Wolfe, 1979) of θ based on W^+, let $W_{ij} = (X_i + X_j)/2$, $1 \leq i \leq j \leq n$ be the Walsh averages. In addition, let W_k be the kth order statistic among $n(n+1)/2$ number of W_{ij}'s. Then the Hodges-Lehmann estimate of θ can be obtained from W^+ as

$$\tilde{\theta} = \text{med}\left\{W_{ij}, 1 \leq i \leq j \leq n\right\}. \tag{2.2}$$

In addition, the corresponding $(1 - \alpha) \times 100\%$ confidence interval (Randles and Wolfe, 1979) can be obtained as

$$\left[W_{\left(\frac{n(n+1)}{2}+1-w^+\left(\frac{\alpha}{2},n\right)\right)}, W_{\left(w^+\left(\frac{\alpha}{2},n\right)\right)}\right). \tag{2.3}$$

Alloway and Raghavachari (1991) proposed median control charts based on (2.2) and (2.3) to determine CL, LCL and UCL by several subgroups with varying the sample sizes. Then by abusing notation, we can summarize the corresponding median control chart proposed by Alloway and Raghavachari (1991) as

(2.2.1) $\text{CL} = \tilde{\theta}$
(2.2.2) $\text{LCL} = W_{(n(n+1)/2+1-w^+(\alpha/2,n))}$ and $\text{UCL} = W_{(w^+(\alpha/2,n))}$.

We note that the equivalence between the test for (2.1) for the signed rank test and control chart may not hold exactly; however, the consistency of those related statistics implies that the equivalence between both would asymptotically hold.

In addition, one may apply the sign test for testing (2.1), whose test statistic B can be defined as

$$B = \sum_{i=1}^{n} I(X_i > \theta_0).$$

We may then propose a new median control chart using the Hodges-Lehmann estimate based on B with its related confidence interval for θ along the way of the idea of Alloway and Raghavachari (1991). For this purpose, let $b(\alpha, n, 1/2)$ be the upper $\alpha \times 100\%$th percentile point for the binomial $(n, 1/2)$ distribution such that $\Pr\{B \geq b(\alpha, n, 1/2)|H_0\} = \alpha$. Then for testing (2.1), we may reject $H_0 : \theta = \theta_0$ for any given significance level α when $B \geq b(\alpha/2, n, 1/2)$ or $B \leq n - b(\alpha/2, n, 1/2)$. Then based on B, we note that the Hodges-Lehmann estimate of θ is

$$\hat{\theta} = \text{med}\{X_1, \ldots, X_n\}, \tag{2.4}$$

which is the usual estimate of median θ. In addition, the $(1 - \alpha) \times 100\%$ nonparametric confidence interval for θ based on B can be obtained as

$$\left[X_{\left(n+1-b\left(\frac{\alpha}{2},n,\frac{1}{2}\right)\right)}, X_{\left(b\left(\frac{\alpha}{2},n,\frac{1}{2}\right)\right)}\right), \tag{2.5}$$

where X_k means the kth order statistic from $X_1, \ldots, X_n$. Then we propose a new median control chart using (2.4) and (2.5) as follows.

(2.2.3) $\quad \mathrm{CL} = \hat{\theta}$
(2.2.4) $\quad \mathrm{LCL} = X_{(n+1-b(\alpha/2,n,1/2))}$ and $\mathrm{UCL} = X_{(b(\alpha/2,n,1/2))}$.

Section 2.2 shows that the confidence intervals are equivalent or at least asymptotically equivalent to the two-sided nonparametric tests and also the nonparametric two-sided tests are equivalent to the median control charts for the location parameters in the sense that if the null hypothesis $H_0 : \theta = \theta_0$ is not rejected under the significance level α, then θ_0 should be contained in the $(1 - \alpha) \times 100\%$ confidence interval and $\theta_0 \in (\mathrm{LCL}, \mathrm{UCL})$ when the $(1 - \alpha) \times 100\%$ control limits is used. Therefore we could construct median control charts with the test statistics, B and W^+. In the following, we review median control charts that use the nonparametric tests directly. Some authors (Chakraborti and Eryilmaz, 2007) denote these median control charts as nonparametric control charts.

2.3. The nonparametric control chart based on the sign test

Farnum and Stanton (1986) proposed a control chart using the sign test to determine the control limits. However we cannot say exactly that the control chart proposed by them should be a nonparametric one since they assume that the underlying process distribution is normal and interested in the mean (not median). In this study, we modify the control chart of Farnum and Stanton (1986) into the nonparametric control chart based on a sign test that can be constructed in the following method with the sample $X_1, \ldots, X_n$. In the following, we note that $n/2$ is the mean of B when $H_0 : \theta = \theta_0$ is true.

(2.3.1) $\quad \mathrm{CL} = n/2$
(2.3.2) $\quad \mathrm{LCL} = n - b(\alpha/2, n, 1/2)$ and $\mathrm{UCL} = b(\alpha/2, n, 1/2)$.

In general, one cannot choose $\alpha = 0.0026$ exactly since the null distribution of B is the binomial $(n, 1/2)$ which is discrete; subsequently, one should choose an as close to 0.0026 as possible.

2.4. The nonparametric control chart based on the signed rank test

Bakir (2004) proposed a nonparametric control chart based on the signed rank statistic W^+. In the following $n(n + 1)/4$ is the mean of W^+ when $H_0 : \theta = \theta_0$ is true. With this, one can construct a nonparametric control chart using the signed rank test as follows.

(2.4.1) $\quad \mathrm{CL} = n(n + 1)/4$
(2.4.2) $\quad \mathrm{LCL} = n(n + 1)/2 - w^+(\alpha/2, n)$ and $\mathrm{UCL} = w^+(\alpha/2, n)$.

The distribution of W^+ is also discrete; therefore, the value of α should be chosen as close to 0.0026 as possible for the $3 - \sigma$ control limits.

Then the nonparametric control charts are equivalent to the corresponding nonparametric tests. In addition, we note that the maintenance of the nonparametric control charts are simple and easy for everyone in the production floor but require some knowledge about a nonparametric test procedure, which makes difficult to implement on the floor.

3. A Numerical Example and Comparison Study

In this section, we first illustrate the construction of several median control charts with the data of the primary thickness problem from the Ford Motor Company (Alloway and Raghavachari, 1991). The

Table 1: Construction of median control charts

Control Chart	CL	LCL	UCL
M_B	1.1225	0.995	1.246
M_W	1.1190	0.950	1.310
M_S	1.1225	0.950	1.310

Table 2: Empirical probability of falling outside of control limits(Normal)

Control Chart	n	θ						
		0.0	0.3	0.6	0.9	1.2	1.5	1.8
Sign-chart	10	0.0021	0.0079	0.0399	0.1250	0.2880	0.4905	0.6789
	15	0.0007	0.0072	0.0503	0.2008	0.4667	0.7308	0.9006
	20	0.0024	0.0220	0.1508	0.4788	0.8053	0.9579	0.9952
Signed rank-chart	10	0.0032	0.0079	0.0399	0.1250	0.2880	0.4905	0.6789
	15	0.0037	0.0205	0.1346	0.4399	0.7773	0.9516	0.9953
	20	0.0015	0.0163	0.1679	0.5620	0.8977	0.9916	0.9998
Median chart	10	0.0012	0.0076	0.0448	0.1923	0.4679	0.7664	0.9956
	15	0.0018	0.0149	0.1138	0.3981	0.7509	0.9475	0.9955
	20	0.0018	0.0267	0.2095	0.6149	0.9210	0.9939	1.0000

data consist of 20 subgroups with sample size 10 each. From the data, we obtained CL, LCL and UCL for the median control chart(M_B) with the bootstrap method, and two median control charts(M_W and M_S) with the Hodges-Lehmann estimates based on the signed rank and sign statistics, respectively. Table 1 summarizes the results. We note that CL is just the usual sample median($\hat{\theta}$) for M_B and M_S while a median of 210 Walsh averages($\tilde{\theta}$) for M_W. For more detailed information and procedure for the construction of median control charts, please refer to Alloway and Raghavachari (1991) and Park (2009).

We compare the efficiency of nonparametric control charts with a median control chart (Park, 2008) whose control limits were derived by the standard bootstrap method by obtaining empirical probabilities that count the number of points which fall outside control limits through a simulation study. The empirical probabilities correspond to the empirical powers if we consider them as tests. For this study, we consider four different distributions- normal, Cauchy, exponential and double exponential distributions-with unit variance (normal, exponential and double exponential) or unit scale parameter (Cauchy) and three different sample sizes, 10, 15 and 20 to investigate the behavior of median control charts for the small sample case. The control limits for the median control chart have been obtained through simulation study using Maritz and Jarrett's estimation procedure. We varied the values of median from 0 to 1.8 with 0.3 increments. The value 0 for θ implies the target value of median in the production process. The fact that the distributions of B and W^+ are discrete makes hard to choose the exact $3-\sigma$ control limits for each control chart. However, we have given efforts to choose as close as possible. Table 2–Table 5 summarizes the results obtained with 10,000 simulations with SAS/IML PC version.

For the normal, Cauchy and double exponential distributions, the median control chart with bootstrap control limits shows a higher performance than other nonparametric control charts. However for the exponential distribution, W^+-chart provides excellent performance. We note that the normal, Cauchy and double exponential distributions are all symmetric. However, for the exponential distribution, we note that the nonparametric control charts show extremely efficient results. Therefore when the process distribution is asymmetric, the nonparametric control charts may be useful alternatives for median control charts.

Table 3: Empirical probability of falling outside of control limits(Cauchy)

Control Chart	n	θ						
		0.0	0.3	0.6	0.9	1.2	1.5	1.8
Sign-chart	10	0.0021	0.0057	0.0174	0.0434	0.0810	0.1229	0.1673
	15	0.0008	0.0035	0.0199	0.0602	0.1208	0.1936	0.2741
	20	0.0020	0.0155	0.0680	0.1768	0.3268	0.4683	0.5896
Signed rank-chart	10	0.0023	0.0057	0.0174	0.0434	0.0810	0.1229	0.1673
	15	0.0030	0.0091	0.0328	0.0786	0.1368	0.1998	0.2594
	20	0.0015	0.0097	0.0446	0.1188	0.2198	0.3262	0.4206
Median chart	10	0.0032	0.0034	0.0062	0.0112	0.0216	0.0446	0.1109
	15	0.0030	0.0060	0.0177	0.0568	0.1575	0.3873	0.6771
	20	0.0051	0.0101	0.0452	0.1699	0.4631	0.7747	0.9405

Table 4: Empirical probability of falling outside of control limits(Exponential)

Control Chart	n	θ						
		0.0	0.3	0.6	0.9	1.2	1.5	1.8
Sign-chart	10	0.0020	0.0206	0.3921	1.0000			
	15	0.0039	0.0253	0.6110	1.0000			
	20	0.0025	0.0722	0.9089	1.0000			
Signed rank-chart	10	0.0021	0.0207	0.3921	1.0000			
	15	0.0027	0.1803	0.9936	1.0000			
	20	0.0027	0.1803	0.9936	1.0000			
Median chart	10	0.0112	0.0492	0.1703	0.4693	0.8639	0.9993	1.0000
	15	0.0086	0.0522	0.2319	0.6662	0.9832	1.0000	
	20	0.0069	0.0740	0.3940	0.8965	1.0000		

Table 5: Empirical probability of falling outside of control limits(Double exponential)

Control Chart	n	θ						
		0.0	0.3	0.6	0.9	1.2	1.5	1.8
Sign-chart	10	0.0020	0.0205	0.0931	0.2223	0.3928	0.5401	0.6672
	15	0.0008	0.0204	0.1365	0.3502	0.5921	0.7677	0.8796
	20	0.0025	0.0699	0.3550	0.6997	0.8937	0.9705	0.9930
Signed rank-chart	10	0.0030	0.0205	0.0931	0.2223	0.3928	0.5401	0.6672
	15	0.0027	0.0388	0.2288	0.5185	0.7637	0.8965	0.9570
	20	0.0026	0.0624	0.3581	0.7268	0.9257	0.9854	0.9977
Median chart	10	0.0040	0.0163	0.1279	0.5233	0.8885	0.9847	0.9981
	15	0.0039	0.0273	0.2538	0.8131	0.9822	0.9987	0.9999
	20	0.0041	0.0596	0.5398	0.9587	0.9991	1.0000	

4. Concluding Remarks

The main difficulty to establish any median control chart is due to the non-availability of the exact distribution of sample median since it would be impossible to determine any reasonable control limits without knowledge of the distribution. The control limits would be different according to subgroups even though they are consistent in the limiting sense even for those which use the Hodges-Lehmann estimates since the decision for the control limits depends solely on the subgroups. However this difficulty disappears and the exact control limits can be obtained if one uses the nonparametric test statistics directly for the control chart.

We note that the procedure of construction of the control charts based on the nonparametric statistics is relatively simple and easy. In addition, the maintenance of the control charts may be relatively easy but the worker on the floor to maintain the control charts needs some advanced knowledge for the statistical tests. Therefore, it would be necessary to educate the workers the statistical inference in order to introduce the median control charts based on B or W^+.

The maintenance of the proposed median control charts assumes the non-variability of scale parameter or variance of the process distribution. Especially, the proposed median control charts would be non-sensitive at the scale change since the statistics B and W^+ only consider the changes of median.

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