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REDUCING SUBSPACES FOR TOEPLITZ OPERATORS ON THE POLYDISK

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Abstract. In this note, we completely characterize the reducing subspaces of $T_{z_1^N z_2^M}$ on $A_\alpha^2(D^2)$ where $\alpha > -1$ and N, M are positive integers with $N \neq M$, and show that the minimal reducing subspaces of $T_{z_1^N z_2^M}$ on the unweighted Bergman space and on the weighted Bergman space are different.

1. Introduction

Let D denote the open unit disk in the complex plane. For $-1 < \alpha < +\infty$, $L^2(D,\ dA_\alpha)$ is the space of functions on D which are square integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$, where dA denotes the normalized Lebesgue area measure on D. $L^2(D,\ dA_\alpha)$ is a Hilbert space with the inner product $\langle f,\ g\rangle_\alpha = \int_D f(z)\overline{g(z)}dA_\alpha$. The weighted Bergman space A_α^2 is the closed subspace of $L^2(D,\ dA_\alpha)$ consisting of analytic functions on D. If $\alpha = 0$, A_0^2 is the Bergman space. We write $A^2{=}A_0^2$. It is known that $\{\frac{z^n}{\|z^n\|_\alpha}\}_{n=0}^{+\infty}$ is an orthogonal basis of $A_\alpha^2(D)$. Let $\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}}$ for $n = 0, 1, 2, \ldots$. Therefore,

$$\|f\|_\alpha^2 = \sum_{n=0}^{+\infty} \gamma_n^2 |a_n|^2 < \infty,$$

with $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A_\alpha^2(D)$.

Denote the unit polydisk by D^n. The weighted Bergman space $A_\alpha^2(D^n)$ is then the space of all holomorphic functions on $L^2(D^n, dv_\alpha)$, where $dv_\alpha(z) = dA_\alpha(z_1)\cdots dA_\alpha(z_n)$. For multi-index $\beta = (\beta_1, \ldots, \beta_n)$, $\beta \succeq 0$ means that

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$\beta_i \geq 0$ for any $i \geq 0$. Denote by $z^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n}$ and

$$e_\beta = \frac{z^\beta}{\gamma_{\beta_1} \cdots \gamma_{\beta_n}},$$

then $\{e_\beta\}_\beta$ is an orthogonal basis in $A^2_\alpha(D^n)$.

Let P be the Bergman orthogonal projection from $L^2(D^n)$ onto $A^2_\alpha(D^n)$. For a bounded measurable function $f \in L^\infty(D^n)$, the Toeplitz operator with symbol f is defined by $T_f h = P(fh)$ for every $h \in A^2_\alpha(D^n)$.

Recall that in a Hilbert space $\mathcal{H}$, a (closed) subspace $\mathcal{M}$ is called a reducing subspace of the operator T if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$. A nontrivial reducing subspace $\mathcal{M}$ is said to be minimal if the only reducing subspaces contained in $\mathcal{M}$ are $\mathcal{M}$ and $\{0\}$. On the Bergman space $A^2_\alpha(D)$, the reducing subspaces of the Toeplitz operators with finite Blaschke product simples are well studied (see [1, 2, 8] for example). On $A^2_\alpha(D^2)$, Y. Lu and X. Zhou [4] characterized the reducing subspaces of Toeplitz operators $T_{z_1^N z_2^N}$, $T_{z_1^N}$ and $T_{z_2^N}$.

In this note, we consider the reducing subspaces of the Toeplitz operators $T_{z_1^N z_2^M}$ on $A^2_\alpha(D^2)$ and $T_{z_i^N z_j^M}$ on $A^2_\alpha(D^n)$, where $N, M \geq 1$ are integers and $1 \leq i < j \leq n$. Usually, the Toeplitz operators on the unweighted Bergman space and the weighted Bergman space have similar properties (see [5, 6, 7, 9] for example). However, we obtain that the minimal reducing subspaces of $T_{z_1^N z_2^M}$ with $N \neq M$ on $A^2_\alpha(D^2)(\alpha \neq 0)$ are less then that on $A^2(D^2)$ (see Theorem 2.4 and Theorem 3.2).

2. The results on the Bergman space

Let M, N be integers with $M, N \geq 1$ and $M \neq N$. In this section, we consider the minimal reducing subspace of $T_{z_1^N z_2^M}$ on $A^2(D^2)$. Here $\gamma_k = \|z^k\|_0 = \sqrt{\frac{1}{k+1}}$. Let $\rho_1(k) = \frac{(k+1)N}{M} - 1$ and $\rho_2(k) = \frac{(k+1)M}{N} - 1$. Let $\mathcal{H}_{nm} = \mathrm{Span}\{z_1^n z_2^m,\ z_1^{\rho_1(m)} z_2^{\rho_2(n)}\}$ and P_{nm} be the orthogonal projection from $A^2_\alpha(D^2)$ onto $\mathcal{H}_{nm}$.

Lemma 2.1. *Let n, m, h be nonnegative integers. Then the following statements hold*:

(a) *if $\rho_1(m)$ is an integer, then $\rho_1(m + hM) = \rho_1(m) + hN$ is an integer for every $h \geq 0$;*

(b) *if $\rho_2(n)$ is an integer, then $\rho_2(n + hN) = \rho_2(n) + hM$ is an integer for every $h \geq 0$;*

(c) *if $\rho_1(m)$ and $\rho_2(n)$ are positive integers, then $\gamma_{\rho_1(m)}\gamma_{\rho_2(n)} = \gamma_m \gamma_n$;*

(d) *$\rho_1(\rho_2(n)) = n$ and $\rho_2(\rho_1(m)) = m$.*

Proof. Notice that if $\rho_1(m)$ and $\rho_2(n)$ are positive integers, then $\gamma_{\rho_1(m)} = \sqrt{\frac{M}{N}}\gamma_m$ and $\gamma_{\rho_2(n)} = \sqrt{\frac{N}{M}}\gamma_n$. So (c) holds. By the direct calculation, (a), (b) and (d) are obvious. $\square$

Theorem 2.2. *Let n, m be integers such that $0 \le n \le N-1$ or $0 \le m \le M-1$, and both of $\rho_1(m)$ and $\rho_2(n)$ are integers. Then for $a, b \in \mathbb{C}$,*

$$\mathcal{M} = \mathrm{Span}\{az_1^{n+hN} z_2^{m+hM} + bz_1^{\rho_1(m+hM)} z_2^{\rho_2(n+hN)}; h = 0, 1, 2, \ldots\}$$

is a minimal reducing subspace of $T_{z_1^N z_2^M}$ on the polydisk.

Proof. By Lemma 2.1(a) and (b), it is easy to check that $T_{z_1^N z_2^M}(\mathcal{M}) \subseteq \mathcal{M}$.

On the other hand,

$$\begin{aligned} T^*_{z_1^N z_2^M}(z_1^k z_2^l) &= \sum_{\beta \succeq 0} \langle T^*_{z_1^N z_2^M} z_1^k z_2^l, e^\beta \rangle e^\beta \\ &= \begin{cases} \frac{\gamma_k^2 \gamma_l^2}{\gamma_{k-N}^2 \gamma_{l-M}^2} z_1^{k-N} z_2^{l-M}, & \text{if} \quad k \ge N, l \ge M, \\ 0, & \text{if} \quad \text{others}. \end{cases} \end{aligned}$$

For each $h \ge 1$,

$$\begin{aligned} & T^*_{z_1^N z_2^M}(z_1^{n+hN} z_2^{m+hM}) \\ = \ & \frac{\gamma^2_{n+hN} \gamma^2_{m+hM}}{\gamma^2_{n+(h-1)N} \gamma^2_{m+(h-1)M}} z_1^{n+(h-1)N} z_2^{m+(h-1)M}, \\ & T^*_{z_1^N z_2^M}(z_1^{\rho_1(m+hM)} z_2^{\rho_2(n+hN)}) \\ = \ & \frac{\gamma^2_{\rho_1(m+hM)} \gamma^2_{\rho_2(n+hN)}}{\gamma^2_{\rho_1(m+hM)-N} \gamma^2_{\rho_2(n+hN)-M}} z_1^{\rho_1(m+hM)-N} z_2^{\rho_2(n+hN)-M}. \end{aligned}$$

Combining this with Lemma 2.1(c), it is easy to check that

$$\begin{aligned} & T^*_{z_1^N z_2^M}(az_1^{n+hN} z_2^{m+hM} + bz_1^{\rho_1(m+hM)} z_2^{\rho_2(n+hN)}) \\ = \ & \mu(az_1^{n+hN-N} z_2^{m+hM-M} + bz_1^{\rho_1(m+hM-M)} z_2^{\rho_2(n+hN-N)}) \in \mathcal{M}, \end{aligned}$$

where $\mu = \frac{\gamma^2_{n+hN} \gamma^2_{m+hM}}{\gamma^2_{n+(h-1)N} \gamma^2_{m+(h-1)M}} = \frac{\gamma^2_{\rho_1(m+hM)} \gamma^2_{\rho_2(n+hN)}}{\gamma^2_{\rho_1(m+hM)-N} \gamma^2_{\rho_2(n+hN)-M}}$.

Since $0 \le n \le N-1$ (or $0 \le m \le M-1$), we get $\rho_2(n) < M$ (or $\rho_1(m) < N$, respectively). Therefore, $T^*_{z_1^N z_2^M}(az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)}) = 0 \in \mathcal{M}$. So $T^*_{z_1^N z_2^M}(\mathcal{M}) \in \mathcal{M}$, which finishes the proof. □

Lemma 2.3. *Suppose $\mathcal{M} \neq 0$ is a reducing subspace of $T_{z_1^N z_2^M}$ in $A^2(D^2)$. Let $f = \sum_{(k,l) \succeq 0} a_{k,l} z_1^k z_2^l \in \mathcal{M}$. For each nonnegative integers n, m with $a_{nm} \neq 0$, the following statements hold:*

(I) *if $\rho_1(m), \rho_2(n)$ are integers and $a_{\rho_1(m)\rho_2(n)} \neq 0$, then*

$$a_{nm} z_1^n z_2^m + a_{\rho_1(m)\rho_2(n)} z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}.$$

(II) *if at least one of $\rho_1(m), \rho_2(n)$ is not an integer, or $a_{\rho_1(m)\rho_2(n)} = 0$, then $z_1^n z_2^m \in M$.*

Proof. For every integer $h \geq 0$, denote by $T_h = T_{z_1^{hN} z_2^{hM}}$. Notice that

$$T_h^* T_h(z_1^n z_2^m) = \frac{\gamma_{hN+n}^2 \gamma_{hM+m}^2}{\gamma_n^2 \gamma_m^2} z_1^n z_2^m \in \mathcal{M}, \forall n, m \geq 0. \tag{2.1}$$

Let $P_{\mathcal{M}}$ be the orthogonal projection from $A_\alpha^2(D)$ onto $\mathcal{M}$, then for nonnegative integers n, m, k, l,

$$\langle P_{\mathcal{M}} T_h^* T_h z_1^n z_2^m, z_1^k z_2^l \rangle = \langle T_h^* T_h P_{\mathcal{M}} z_1^n z_2^m, z_1^k z_2^l \rangle = \langle P_{\mathcal{M}} z_1^n z_2^m, T_h^* T_h z_1^k z_2^l \rangle.$$

Thus $\frac{\gamma_{hN+k}^2 \gamma_{hM+l}^2}{\gamma_k^2 \gamma_l^2} = \frac{\gamma_{hN+n}^2 \gamma_{hM+m}^2}{\gamma_n^2 \gamma_m^2}$. Equivalently,

$$\frac{(k+1)(l+1)}{(n+1)(m+1)} = \frac{(k+hN+1)(l+hM+1)}{(n+hN+1)(m+hM+1)}, \ h \geq 0. \tag{2.2}$$

We claim that $(k, l) = (n, m)$ or $(k, l) = (\rho_1(m), \rho_2(n))$. In fact, let $h \to +\infty$, then

$$(k+1)(l+1) = (n+1)(m+1). \tag{2.3}$$

It follows that $(k+hN+1)(l+hM+1) = (n+hN+1)(m+hM+1)$. Since $g(\lambda) = (k+\lambda N+1)(l+\lambda M+1) - (n+\lambda N+1)(m+\lambda M+1)$ is an analytic polynormal on $\mathbb{C}$, $g(\lambda) = 0$ for any $\lambda \in \mathbb{C}$. The coefficient of λ must be zero. We get

$$M(n-k) = N(l-m). \tag{2.4}$$

This together with (2.3) implies the claim.

Therefore, $P_{\mathcal{M}}(z_1^n z_2^m) \in \mathcal{H}_{nm}$. Hence,

$$P_{nm} P_{\mathcal{M}}(z_1^n z_2^m) = P_{\mathcal{M}}(z_1^n z_2^m).$$

Since $P_{\mathcal{M}} f = f$ for every $f \in \mathcal{M}$, we arrive to

$$\langle P_{\mathcal{M}} P_{nm} f, z_1^n z_2^m \rangle = \langle f, P_{nm} P_{\mathcal{M}} z_1^n z_2^m \rangle = \langle f, P_{\mathcal{M}} z_1^n z_2^m \rangle = \langle P_{nm} f, z_1^n z_2^m \rangle.$$

Notice that $\rho_2(\rho_1(m)) = m$, $\rho_1(\rho_2(n)) = n$ and $\mathcal{H}_{\rho_1(m)\rho_2(n)} = \mathcal{H}_{nm}$. Replacing n, m by $\rho_1(m)$ and $\rho_2(n)$, respectively, it is easy to get that

$$\langle P_{\mathcal{M}} P_{nm} f, z_1^{\rho_1(m)} z_2^{\rho_2(n)} \rangle = \langle P_{nm} f, z_1^{\rho_1(m)} z_2^{\rho_2(n)} \rangle.$$

Moreover, $\langle P_{\mathcal{M}} P_{nm} f, z_1^k z_2^l \rangle = \langle P_{nm} f, z_1^k z_2^l \rangle = 0$ for any $(k, l) \neq (\rho_1(m), \rho_2(n))$ and $(k, l) \neq (n, m)$. Hence $P_{nm} f = P_{\mathcal{M}} P_{nm}(f) \in \mathcal{M}$. So we get the result. □

Theorem 2.4. *Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace of $T_{z_1^N z_2^M}$ in the Bergman space $A^2(D^2)$. Then there exist $a, b \in \mathbb{C}$ and nonnegative integers m, n with $0 \leq n \leq N-1$ or $0 \leq m \leq M-1$, such that $\mathcal{M}$ contains a reducing subspace as follows*

$$\mathcal{M}_{n,m,a,b} = \mathrm{Span}\{a z_1^{hN+n} z_2^{hM+m} + b z_1^{\rho_1(m+hN)} z_2^{\rho_2(n+hM)} : h = 0, 1, 2, \ldots\},$$

where $\rho_1(m+hN) = \frac{(m+hN+1)M}{N} - 1$ and $\rho_2(n+hM) = \frac{(n+hM+1)N}{M} - 1$. In particular, if $\rho_1(m)$ (or $\rho_2(n)$) is not a positive integer, then $b = 0$. Moreover, $\mathcal{M}$ is minimal if and only if $\mathcal{M} = \mathcal{M}_{n,m,a,b}$.

Proof. (I) If $\mathcal{M} \neq 0$, there exist nonzero function $f \in \mathcal{M}$ and k, l, such that $P_{kl} f \neq 0$. Lemma 2.3 implies that

$$g_{kl} = P_{kl} f = a z_1^k z_2^l + b z_1^{\rho_1(l)} z_2^{\rho_2(k)} \in \mathcal{M}.$$

Observe that there is a positive integer h_0 such that $a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)} = (T^*_{z_1^N z_2^M})^{h_0}(g_{kl}) \neq 0$, $(T^*_{z_1^N z_2^M})^{h_0+1}(g_{kl}) = 0$, where $n = k - h_0 N$, $m = l - h_0 M$. Clearly, $0 \leq n \leq N-1$ or $0 \leq m \leq M-1$. So Theorem 2.2 shows that $a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}_{n,m,a,b} \subseteq \mathcal{M}$.

(II) Suppose $\mathcal{M}$ is minimal. As in (I), there is a nonzero function $a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}$. Then the following statements hold:

(a) if $z_1^n z_2^m \in \mathcal{M}$, then $\mathcal{M} = \mathrm{Span}\{z_1^{n+hN} z_2^{m+hM}, h \geq 0\}$;

(b) if $\rho_1(m)$, $\rho_2(n)$ are integers, and $z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}$, then

$$\mathcal{M} = \mathrm{Span}\{z_1^{\rho_1(m)+hN} z_2^{\rho_2(n)+hM}, h \geq 0\};$$

(c) if none of $z_1^n z_2^m$ and $z_1^{\rho_1(m)} z_2^{\rho_2(n)}$ is in $\mathcal{M}$, then $\mathcal{M} = \mathcal{M}_{n,m,a,b}$ with $ab \neq 0$.

So we finish the proof. $\square$

Remark 2.5. Note that

$$z_1^{\rho_1(m+hN)} z_2^{\rho_2(n+hM)} = z_1^{m+\frac{(M-N)(m+1)}{N}} z_2^{n+\frac{(N-M)(n+1)}{M}} z_1^{hM} z_2^{hN}.$$

If $N = M$, then $\rho_1(n) = m$ and $\rho_2(m) = n$. Y. Lu and X. Zhou [4] showed that $\mathrm{Span}\{(z_1^k z_2^m + z_1^m z_2^k)(z_1 z_2)^{hN} : h = 0, 1, 2, \ldots\}$ and $\mathrm{Span}\{z_1^k z_2^m (z_1 z_2)^{hN} : h = 0, 1, 2, \ldots\}$ are the only minimal reducing subspaces of $T_{z_1^N z_2^N}$. Let $ab \neq 0$ with $a \neq b$. Then $\mathcal{M}_{n,m,a,b}$ is a reducing subspace of $T_{z_1^N z_2^M}$ when $N \neq M$, but is not a reducing subspace of $T_{z_1^N z_2^N}$.

3. The results on the weighted Bergman space

Let $-1 < \alpha < +\infty$ with $\alpha \neq 0$. In this section, we consider the reducing subspace of $T_{z_1^N z_2^M}$ on the weighted Bergman Space $A^2_\alpha(D)$. Here $\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}}$. We begin with a useful lemma.

Lemma 3.1. *Let M, N, n, m, k, l be nonnegative integers with $l > m, n > k$ and $M, N \geq 1$. If*

$$\gamma^2_{hN+k} \gamma^2_{hM+l} = \gamma^2_{hN+n} \gamma^2_{hM+m}, \ h \geq 0, \tag{3.1}$$

then $N = M$, $l = n$ and $m = k$.

Proof. First, note that the equality (3.1) holds if and only if for any $\lambda \in \mathbb{C}$ the following equality holds:

$$\prod_{j=1}^{n-k} (\lambda N + j + k) \prod_{j=1}^{l-m} (\lambda M + 2 + \alpha + l - j) \tag{3.2}$$

$$= \prod_{j=1}^{n-k}(\lambda N + 2 + \alpha + n - j) \prod_{j=1}^{l-m}(\lambda M + j + m).$$

By computing the coefficient of $\lambda^{n-k+l-m-1}$ in the equality (3.2), we obtain $M\sum_{j=1}^{n-k}(j+k)+N\sum_{j=1}^{l-m}(2+\alpha+l-j) = M\sum_{j=1}^{n-k}(2+\alpha+n-j)+N\sum_{j=1}^{l-m}(j+m)$. It follows that $M(n-k) = N(l-m)$.

Second, we prove that if α is not an integer, then the following statements hold:

$$(3.3) \qquad (m+1)N = (k+1)M \text{ and } (l+1+\alpha)N = (n+1+\alpha)M.$$

(a) Let $\lambda_1 = -\frac{k+1}{N}$. Then $\lambda_1 N + k + 1 = 0$ and $\lambda_1 N + 2 + \alpha + n - j \neq 0$ for any $1 \leq j \leq n-k$, because $\lambda_1 N + 2 + \alpha + n - j$ is not an integer. Therefore, the equality (3.2) implies that $\prod_{j=1}^{l-m}(\lambda_1 M + j + m) = 0$. That is, there exists $1 \leq h_1 \leq l-m$ such that $\lambda_1 M + m + h_1 = 0$. So, $h_1 = \frac{k+1}{N}M - m \geq 1$. It follows that $(m+1)N \leq (k+1)M$.

(b) Let $\lambda_2 = -\frac{m+1}{M}$. Then $\lambda_2 M + m + 1 = 0$. Similarly, we can get an integer h_2 such that $1 \leq h_2 \leq l-m$ and $\lambda_2 N + k + h_2 = 0$, which implies that $h_2 = \frac{m+1}{M}N - k \geq 1$. Thus $(m+1)N \geq (k+1)M$.

Comparing (a) with (b), we arrive at $(m+1)N = (k+1)M$.

(c) Let $\mu_1 = -\frac{n+1+\alpha}{N}$. Then $\mu_1 N + n + 1 + \alpha = 0$, $\mu_1 N + k + j \neq 0$ for any $1 \leq j \leq n-k$. Therefore, $\prod_{j=1}^{l-m}(\mu_1 M + 2 + \alpha + l - j) = 0$. That is, there exists $1 \leq h_3 \leq l-m$ such that $\mu_1 M + 2 + \alpha + l - h_3 = 0$. So, $h_3 = -\frac{n+1+\alpha}{N}M + (2+\alpha+l) \geq 1$, i.e., $(l+1+\alpha)N \geq (n+1+\alpha)M$.

(d) Let $\mu_2 = -\frac{l+1+\alpha}{M}$. Then $\mu_2 M + l + 1 + \alpha = 0$. As in (c), there exists $1 \leq h_4 \leq n-k$ such that $\mu_2 N + \alpha + 2 + n - h_4 = 0$. So, $1 \leq h_4 = -\frac{l+1+\alpha}{M}N + (2+\alpha+n) \leq n-k$ and $(l+1+\alpha)N \leq (n+1+\alpha)M$.

Comparing (c) with (d), we arrive at $(l+1+\alpha)N = (n+1+\alpha)M$.

Third, we prove that if α is an positive integer, then (3.3) holds. In fact, if $1+\alpha \geq 2$ is an integer, then (3.2) can be simplified into

$$\prod_{j=1}^{k_1}(\lambda N + j + k) \prod_{j=1}^{m_1}(\lambda M + 2 + \alpha + l - j)$$
$$= \prod_{j=1}^{k_1}(\lambda N + 2 + \alpha + n - j) \prod_{j=1}^{m_1}(\lambda M + j + m), \forall \lambda \in \mathbb{C},$$

where $2 \leq k_1 \leq n-k$, $2 \leq m_1 \leq l-m$, $2+\alpha+n-k_1 > k_1+k$ and $2+\alpha+l-m_1 > m_1+m$. By the same technique as in second part of the proof, we can get the equalities in (3.3).

Finally, combining the equalities (3.3) with $M(n-k) = N(l-m)$, it is easy to get $\alpha N = \alpha M$. Since $\alpha \neq 0$, we have $N = M$, $l = n$, $k = m$. □

Theorem 3.2. *Let $\alpha \neq 0$, $M, N \geq 1$ with $M \neq N$. Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace of $T_{z_1^N z_2^M}$ in the weighted Bergman space $A^2_\alpha(D^2)$. Then* *there exist nonnegative integers* n, m *with* $0 \leq n \leq N-1$ *or* $0 \leq m \leq M-1$ *such that*

$$\mathcal{M}_{nm} = \operatorname{Span}\{z_1^{hN+n} z_2^{hM+m} : h = 0, 1, 2, \ldots\} \subseteq \mathcal{M}.$$

In particular, $\mathcal{M}$ *is minimal if and only if there exist* n, m *as in assumption such that* $\mathcal{M} = \mathcal{M}_{nm}$.

Proof. Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace. As in the proof of Lemma 2.3, there exist integers n, m such that $P_{\mathcal{M}}(z_1^n z_2^m) \neq 0$ and

$$\frac{\gamma_{hN+k}^2 \gamma_{hM+l}^2}{\gamma_k^2 \gamma_l^2} = \frac{\gamma_{hN+n}^2 \gamma_{hM+m}^2}{\gamma_n^2 \gamma_m^2}, \forall h \geq 0,$$

whenever $\langle P_{\mathcal{M}}(z_1^n z_2^m), z_1^k z_2^l \rangle \neq 0$. Considering that $\{\gamma_j\}_{j=0}^{+\infty}$ is strictly decreasing and $\frac{\gamma_{hN+k}^2 \gamma_{hM+l}^2}{\gamma_{hN+n}^2 \gamma_{hM+m}^2} \to 1$ as $h \to +\infty$ [3], we obtain that $\gamma_k^2 \gamma_l^2 = \gamma_n^2 \gamma_m^2$ and $\gamma_{hN+k}^2 \gamma_{hM+l}^2 = \gamma_{hN+n}^2 \gamma_{hM+m}^2$, $h \geq 0$. This means that one of the following statements holds:

(1) $l = m$, $n = k$;
(2) $l > m$ and $n > k$;
(3) $l < m$ and $n < k$.

Since $N \neq M$, Lemma 3.1 implies that (2) does not hold. By the same technique, (3) does not hold. So, (1) holds, that is, there exists $c_{nm} \in \mathbb{C}$ such that $P_{\mathcal{M}}(z_1^n z_2^m) = c_{nm} z_1^n z_2^m$. For $f = \sum_{(k,l) \succeq 0} a_{kl} z_1^k z_2^l \in \mathcal{M}$, we claim that if $a_{nm} \neq 0$, then $c_{nm} \neq 0$. In fact,

$$\begin{aligned} Q_{nm} f = Q_{nm} P_{\mathcal{M}}(f) &= Q_{nm}\Big(\sum_{(k,l) \succeq 0} P_{\mathcal{M}}(a_{kl} z_1^k z_2^l)\Big) \\ &= c_{nm} a_{nm} z_1^n z_2^m = c_{nm} Q_{nm} f, \end{aligned}$$

where Q_{nm} is the orthogonal projection from $A_\alpha^2(D^2)$ onto $\operatorname{Span}\{z_1^n z_2^m\}$. Therefore, $c_{nm} = 1 \neq 0$.

Hence $z_1^n z_2^m \in \mathcal{M}$. Choose an integer h_0 such that $0 \leq n - h_0 N \leq N-1$, $m - h_0 M \geq 0$ or $0 \leq m - h_0 M \leq M-1$, $n - h_0 N \geq 0$. As in the proof of Theorem 2.4, $\operatorname{Span}\{z_1^{n+(h-h_0)N} z_2^{m+(h-h_0)M} : h = 0, 1, 2, \ldots\} \subseteq \mathcal{M}$ is a minimal reducing subspace of $T_{z_1^N z_2^M}$. The proof is complete. $\square$

Remark 3.3. By the proof of above theorem, we know that on the weighted Bergman space, either $\operatorname{Span}\{z_1^n z_2^m\} \subseteq \mathcal{M}$ or $\operatorname{Span}\{z_1^n z_2^m\} \subseteq \mathcal{M}^\perp$ holds.

Theorem 3.4. *Let* $N, M \geq 1$ *and* $N \neq M$. *Every nonzero reducing subspace* $\mathcal{M}$ *of* $T_{z_1^N z_2^M}$ *in* $A_\alpha^2(D^2)$ *for every* $\alpha > -1$ *is a direct* (*orthogonal*) *sum of some minimal reducing subspaces.*

Proof. We prove the theorem in two cases.

Case one: $\alpha \neq 0$. Let us denote

$$\mathcal{M}_{nm} = \operatorname{Span}\{z_1^{hN+n} z_2^{hM+m} : h = 0, 1, 2, \ldots\},$$

where $0 \leq n \leq N-1$ or $0 \leq m \leq M-1$. By Lemma 3.1, we have $\mathcal{M}_{nm} \subseteq \mathcal{M}$ if and only if there exist some $f \in \mathcal{M}$ with $\langle f, z_1^n z_2^m \rangle \neq 0$. Let $E_1 = \{(n,m) \succeq 0;\ n \leq N-1 \text{ or } m \leq M-1,\ \langle f, z_1^n z_2^m \rangle \neq 0 \text{ for some } f \in \mathcal{M}\}$. Then $\mathcal{M} = \bigoplus_{(n,m)\in E_1} \mathcal{M}_{nm}$.

Case two: $\alpha = 0$. For $n, m \geq 0$, there exist $a, b \in \mathbb{C}$ such that $\mathcal{M}$ contains the minimal reducing subspace of $T_{z_1^N z_2^M}$ defined by

$$\mathcal{M}_{n,m,a,b} = \operatorname{Span}\{a z_1^{hN+n} z_2^{hM+m} + b z_1^{\rho_1(m+hN)} z_2^{\rho_2(n+hM)} : h = 0, 1, 2, \ldots\}.$$

In fact,

(1) If $z_1^n z_2^m \in \mathcal{M}$, then $\mathcal{M}_{n,m,1,0} = \mathcal{M}_{nm}$.

(2) If $z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}$, then $\mathcal{M}_{n,m,0,1} = \mathcal{M}_{\rho_1(m)\rho_2(n)}$.

(3) If neither $z_1^n z_2^m$ nor $z_1^{\rho_1(m)} z_2^{\rho_2(n)}$ are in $\mathcal{M}$, and there exists $f \in \mathcal{M}$ such that $P_{nm} f \neq 0$, then Theorem 2.4 implies that $\mathcal{M}_{n,m,a,b} \subseteq \mathcal{M}$ is a minimal reducing subspace of $T_{z_1^N z_2^M}$, where $P_{nm} f = a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)}$. It follows that $P_{nm} g = \lambda(a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)})$ for every $g \in \mathcal{M}$ with $P_{nm} g \neq 0$.

(4) If $P_{nm} f = 0$ for any $f \in \mathcal{M}$, then $\mathcal{M}_{n,m,a,b} \subseteq \mathcal{M}$ if and only if $a = 0, b = 0$, i.e., $\mathcal{M}_{n,m,0,0} = \{0\}$.

Let $\mathcal{M}' = \mathcal{M} \ominus \mathcal{M}_{n,m,a,b}$. Then $\mathcal{M}'$ is a reducing subspace. Continuing this process, since $A^2(D^2) = \bigoplus_{(n,m)\succeq 0} z_1^n z_2^m$, it is not different to prove that $\mathcal{M}$ is the direct (orthogonal) sum of some minimal reducing subspaces as $\mathcal{M}_{n,m,a,b}$. $\square$

In [8], Kehe Zhu shows that a reducing subspace of T_{z^N} on $A^2(D)$ is the direct (orthogonal) sum of at most N minimal reducing subspaces. However, the reducing subspace of $T_{z_1^N z_2^M}$ on $A^2(D^2)$ may be the direct (orthogonal) sum of infinity numbers of minimal reducing subspaces. For example, $\mathcal{M} = \operatorname{Span}\{z_1^{1+2h} f(z_2); f \in A_\alpha^2(D),\ h = 0, 1, 2, \ldots\}$ is a reducing subspace of $T_{z_1^2 z_2^3}$ and $\mathcal{M} = \bigoplus_{n=0}^{+\infty} \mathcal{M}_n$, where $\mathcal{M}_n = \operatorname{Span}\{z_1^{1+2h} z_2^{n+3h};\ h = 0, 1, 2, \ldots\}$.

4. The results on the polydisk $A_\alpha^2(D^n)$

In this section, we consider the reducing subspace of $T_{z_i^N z_j^M}$ in the weighted Bergman space $A_\alpha^2(D^n)$ with $N \neq M$.

Theorem 4.1. *Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace of $T_{z_i^N z_j^M}$ $(N, M \geq 1, N \neq M,\ i \neq j)$ in the weighted Bergman space $A_\alpha^2(D^n)$. Then the following statements hold*:

(a) *if $\alpha = 0$, then there exist functions $g_1, g_2 \in A_\alpha^2(D^{n-2})$ and integers l, m with $0 \leq l \leq N-1$ or $0 \leq m \leq M-1$, such that $\mathcal{M}$ contains the reducing subspace*

$$\mathcal{M}' = \operatorname{Span}\{(g_1(z') z_1^{hN+l} z_2^{hM+m} + g_2(z') z_1^{\rho_1(l+hN)} z_2^{\rho_2(m+hM)}); h \geq 0\};$$

(b) *if $\alpha \neq 0$, then there exist a function $g \in A^2_\alpha(D^{n-2})$ and integers l, m with $0 \leq l \leq N-1$ or $0 \leq m \leq M-1$ such that $\mathcal{M}$ contains the reducing subspace*

$$\mathcal{M}_{lmg} = \mathrm{Span}\{z_i^{hN+l} z_j^{hM+m} g(z') : h = 0, 1, 2, \ldots\},$$

where $z' = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$.

Moreover, $\mathcal{M}'$ is the only minimal reducing subspace of $T_{z_i^N z_j^M}$ on $A^2(D^n)$ and $\mathcal{M}_{lmg}$ is the only minimal reducing subspace of $T_{z_i^N z_j^M}$ on $A^2_\alpha(D^n)$ with $\alpha \neq 0$.

Proof. Without loss of generality, let $i = 1$ and $j = 2$. Denote by $P_\mathcal{M}$ the orthogonal projection from $A^2_\alpha(D^n)$ onto $\mathcal{M}$. Let $z^K = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ with $P_\mathcal{M}(z^K) \neq 0$. Let $T_h = T_{z_1^{hN} z_2^{hM}}$. Then $\langle T_h^* T_h P_\mathcal{M} z^K, z^L \rangle = \langle P_\mathcal{M} T_h^* T_h z^K, z^L \rangle$ for any $z^L = z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n}$. Observe that

$$\langle P_\mathcal{M} z^K, T_h^* T_h z^L \rangle = \frac{\gamma^2_{hN+l_1} \gamma^2_{hM+l_2}}{\gamma^2_{l_1} \gamma^2_{l_2}} \langle P_\mathcal{M} z^K, z^L \rangle,$$

and

$$\langle T_h^* T_h z^K, P_\mathcal{M} z^L \rangle = \frac{\gamma^2_{hN+k_1} \gamma^2_{hM+k_2}}{\gamma^2_{k_1} \gamma^2_{k_2}} \langle z^K, P_\mathcal{M} z^L \rangle.$$

Therefore,

$$\frac{\gamma^2_{hN+k_1} \gamma^2_{hM+k_2}}{\gamma^2_{k_1} \gamma^2_{k_2}} = \frac{\gamma^2_{hN+l_1} \gamma^2_{hM+l_2}}{\gamma^2_{l_1} \gamma^2_{l_2}}, \forall h \geq 0,$$

whenever $\langle P_\mathcal{M} z^K, z^L \rangle \neq 0$.

If $\alpha = 0$, then as in Lemma 2.3 we have $(l_1, l_2) = (k_1, k_2)$ or $(l_1, l_2) = (\rho_1(k_2), \rho_2(k_1))$ where $\rho_1(k_2), \rho_2(k_1)$ are integers. Thus $P_\mathcal{M} z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} z'^{K'}$ and $P_\mathcal{M} z^K$ are in $z_1^{k_1} z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} A^2(D^{n-2})$, where $z' = (z_3, \ldots, z_n)$ and $K' = (k_3, \ldots, k_n)$. Let $P_{k_1 k_2}$ be the orthogonal projection from $A^2(D^n)$ onto

$$\mathrm{Span}\{z_1^{k_1} z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} A^2(D^{n-2}); h = 0, 1, 2, \ldots\}.$$

Then $P_{k_1 k_2} P_\mathcal{M} z^K = P_\mathcal{M} P_{k_1 k_2} z^K$. For each $f \in \mathcal{M}$ with $f \neq 0$, there are integers $l, m \geq 0$ such that $P_{lm} f \neq 0$. By the similar technique, we can proof that $\langle P_\mathcal{M} P_{ml} f, z^K \rangle = \langle P_{ml} f, z^K \rangle$ for any $K \succeq 0$, i.e., $P_\mathcal{M} P_{ml} f = P_{ml} f$. So, there exist $f_1(z')$ and $g_2(z') \in A^2(D^{n-2})$ such that $P_{ml} f = g_1(z') z_1^m z_2^l + g_2(z') z_1^{\rho_1(l)} z_2^{\rho_2(m)} \in \mathcal{M}$, which implies that (a) holds.

If $\alpha \neq 0$, then we arrive at $P_\mathcal{M} z^K \in z_1^{k_1} z_2^{k_2} A^2_\alpha(D^{n-2})$. Denote by $P'_{k_1 k_2}$ the orthogonal projection from $A^2_\alpha(D^n)$ onto

$$\mathrm{Span}\{z_1^{k_1} z_2^{k_2} A^2(D^{n-2}); h = 0, 1, 2, \ldots\}.$$

Then $P'_{k_1 k_2}(f) = P'_{k_1 k_2} P_\mathcal{M}(f) = P_\mathcal{M} P'_{k_1 k_2}(f) \in \mathcal{M}$ for each $f \in \mathcal{M}$. Hence (b) holds. The rest of the proof is obvious. $\square$

References

[1] K. Guo and H. Huang, *On Multiplication operators on the Bergman space: similarity, unitary equivalence and reducing subspaces*, J. Operator Theory **65** (2011), no. 2, 355–378.

[2] K. Guo, S. Sun, D. Zheng, and C. Zhong, *Multiplication operators on the Bergman space via the Hardy space of the bidisk*, J. Reine Angew. Math. **628** (2009), 129–168.

[3] Y. Lu and Y. Shi, *Hyponormal Toeplitz operators on the weighted Bergman space*, Integral Equations Operator Theory **65** (2009), no. 1, 115–129.

[4] Y. Lu and X. Zhou, *Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk*, J. Math. Soc. Japan **62** (2010), no. 3, 745–765.

[5] S. Shimorin, *On Beurling-type theorems in weighted l^2 and Bergman spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 6, 1777–1787.

[6] L. Trieu, *On Toeplitz operators on Bergman spaces of the unit polydisk*, Proc. Amer. Math. Soc. **138** (2010), no. 1, 275–285.

[7] X. Zhou, Y. Shi, and Y. Lu, *Invariant subspaces and reducing subspaces of weighted Bergman space over polydisc*, Sci. Sin. Math. **41** (2011), no. 5, 427–438.

[8] K. Zhu, *Reducing subspaces for a class of multiplication operators*, J. London Math. Soc. **62** (2000), no. 2, 553–568.

[9] ———, *Operator Theory in Function Spaces*, 2nd ed. Providence, R.I.: American Mathematical Society, 2007.

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