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A NOTE ON THE TWISTED LERCH TYPE EULER ZETA FUNCTIONS

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Abstract. In this note, the q-extension of the twisted Lerch Euler zeta functions considered by Jang [Bull. Korean Math. Soc. **47** (2010), no. 6, 1181–1188] is further investigated, and the generalized multiplication theorem for the q-extension of the twisted Lerch Euler zeta functions is given. As applications, some well-known results in the references are deduced as special cases.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{\frac{1}{1-p}}$, so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is the uniformly differentiable function}\}$, the p-adic q-integral (also be called as q-Volkenborn integration) is defined by (see [6, 13])

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{j=0}^{p^N-1} f(j) q^j \tag{1.1}$$

with $[x]_q = [x : q] = (1-q^x)/(1-q)$. For some applications of the p-adic q-integral, we infer to [4, 7, 8, 10, 12, 16, 17, 18, 19].

Recently, based on the work of Kim [11], Jang [5] investigated the twisted q-Euler polynomials $E_{m,q,\xi}^{(-m,k)}(x)$ of order k in the variable x in $\mathbb{C}_p$ given by

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$$(1.2)\quad E_{m,q,\xi}^{(-m,k)}(x)=\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_q^m\xi^{x_1+\cdots+x_k}\times q^{-x_1(m+1)-\cdots-x_k(m+k)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k),$$

where k,m are positive integers and $\xi\in\mathbb{T}_p=\bigcup_{n\geq 1}\mathbb{C}_{p^n}$ is the locally constant space with $\mathbb{C}_{p^n}=\{\xi\mid\xi^{p^n}=1\}$ being the cyclic group of order p^n, and gave several explicit expressions of the twisted q-Euler polynomials of order k by using the p-adic q-integral and some transformation techniques. In particular, he constructed a new complex q-analogue of twisted Lerch type Euler zeta function at negative integers which interpolate the above twisted q-Euler polynomials.

The aim of the present note is to perform a further investigation for the q-extension of the twisted Lerch Euler zeta functions considered by Jang [5]. By using some elementary methods and techniques, we derive the generalized multiplication theorem for the q-extension of the twisted Lerch Euler zeta functions. It turns out that some well-known results, for example, Jang [5], Kim [9], etc., are reobtained.

2. The restatement of results

We firstly recall the q-extension of the twisted Lerch Euler zeta functions which is given by (see [5])

$$(2.1)\quad \zeta_{q,E,\xi}(s,x)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\xi^nq^{ns}}{[x+n]_q^s},$$

where $q,s\in\mathbb{C}$ with $|q|<1$ and $\mathrm{Re}(s)>1$, $\xi\in\mathbb{T}_p$ and x is a positive real number. Obviously, the case $\xi=1$ in (2.1) leads to the q-extension of Hurwitz's type Euler zeta function due to Kim [11]. Now, let a,b be positive integers and j be a non-negative integer. If substituting $bx+bj/a$ for x in (2.1), we have

$$(2.2)\quad \zeta_{q,E,\xi}\left(s,bx+\frac{bj}{a}\right)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\xi^nq^{ns}}{[bx+bj/a+n]_q^s}.$$

It is easy to see that for any complex numbers x and y, $[xy]_q=[x]_q[y]_{q^x}$. Hence, in view of replacing q by q^a and ξ by ξ^a in (2.2), we derive

$$(2.3)\quad \zeta_{q^a,E,\xi^a}\left(s,bx+\frac{bj}{a}\right)=[2]_{q^a}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{an}q^{ans}}{[bx+bj/a+n]_{q^a}^s}=[2]_{q^a}[a]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{an}q^{ans}}{[abx+bj+an]_q^s}.$$

Since for any non-negative integer n and positive integer b, there exist unique non-negative integers r and i such that $n=br+i$ with $0\leq i\leq b-1$. So the above identity (2.3) can be rewritten as follows

$$\zeta_{q^a,E,\xi^a}\left(s, bx+\frac{bj}{a}\right)=[2]_{q^a}[a]_q^s\sum_{i=0}^{b-1}\sum_{n=0}^{\infty}\frac{(-1)^{bn+i}\xi^{a(bn+i)}q^{as(bn+i)}}{[abx+bj+a(bn+i)]_q^s}. \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned}&\frac{[b]_q^s}{[2]_{q^a}}\sum_{j=0}^{a-1}(-1)^j\xi^{bj}q^{bsj}\zeta_{q^a,E,\xi^a}\left(s,bx+\frac{bj}{a}\right)\\&=([a]_q[b]_q)^s\sum_{j=0}^{a-1}(-1)^j\xi^{bj}q^{bjs}\sum_{i=0}^{b-1}(-1)^i\xi^{ai}q^{ais}\sum_{n=0}^{\infty}\frac{(-1)^{bn}\xi^{abn}q^{abns}}{[ab(x+n)+ai+bj]_q^s}.\end{aligned} \tag{2.5}$$

In the same way,

$$\begin{aligned}&\frac{[a]_q^s}{[2]_{q^b}}\sum_{j=0}^{b-1}(-1)^j\xi^{aj}q^{asj}\zeta_{q^b,E,\xi^b}\left(s,ax+\frac{aj}{b}\right)\\&=([a]_q[b]_q)^s\sum_{j=0}^{b-1}(-1)^j\xi^{aj}q^{ajs}\sum_{i=0}^{a-1}(-1)^i\xi^{bi}q^{bis}\sum_{n=0}^{\infty}\frac{(-1)^{an}\xi^{abn}q^{abns}}{[ab(x+n)+bi+aj]_q^s}.\end{aligned} \tag{2.6}$$

Thus, if a and b in (2.5) and (2.6) satisfy $a \equiv b \pmod 2$, then we immediately obtain:

Theorem 2.1. *Let $s, q \in \mathbb{C}$ with $|q| < 1$. Then for positive integers a and b with the same parity,*

$$\begin{aligned}&\frac{[b]_q^s}{[2]_{q^a}}\sum_{j=0}^{a-1}(-1)^j\xi^{bj}q^{bsj}\zeta_{q^a,E,\xi^a}\left(s,bx+\frac{bj}{a}\right)\\&=\frac{[a]_q^s}{[2]_{q^b}}\sum_{j=0}^{b-1}(-1)^j\xi^{aj}q^{asj}\zeta_{q^b,E,\xi^b}\left(s,ax+\frac{aj}{b}\right).\end{aligned} \tag{2.7}$$

Next, we discuss some special cases of Theorem 2.1. Setting $b = 1$ in Theorem 2.1, we have the following distribution formula

$$\zeta_{q,E,\xi}(s, ax)=\frac{[2]_q}{[2]_{q^a}[a]_q^s}\sum_{j=0}^{a-1}(-1)^j\xi^j q^{sj}\zeta_{q^a,E,\xi^a}\left(s,x+\frac{j}{a}\right). \tag{2.8}$$

Especially, setting $a = 2$ in (2.8), we have the duplication formula

$$\zeta_{q,E,\xi}(s,2x)=\frac{1}{[2]_{q^2}[2]_q^{s-1}}\left(\zeta_{q^2,E,\xi^2}(s,x)-\xi q^s\zeta_{q^2,E,\xi^2}\left(s,x+\frac{1}{2}\right)\right). \tag{2.9}$$

On the other hand, since the twisted q-Euler polynomials can be expressed in following way (see [5, Theorem 4])

$$E_{m,q,\xi}^{(-m,1)}(x)=[2]_q\sum_{n=0}^{\infty}(-1)^n q^{-mn}\xi^n[x+n]_q^m, \tag{2.10}$$

then by (2.1), (2.10) and the analytic continuation of $\zeta_{q,E,\xi}(s,x)$, one can easily obtain

$$E^{(-m,1)}_{m,q,\xi}(x)=\zeta_{q,E,\xi}(-m,x). \tag{2.11}$$

In fact, using the relation

$$[x+n]_q^m=\frac{1}{(1-q)^m}\sum_{i=0}^m\binom{m}{i}(-1)^i q^{(x+n)i}, \tag{2.12}$$

the above identity (2.10) can be reduced in the following way

$$E^{(-m,1)}_{m,q,\xi}(x)=\frac{[2]_q}{(1-q)^m}\sum_{i=0}^m\binom{m}{i}(-1)^i\frac{q^{xi}}{1+\xi q^{i-m}}, \tag{2.13}$$

which means the symmetric distribution of the twisted q-Euler polynomials

$$E^{(-m,1)}_{m,q,\xi}(x)=(-1)^{m+1}q\xi E^{(-m,1)}_{m,q^{-1},\xi^{-1}}(1-x). \tag{2.14}$$

Thus, by applying (2.11) to Theorem 2.1, we state:

Theorem 2.2. *Let a,b,m be positive integers with $a\equiv b \pmod 2$. Then*

$$\begin{aligned}\frac{[a]_q^m}{[2]_{q^a}}&\sum_{j=0}^{a-1}(-1)^j\xi^{bj}q^{-bmj}E^{(-m,1)}_{m,q^a,\xi^a}\left(bx+\frac{bj}{a}\right)\\&=\frac{[b]_q^m}{[2]_{q^b}}\sum_{j=0}^{b-1}(-1)^j\xi^{aj}q^{-amj}E^{(-m,1)}_{m,q^b,\xi^b}\left(ax+\frac{aj}{b}\right).\end{aligned} \tag{2.15}$$

It follows that we show some special cases of Theorem 2.2. Setting $b=1$ and replacing x by x/a in Theorem 2.2, we have the following multiplication formula of the twisted q-Euler polynomials due to Jang (see [5, Theorem 3])

$$E^{(-m,1)}_{m,q,\xi}(x)=\frac{[2]_q[a]_q^m}{[2]_{q^a}}\sum_{j=0}^{a-1}(-1)^j\xi^j q^{-mj}E^{(-m,1)}_{m,q^a,\xi^a}\left(\frac{x+j}{a}\right)\quad(2\nmid a). \tag{2.16}$$

If multiplying $\sum_{m=0}^\infty t^m/m!$ in both sides of (2.10), one can easily derive

$$\begin{aligned}\sum_{m=0}^\infty E^{(-m,1)}_{m,q,\xi}(x)\frac{t^m}{m!}&=[2]_q\sum_{n=0}^\infty(-1)^n\xi^n\sum_{m=0}^\infty q^{-mn}[x+n]_q^m\frac{t^m}{m!}\\&=[2]_q\sum_{n=0}^\infty(-1)^n\xi^n e^{q^{-n}[x+n]_q t}.\end{aligned} \tag{2.17}$$

It follows from (2.17) that

$$\lim_{q\to1}E^{(-m,1)}_{m,q,1}(x)=E_m(x), \tag{2.18}$$

where $E_n(x)$ denotes the classical Euler polynomials given by (see [1, 2, 3])

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^\infty E_n(x)\frac{t^n}{n!}\quad(|t|<\pi). \tag{2.19}$$

Hence, by setting $\xi = 1$ and letting $q \to 1$ in Theorem 2.2, we obtain that for positive integers a, b and non-negative integer n,

(2.20)
$$a^n \sum_{j=0}^{a-1} (-1)^j E_n\left(bx + \frac{bj}{a}\right) = b^n \sum_{j=0}^{b-1} (-1)^j E_n\left(ax + \frac{aj}{b}\right) \quad (a \equiv b \ (\mathrm{mod}\ 2)),$$

which was rediscovered by many authors; see for example [14, 9]. For the generalization of (2.20) in other direction, see [15] for a detail introduction. If substituting $x+y$ for x in (2.17), then by using the relation $[x+y]_q = [x]_q + q^x[y]_q$ for any complex numbers x and y, we get

$$\sum_{m=0}^{\infty} E_{m,q,\xi}^{(-m,1)}(x+y)\frac{t^m}{m!} = [2]_q \sum_{n=0}^{\infty} (-1)^n \xi^n e^{q^{-n}[y+n]_q q^x t} e^{q^{-n}[x]_q t}. \tag{2.21}$$

Putting the exponential series $e^{xt} = \sum_{n=0}^{\infty} x^n t^n / n!$ and (2.17) to (2.21), with help of the Cauchy product, we derive

$$\begin{aligned} \sum_{m=0}^{\infty} E_{m,q,\xi}^{(-m,1)}(x+y)\frac{t^m}{m!} &= \left(\sum_{m=0}^{\infty} [x]_q^m \frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty} q^{mx} E_{m,q,q^{-m}\xi}^{(-m,1)}(y)\frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{m}\binom{m}{i} q^{ix} E_{i,q,q^{-i}\xi}^{(-i,1)}(y)[x]_q^{m-i}\right)\frac{t^m}{m!}. \end{aligned} \tag{2.22}$$

Hence, by comparing the coefficients of $t^m/m!$ in (2.22), we obtain the addition theorem of the twisted q-Euler polynomials as follows

$$E_{m,q,\xi}^{(-m,1)}(x+y) = \sum_{i=0}^{m}\binom{m}{i} q^{ix} E_{i,q,q^{-i}\xi}^{(-i,1)}(y)[x]_q^{m-i}. \tag{2.23}$$

In light of applying (2.23) to Theorem 2.2, we immediately derive after some calculation.

Theorem 2.3. *Let a, b, m be positive integers with $a \equiv b$ (mod 2). Then*

$$\begin{aligned} &[2]_{q^b} \sum_{i=0}^{m}\binom{m}{i}[a]_q^i [b]_q^{m-i} E_{i,q^a,q^{-ia}\xi^a}^{(-i,1)}(bx) S_{m-i,\xi^b;q^b}(a) \\ &= [2]_{q^a} \sum_{i=0}^{m}\binom{m}{i}[b]_q^i [a]_q^{m-i} E_{i,q^b,q^{-ib}\xi^b}^{(-i,1)}(ax) S_{m-i,\xi^a;q^a}(b), \end{aligned} \tag{2.24}$$

where $S_{m,\xi;q}(a) = \sum_{j=0}^{a-1}(-\xi)^j q^{-mj}[j]_q^m$.

If taking $\xi = 1$ and letting $q \to 1$ in Theorem 2.3, then we have the following identity between the classical Euler polynomials and alternating sum (see [14, 9])

$$\sum_{i=0}^{n}\binom{n}{i} a^{n-i} b^i E_{n-i}(bx) S_i(a) = \sum_{i=0}^{m}\binom{n}{i} b^{n-i} a^i E_{n-i}(ax) S_i(b), \tag{2.25}$$

where n is a non-negative integer, a, b are positive integers with $a \equiv b \pmod 2$ and $S_n(a) = \sum_{j=0}^{a-1}(-1)^j j^n$. For the generalization of the above identity (2.25) in the Apostol-type direction, the interested readers may consult to [15].

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References

[1] Y. He and Q. Y. Liao, *Some congruences invloving Euler numbers*, Fibonacci Quart. **46/47** (2008/2009), 225–234.

[2] Y. He and W. P. Zhang, *Some symmetric identities involving a sequence of polynomials*, Electron. J. Combin. **17** (2010), no. 1, Note 7, 7 pp.

[3] ———, *Some sum relations involving Bernoulli and Euler polynomials*, Integral Transforms Spec. Funct. **22** (2011), no. 3, 207–215.

[4] L. Jang, *On a q-analogue of the p-adic generalized twisted L-functions and p-adic q-integrals*, J. Korean Math. Soc. **44** (2007), no. 1, 1–10.

[5] ———, *The q-analogue of twisted Lerch type Euler zeta functions*, Bull. Korean Math. Soc. **47** (2010), no. 6, 1181–1188.

[6] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), no. 3, 288–299.

[7] ———, *q-generalized Euler numbers and polynomials*, Russ. J. Math. Phys. **13** (2006), no. 3, 293–298.

[8] ———, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$*, J. Math. Anal. Appl. **331** (2007), no. 2, 779–792.

[9] ———, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14** (2008), no. 12, 1267–1277.

[10] ———, *p-adic interpolating function for q-Euler numbers and its derivatives*, J. Math. Anal. Appl. **339** (2008), no. 1, 598–608.

[11] ———, *Note on the Euler q-zeta functions*, J. Number Theory **129** (2009), no. 7, 1798–1804.

[12] ———, *Some identities on the q-Euler polynomials of higher order and q-Stirling umbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **16** (2009), no. 4, 484–491.

[13] ———, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory **76** (1999), no. 2, 320–329.

[14] H. M. Liu and W. P. Wang, *Some identities on the Bernoulli, Euler and Genocchi polynomials via power sums and alternate power sums*, Discrete Math. **309** (2009), no. 10, 3346–3363.

[15] D. Q. Lu and H. M. Srivastava, *Some series identities involving the generalized Apostol type and related polynomials*, Comput. Math. Appl. **62** (2011), no. 9, 3591–3602.

[16] H. Ozden and Y. Simsek, *A new extension of q-Euler numbers and polynomials related to their interpolation functions*, Appl. Math. Lett. **21** (2008), no. 9, 934–939.

[17] S. H. Rim and T. Kim, *A note on p-adic Euler measure on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **13** (2006), no. 3, 358–361.

[18] Y. Simsek, *On twisted q-Hurwitz zeta function and q-two-variable L-function*, Appl. Math. Comput. **187** (2007), no. 1, 466–473.

[19] Y. Simsek, V. Kurt, and D. Kim, *New approach to the complete sum of products of the twisted $(h;q)$-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys. **14** (2007), no. 1, 44–56.

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