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FINITENESS PROPERTIES OF EXTENSION FUNCTORS OF COFINITE MODULES

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R and T be a non-zero I-cofinite R-module with $\dim(T) \leq 1$. In this paper, for any finitely generated R-module N with support in $V(I)$, we show that the R-modules $\mathrm{Ext}^i_R(T, N)$ are finitely generated for all integers $i \geq 0$. This immediately implies that if I has dimension one (i.e., $\dim R/I = 1$), then $\mathrm{Ext}^i_R(H^j_I(M), N)$ is finitely generated for all integers $i, j \geq 0$, and all finitely generated R-modules M and N, with $\mathrm{Supp}(N) \subseteq V(I)$.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n\geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

We refer the reader to [9] or [5] for more details about local cohomology. Hartshorne [10] defined an R-module M to be I-cofinite if $\mathrm{Supp}\, M \subseteq V(I)$ and $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all j and asked:

For which rings R and ideals I are the modules $H^i_I(M)$ I-cofinite for all i and all finitely generated modules M?

This question has been studied by several authors; see, for example, Hartshorne [10], Chiriacescu [6], Huneke-Koh [11], Delfino [7], Delfino and Marley [8], Yoshida [15], Bahmanpour and Naghipour [2], Abazari and Bahmanpour [1], Bahmanpour, Naghipour and Sedghi [3].

In this paper we consider the following question:

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Let I be an ideal of a Noetherian ring R and M be a non-zero I-cofinite R-module with $\dim(M) \leq 1$. *Are the R-modules* $\mathrm{Ext}^i_R(M, R/I)$ *finitely generated for all integers $i \geq 0$?*

The main purpose of this paper is to provide an affirmative answer to this question. In fact the main result of this paper states that, for any finitely generated R-module N with support in $V(I)$, instead of R/I, the assertion holds.

More precisely, we shall show that:

Theorem 1.1. *Let R be a Noetherian ring, I a proper ideal of R and M be a non-zero I-cofinite R-module such that* $\dim(M) \leq 1$. *Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules* $\mathrm{Ext}^i_R(M, N)$ *are finitely generated for all integers $i \geq 0$.*

As an immediately consequence of above theorem we derive the following new insight on the cofiniteness properties of local cohomology modules.

Theorem 1.2. *Let R be a Noetherian ring, I be an ideal of R and M be a non-zero finitely generated R-module such that* $\dim M/IM \leq 1$ (*e.g.,* $\dim R/I \leq 1$). *Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules* $\mathrm{Ext}^i_R(H^j_I(M), N)$ *are finitely generated for all integers $i \geq 0$ and $j \geq 0$.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. In [16] H. Zöschinger, introduced the interesting class of minimax modules. The R-module N is said to be a *minimax module*, if there is a finitely generated submodule L of N, such that N/L is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules. For an Artinian R-module A we denote by $\mathrm{Att}_R(A)$ the set of attached prime ideals of A. For each R-module L, we denote by $\mathrm{Assh}_R L$, the set $\{\mathfrak{p} \in \mathrm{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. We shall use $\mathrm{Max}\, R$ to denote the set of all maximal ideals of R. Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [5] and [13].

2. Main results

The following well known lemma will be quite useful in this paper.

Lemma 2.1. *Let R be a Noetherian ring, I an ideal of R and k be a positive integer. Then, for any R-module T, the followings conditions are equivalent*:

(i) $\mathrm{Ext}^n_R(R/I, T)$ *is finitely generated for each integer* $0 \leq n < k$,

(ii) *for any finitely generated R-module N with support in $V(I)$,* $\mathrm{Ext}^n_R(N, T)$ *is finitely generated for each integer* $0 \leq n < k$.

Proof. See [12, Lemma 1]. □

The following lemma is needed in the proof of Theorem 2.3.

Lemma 2.2. *Let R be a Noetherian ring, I a proper ideal of R and A be a non-zero Artinian I-cofinite R-module. Then $V(I) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R)$.*

Proof. Let $\mathfrak{p} \in V(I) \cap \operatorname{Att}_R A$, and let

$$A = T + S_1 + \cdots + S_n,$$

be a minimal secondary representation of A, where T is $\mathfrak{p}$-secondary and S_i is $\mathfrak{p}_i$-secondary for every $i = 1, \ldots, n$. Since T is $\mathfrak{p}$-secondary, it follows that there exists a positive integer k such that $\mathfrak{p}^k T = 0$, and so $I^k T = 0$ (note that $I \subseteq \mathfrak{p}$). On the other hand, since $\operatorname{Hom}_R(R/I, A)$ is finitely generated, it follows from Lemma 2.1 that $\operatorname{Hom}_R(R/I^k, A) \cong (0 :_A I^k)$ is also a finitely generated R-module. Hence from $T \subseteq (0 :_A I^k)$, it follows that T has finite length, and so $\operatorname{Supp}(T) \subseteq \operatorname{Max}(R)$. By definition we have $\mathfrak{p} = \operatorname{Rad}(\operatorname{Ann}_R(T))$ and hence $\mathfrak{p} \in V(\operatorname{Ann}_R(T)) = \operatorname{Supp}(T)$. Therefore, $\mathfrak{p} \in \operatorname{Max}(R)$ and so $V(I) \cap \operatorname{Att}_R A \subseteq \operatorname{Max}(R)$, as required. □

Before bringing of the next result, recall the important concept of the arithmetic rank of an ideal. The *arithmetic rank* of an ideal $\mathfrak{b}$ in a commutative Noetherian ring R, denoted by $\operatorname{ara}(\mathfrak{b})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{b}$, i.e.,

$$\operatorname{ara}(\mathfrak{b}) = \min\{n \in \mathbb{N}_0 : \exists b_1, \ldots, b_n \in R \text{ with } \operatorname{Rad}(b_1, \ldots, b_n) = \operatorname{Rad}(\mathfrak{b})\}.$$

Let K be a R-module. The arithmetic rank of an ideal $\mathfrak{b}$ of R with respect to K, denoted by $\operatorname{ara}_K(\mathfrak{b})$, is defined the arithmetic rank of the ideal $\mathfrak{b} + \operatorname{Ann}_R(K)/\operatorname{Ann}_R(K)$ in the ring $R/\operatorname{Ann}_R(K)$.

Theorem 2.3. *Let R be a Noetherian ring, I a proper ideal of R and A be a non-zero Artinian I-cofinite R-module. Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules $\operatorname{Ext}^i_R(A, N)$ have finite length for all integers $i \geq 0$.*

Proof. Since N is finitely generated with support in $V(I)$, it follows that $I^k N = 0$ for some positive integers k. On the other hand, as $V(I^k) = V(I)$, using Lemma 2.1 we deduce that the R-module A is I^k-cofinite, too. Consequently, without loss of generality, by replacing I by I^k, we may assume that $IN = 0$. Now we use induction on $t := \operatorname{ara}_A(I) = \operatorname{ara}(I + \operatorname{Ann}_R(A)/\operatorname{Ann}_R(A))$. If $t = 0$, then, it follows from the definition that $I^n \subseteq \operatorname{Ann}_R(A)$ for some positive integer n, and so $A = (0 :_A I^n)$. But since by hypothesis the R-module $\operatorname{Hom}_R(R/I, A)$ is finitely generated it follows from Lemma 2.1 that the R-module $\operatorname{Hom}_R(R/I^n, A) \cong (0 :_A I^n) = A$ is also finitely generated with support in $\operatorname{Max}(R)$ and hence is of finite length. Therefore, the assertion holds for the case $t = 0$. So assume that $t > 0$ and the result has been proved for $0, 1, \ldots, t-1$. By definition there exist elements $a_1, \ldots, a_t \in I$, such that

$$\operatorname{Rad}(I + \operatorname{Ann}_R(A)/\operatorname{Ann}_R(A)) = \operatorname{Rad}((a_1, \ldots, a_t) + \operatorname{Ann}_R(A)/\operatorname{Ann}_R(A)).$$

From the Lemma 2.2, we have $V(I) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R)$ and so
$$V(I + \operatorname{Ann}_R(A)) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R).$$
Therefore,
$$V((a_1, \ldots, a_t) + \operatorname{Ann}_R(A)) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R).$$
Hence, using the fact that $\operatorname{Att}_R(A) \subseteq V(\operatorname{Ann}_R(A))$, it follows that
$$V((a_1, \ldots, a_t)) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R).$$
Consequently, we have
$$(a_1, \ldots, a_t) \nsubseteq \bigcup_{\mathfrak{p} \in (\operatorname{Att}_R(A) \setminus \operatorname{Max}(R))} \mathfrak{p}.$$

Therefore, by [13, Exercise 16.8] there is $c \in (a_2, \ldots, a_t)$ such that
$$a_1 + c \notin \bigcup_{\mathfrak{p} \in (\operatorname{Att}_R(A) \setminus \operatorname{Max}(R))} \mathfrak{p}.$$
Let $b := a_1 + c$. Then $b \in I$ and $V(Rb) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R)$. Also, we have
$$\operatorname{Rad}(I + \operatorname{Ann}_R(A)/\operatorname{Ann}_R(A)) = \operatorname{Rad}((b, a_2, \ldots, a_t) + \operatorname{Ann}_R(A)/\operatorname{Ann}_R(A)).$$
Now, let $B := (0 :_A b)$. Then it is easy to see that
$$\operatorname{ara}_B(I) = \operatorname{ara}(I + \operatorname{Ann}_R(B)/\operatorname{Ann}_R(B)) \leq t - 1$$
(Note that $b \in \operatorname{Ann}_R(B)$ and hence
$$\operatorname{Rad}(I + \operatorname{Ann}_R(B)/\operatorname{Ann}_R(B)) = \operatorname{Rad}((a_2, \ldots, a_t) + \operatorname{Ann}_R(B)/\operatorname{Ann}_R(B))).$$
On the other hand, since A is I-cofinite and Artinian, it follows from [14, Corollary 4.4] that B is also an Artinian I-cofinite R-module. Therefore, by inductive hypothesis for all integers $i \geq 0$, the R-modules $\operatorname{Ext}^i_R(B, N)$ are of finite length. Next, let $C := A/bA$. Then it is easy to see that $\operatorname{Att}_R(C) \subseteq V(Rb) \cap \operatorname{Att}_R(A) \subseteq \operatorname{Max}(R)$. Next, let $\operatorname{Att}_R(C) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. Then as $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \cap_{j=1}^n \mathfrak{m}_j = \operatorname{Rad}(\operatorname{Ann}_R(C))$, it follows that $(\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n)^k C = 0$ for some positive integer k. Now as C is an Artinian R-module and $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \operatorname{Max}(R)$, it follows that the R-module C has finite length. Hence, for all integers $i \geq 0$, the R-modules $\operatorname{Ext}^i_R(C, N)$ are of finite length. We have two exact sequences
$$0 \to B \to A \xrightarrow{f} bA \to 0 \quad \text{and} \quad 0 \to bA \xrightarrow{g} A \to C \to 0,$$
where $g \circ f$ is the map $A \xrightarrow{b} A$. Since $b \in I$ and by hypothesis we have $IN = 0$ it follows that $bN = 0$. In particular, for each $i \geq 0$, the map $\operatorname{Ext}^i_R(A, N) \xrightarrow{b} \operatorname{Ext}^i_R(A, N)$ is the zero R-homomorphism. Therefore, $\operatorname{Coker}(\operatorname{Ext}^i_R(g \circ f, N)) \cong \operatorname{Ext}^i_R(A, N)$ for all integers $i \geq 0$. As,
$$\operatorname{Ext}^i_R(g \circ f, N) = \operatorname{Ext}^i_R(f, N) \circ \operatorname{Ext}^i_R(g, N),$$
it follows that, for all integers $i \geq 0$, there is an exact sequence
$$\operatorname{Coker}(\operatorname{Ext}^i_R(g, N)) \to \operatorname{Coker}(\operatorname{Ext}^i_R(g \circ f, N)) \to \operatorname{Coker}(\operatorname{Ext}^i_R(f, N)) \to 0.$$

Since, for all integers $i \geq 0$, the R-modules

$$\operatorname{Coker}(\operatorname{Ext}^i_R(g, N)) \text{ and } \operatorname{Coker}(\operatorname{Ext}^i_R(f, N))$$

have finite length, it follows that the R-module

$$\operatorname{Coker}(\operatorname{Ext}^i_R(g \circ f, N)) = \operatorname{Ext}^i_R(A, N)$$

has finite length for any $i \geq 0$. This completes the inductive step. □

The following proposition, which generalizes the argument of Theorem 2.3 to the larger class of minimax modules, is an immediately consequence of Theorem 2.3 and will be useful in the proof of the main result of this paper.

Proposition 2.4. *Let R be a Noetherian ring, I a proper ideal of R and M be a non-zero minimax I-cofinite R-module. Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules $\operatorname{Ext}^i_R(M, N)$ are finitely generated for all integers $i \geq 0$.*

Proof. By definition M has a finitely generated submodule H such that the R-module M/H is Artinian. According to [14, Corollary 4.4], the R-module M/H is I-cofinite. Now the assertion follows from the exact sequence

$$0 \to H \to M \to M/H \to 0,$$

using Theorem 2.3. □

Now we need the following well known result.

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and A be a non-zero Artinian R-module. Suppose that x is an element in $\mathfrak{m}$ such that $V(Rx) \cap \operatorname{Att}_R A \subseteq \{\mathfrak{m}\}$. Then the R-module A/xA has finite length.*

Proof. See [2, Lemma 2.4]. □

Before bringing the main result of this paper we need the following well known result and its corollary.

Theorem 2.6. *Let R be a Noetherian ring and I be a proper ideal of R. Let M and N be two non-zero I-cofinite R-modules such that $\dim(M) \leq 1$ and $\dim(N) \leq 1$ and $f : M \to N$ be an R-homomorphism. Then the R-modules $\operatorname{Ker}(f)$ and $\operatorname{Coker}(f)$ are I-cofinite.*

Proof. See [4, Theorem 2.7]. □

Corollary 2.7. *Let R be a Noetherian ring, I a proper ideal of R and M be a non-zero I-cofinite R-modules such that*

$$\dim(M) = 1 \text{ and } \operatorname{ara}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = t \geq 1.$$

Then there exists an element $x \in I$ such that the following conditions hold:

(i) *The R-module $T := (0 :_M x)$ is I-cofinite and*

$$\operatorname{ara}_T(I) = \operatorname{ara}(I + \operatorname{Ann}_R(T)/\operatorname{Ann}_R(T)) \leq t - 1.$$

(ii) *The R-module $L := M/xM$ is I-cofinite and minimax.*

Proof. Let

$$\mathcal{T} := \{\mathfrak{p} \in \operatorname{Supp} M \mid \dim R/\mathfrak{p} = 1\}.$$

It is easy to see that $\mathcal{T} = \operatorname{Assh}_R M$. As $\operatorname{Ass}_R \operatorname{Hom}_R(R/I, M) = V(I) \cap \operatorname{Ass}_R M = \operatorname{Ass}_R M$, it follows that the set $\operatorname{Ass}_R M$ is finite. Hence $\mathcal{T}$ is finite. Moreover, since for each $\mathfrak{p} \in \mathcal{T}$ the $R_{\mathfrak{p}}$-module $\operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})$ is finitely generated, by [13, Exercise 7.7], and $M_{\mathfrak{p}}$ is an $IR_{\mathfrak{p}}$-torsion $R_{\mathfrak{p}}$-module with $\operatorname{Supp} M_{\mathfrak{p}} \subseteq V(\mathfrak{p}R_{\mathfrak{p}})$, it follows that the $R_{\mathfrak{p}}$-module $\operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})$ is Artinian and so is of finite length. Consequently, according to Melkersson's results [14, Proposition 4.1], $M_{\mathfrak{p}}$ is an Artinian and $IR_{\mathfrak{p}}$-cofinite $R_{\mathfrak{p}}$-module. Let

$$\mathcal{T} := \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

By Lemma 2.1, we have

$$V(IR_{\mathfrak{p}_j}) \cap \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq V(\mathfrak{p}_j R_{\mathfrak{p}_j})$$

for all $j = 1, 2, \dots, n$. Next, let

$$\mathcal{U} := \bigcup_{j=1}^{n} \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})\}.$$

Then it is easy to see that $\mathcal{U} \cap V(I) \subseteq \mathcal{T}$.

On the other hand, since $t = \operatorname{ara}_M(I) \geq 1$, there exist elements $y_1, \dots, y_t \in I$ such that

$$\operatorname{Rad}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((y_1, \dots, y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Now, as

$$I \not\subseteq \bigcup_{\mathfrak{q} \in \mathcal{U} \setminus V(I)} \mathfrak{q},$$

it follows that

$$(y_1, \dots, y_t) + \operatorname{Ann}_R(M) \not\subseteq \bigcup_{\mathfrak{q} \in \mathcal{U} \setminus V(I)} \mathfrak{q}.$$

On the other hand, for each $\mathfrak{q} \in \mathcal{U}$ we have

$$\mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})$$

for some integer $1 \leq j \leq n$. Whence,

$$\operatorname{Ann}_R(M) R_{\mathfrak{p}_j} \subseteq \operatorname{Ann}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq \mathfrak{q}R_{\mathfrak{p}_j},$$

which implies $\operatorname{Ann}_R(M) \subseteq \mathfrak{q}$. Consequently, it follows from

$$\operatorname{Ann}_R(M) \subseteq \bigcap_{\mathfrak{q} \in \mathcal{U} \setminus V(I)} \mathfrak{q},$$

that

$$(y_1, \dots, y_t) \not\subseteq \bigcup_{\mathfrak{q} \in \mathcal{U} \setminus V(I)} \mathfrak{q}.$$

Therefore, by [13, Exercise 16.8] there is $a \in (y_2, \dots, y_t)$ such that

$$y_1 + a \notin \bigcup_{\mathfrak{q} \in \mathcal{U} \setminus V(I)} \mathfrak{q}.$$

Let $x := y_1 + a$. Then $x \in I$ and

$$\mathrm{Rad}(I + \mathrm{Ann}_R(M)/\mathrm{Ann}_R(M)) = \mathrm{Rad}((x, y_2, \dots, y_t) + \mathrm{Ann}_R(M)/\mathrm{Ann}_R(M)).$$

Next, let $N := (0 :_M x)$. Now, it is easy to see that

$$\mathrm{ara}_N(I) = \mathrm{ara}(I + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N)) \leq t - 1$$

(Note that $x \in \mathrm{Ann}_R(N)$ and hence

$$\mathrm{Rad}(I + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N)) = \mathrm{Rad}((y_2, \dots, y_t) + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N))).$$

According to Theorem 2.6, the R-modules $T = \mathrm{Ker}(M \xrightarrow{x} M)$ and $L = \mathrm{Coker}(M \xrightarrow{x} M)$ are I-cofinite. But, from Lemma 2.5, it is easy to see that $(M/xM)_{\mathfrak{p}_j}$ is of finite length for all $j = 1, \dots, n$. Therefore, there exists a finitely generated submodule L_j of M/xM such that $(M/xM)_{\mathfrak{p}_j} = (L_j)_{\mathfrak{p}_j}$. Let $L := L_1 + \cdots + L_n$. Then L is a finitely generated submodule of M/xM such that

$$\mathrm{Supp}_R(M/xM)/L \subseteq \mathrm{Supp}(M) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \mathrm{Max}\, R.$$

Now, from the sequence

$$0 \longrightarrow L \longrightarrow M/xM \longrightarrow (M/xM)/L \longrightarrow 0,$$

we get the following exact sequence:

$$\mathrm{Hom}_R(R/I, M/xM) \longrightarrow \mathrm{Hom}_R(R/I, (M/xM)/L) \longrightarrow \mathrm{Ext}^1_R(R/I, L),$$

which implies that the R-module $\mathrm{Hom}_R(R/I, (M/xM)/L)$ is finitely generated. We must show that M/xM is a minimax R-module. To do this, since $\mathrm{Supp}(M/xM)/L \subseteq \mathrm{Max}\, R$ and $(M/xM)/L$ is I-torsion, so that, according to Melkersson [14, Proposition 4.1], $(M/xM)/L$ is an Artinian R-module. That is, M/xM is a minimax R-module. This completes the proof. □

Now we are ready to state and prove the main result of this paper.

Theorem 2.8. *Let R be a Noetherian ring, I a proper ideal of R and M be a non-zero I-cofinite R-module such that* $\dim(M) \leq 1$. *Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules* $\mathrm{Ext}^i_R(M, N)$ *are finitely generated for all integers $i \geq 0$.*

Proof. As in the proof of Theorem 2.3, we may assume that $IN = 0$. If $\dim(M) = 0$, then it follows from hypothesis that the R-module

$$\mathrm{Hom}_R(R/I, M) \cong (0 :_M I)$$

is finitely generated with support in $\mathrm{Max}(R)$ and so has finite length. Therefore, it follows from [14, Prposition 4.1] that M is Artinian and hence the assertion follows from Theorem 2.3. So we may assume $\dim M = 1$. We prove the assertion by induction on $t := \mathrm{ara}_M(I) = \mathrm{ara}(I + \mathrm{Ann}_R M/\,\mathrm{Ann}_R M)$. If

$t = 0$, then it follows from definition that $I^n \subseteq \mathrm{Ann}_R(M)$ for some positive integer n, and so $M = (0 :_M I^n)$. Therefore the assertion follows from Lemma 2.2. So assume that $t > 0$ and the result has been proved for all $i \leq t-1$. Then by Corollary 2.7 there exists an element $x \in I$ such that the R-module $T := (0 :_M x)$ is I-cofinite and

$$\mathrm{ara}_T(I) = \mathrm{ara}(I + \mathrm{Ann}_R(T)/\mathrm{Ann}_R(T)) \leq t-1,$$

and the R-module $L := M/xM$ is I-cofinite and minimax. So it follows from inductive hypothesis that the R-modules $\mathrm{Ext}^i_R(T, N)$ are finitely generated for all integers $i \geq 0$. Also, it follows from Proposition 2.4, that the R-modules $\mathrm{Ext}^i_R(L, N)$ are finitely generated for all integers $i \geq 0$. Now, using the fact that $xN = 0$ and applying the method used in the proof of Theorem 2.3, it follows that the R-modules $\mathrm{Ext}^i_R(M, N)$ are finitely generated for all integers $i \geq 0$. This completes the inductive step. □

The following result is an immediate consequence of Theorem 2.8.

Theorem 2.9. *Let R be a Noetherian ring, I be an ideal of R and M be a non-zero finitely generated R-module such that* $\dim M/IM \leq 1$ (*e.g.,* $\dim R/I \leq 1$). *Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules* $\mathrm{Ext}^i_R(H^j_I(M), N)$ *are finitely generated for all integers $i \geq 0$ and $j \geq 0$.*

Proof. The assertion follows from [2, Corollary 2.7] and Theorem 2.8. □

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