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ON FINSLER METRICS OF CONSTANT S-CURVATURE

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Abstract. In this paper, we study Finsler metrics of constant S-curvature. First we produce infinitely many Randers metrics with non-zero (constant) S-curvature which have vanishing H-curvature. They are counterexamples to Theorem 1.2 in [20]. Then we show that the existence of (α, β)-metrics with arbitrary constant S-curvature in *each* dimension which is not Randers type by extending Li-Shen' construction.

1. Introduction

The S-curvature is one of most important non-Riemannian quantities in Finsler geometry [15]. It vanishes on a Riemannian manifold. So we call it *non-Riemannian quantity*.

In fact, all Berwald manifolds have zero S-curvature [15]. Locally Minkowski manifolds and Riemannian manifolds are all Berwald manifolds.

An n-dimensional Finsler metric F on a manifold M is said to have *constant* S-curvature if $\mathbf{S}(x, y) = (n+1)cF(x, y)$ for some constant c. For example, the following Finsler metric F on the unit ball has constant S-curvature $S = \pm\frac{1}{2}(n+1)F$,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x, y\rangle}{1-|x|^2} \pm \frac{\langle a, y\rangle}{1+\langle a, x\rangle}, \quad y \in T_x\mathbb{R}^n,$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$ [3, 16]. Randers metric of constant flag curvature (or R-quadratic [8]) is of constant S-curvature [2]. Recently, S. Ohta shows that a Randers space (M, F) admits a measure m with $S \equiv 0$ if and only if β is a Killing form of constant length [13]. Shen and Mo-Yu established some global rigidity theorems for Finsler manifolds with constant S-curvature [18, 11].

The aim of this paper is to study a special class of Finsler metrics $-(\alpha, \beta)$-metrics of constant S-curvature. Finsler metrics in the form $F := \alpha\phi(\frac{\beta}{\alpha})$ are called (α, β)-*metrics* (for definition, see Section 2). In particular, when

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$\phi(s) = 1 + s$, $F = \alpha + \beta$ is called a *Randers metrics* [14]. We first produce infinitely many Randers metrics with non-zero (constant) S-curvature which have vanishing H-curvature (see Theorem 5.1). They are counterexamples to Theorem 1.2 in [20]. Note that H-curvature is another interesting non-Riemannian quantity and it is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. Meanwhile Theorem 5.1 means that there exists a Randers metric of *any* constant S-curvature in *each* dimension. After noting this interesting fact, we investigate the existence of non-Randers (α, β)-metrics with arbitrary constant S-curvature. By extending Li-Shen' construction [7] we prove the following:

Theorem 1.1. *For arbitrary real number k and arbitrary natural number n, there exists an (α, β)-metric F defined on an open subset in $\mathbb{R}^n$ which is not Randers type such that F has constant S-curvature k.*

The above theorem tells us that Finsler metrics of constant S-curvature form a rich class of Finsler metrics. For interesting results of H-curvature, we refer the reader to [9, 12, 19].

2. Preliminaries

A Finsler metric is a Riemannian metric without quadratic restriction. Precisely, a function $F(x, y)$ on TM is called a *Finsler metric* on a manifold M if it has the following properties:

(a) $F(x, y)$ is C^∞ on $TM\backslash\{0\}$;

(b) $F_x(y) := F(x, y)$ is a Minkowski norm on T_xM for any $x \in M$.

Define the (*mean*) *distortion* $\tau : SM \to R$ by [15]

$$\tau(x, [y]) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma(x)},$$

where SM is the projective sphere bundle of M, obtained from TM by identifying non-zero vectors which differ from each other by a positive multiplicative factor and

$$\sigma(x) = \frac{\mathrm{Vol}\,(\mathrm{B}^n)}{\mathrm{Vol}\,\{(y^i) \in \mathbb{R}^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}},$$

where B^n denotes the unit ball in $\mathbb{R}^n$ and Vol denotes the Euclidean measure on $\mathbb{R}^n$. To measure the rate of changes of the distortion along geodesics, we define

$$\mathbf{S}(x, y) := \frac{d}{dt}[\tau(\dot{c}(t))]_{t=0},$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. We call the scalar function $\mathbf{S}$ the *S-curvature*. $\mathbf{S}$ is said to be *isotropic* if there is a scalar function $c(x)$ on M such that

$$\mathbf{S}(x, y) = (n + 1)c(x)F(x, y).$$

In particular, $\mathbf{S}$ is said to be of *constant* c if $c =$ constant.

In [10], the authors constructed many new examples of Finsler manifolds of isotropic S-curvature.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on a manifold M. Consider the following function

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $[-r, r]$ satisfying

$$\phi(s) - s\phi'(s) > 0, \qquad \phi''(s) > 0, \qquad |s| \leq r.$$

Then F is a Finsler metric if $\|\beta_x\|_\alpha \leq r$ for any $x \in M$ [17]. A Finsler metric in the form (2.1) is called an (α, β)-*metric*.

Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_i := r_{ij}b^j, \qquad s_i := s_{ij}b^j,$$

where $b_{i|j}$ denote covariant derivative of β with respect to α.

For a positive C^∞ function $\phi = \phi(s)$ on $[-r, r]$ and a number $b \in [0, r]$, let

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \tag{2.2}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q', \qquad Q := \frac{\phi'}{\phi - s\phi'}. \tag{2.3}$$

Recently, Cheng-Shen proved the following [4]:

Theorem 2.1. *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that ϕ is not Randers type. Then F is of isotropic S-curvature if and only if one of the following holds*

(i) *β satisfies*

$$r_j + s_j = 0$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0.$$

In this case, $\mathbf{S} = 0$.

(ii) *β satisfies*

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{2.4}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.5}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\epsilon$.

(iii) *β satisfies*

$$r_{ij} = 0, \quad s_j = 0.$$

In this case, $\mathbf{S} = 0$, regardless of the choice of a particular ϕ.

By using Theorem 2.1, we will show that the existence of $(\alpha,\ \beta)$-metrics with arbitrary constant S-curvature in *each* dimension which is not Randers type in Section 4. We are going to simplify the equation (2.5) in the next section.

The *H-curvature* $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined by $H_{ij} = E_{ij|k}y^k$ where "$|$" denote the covariant horizontal derivatives and E_{ij} denote the mean Berwald curvature of F [9, 12].

3. Third order nonlinear ODE

In this section we are going to give the normal type of (2.5).

Lemma 3.1. *Let* $\psi := \phi - s\phi'$. *Then we have*

$$1 + sQ = \frac{\phi}{\psi}, \tag{3.1}$$

$$Q = \frac{\phi'}{\psi}, \tag{3.2}$$

$$Q' = \frac{\phi\phi''}{\psi^2}, \tag{3.3}$$

$$Q'' = \frac{1}{\psi^3}\left[(\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2\right], \tag{3.4}$$

where Q *is given in* (2.3).

Proof. (3.2) is obvious. By using (3.2) we obtain

$$1 + sQ = 1 + s\frac{\phi'}{\psi} = \frac{1}{\psi}(\psi + s\phi') = \frac{\phi}{\psi}.$$

This gives (3.1). From the definition of ψ, ones get $\psi' = -s\phi''$. Together with (3.2) we get

$$Q' = \frac{\phi''\psi - \phi'\psi'}{\psi^2} = \frac{\phi''(\phi - s\phi') - \phi'(-s\phi'')}{\psi^2} = \frac{\phi\phi''}{\psi^2}$$

which implies (3.3). By a similar calculation, we get

$$Q'' = \frac{(\phi'\phi'' + \phi\phi''')\psi^2 - 2\phi\phi''\psi\psi'}{\psi^4} = \frac{1}{\psi^3}\left[(\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2\right]. \qquad \square$$

Lemma 3.2. *We have the following*

$$\Delta = \phi \cdot \frac{\psi + (b^2 - s^2)\phi''}{\psi^2}, \tag{3.5}$$

$$Q - sQ' = \frac{\phi\phi' - s(\phi'^2 + \phi\phi'')}{\psi^2}, \tag{3.6}$$

$$n\Delta + 1 + sQ = \frac{\phi}{\psi}\left(n + 1 + n\phi''\frac{b^2 - s^2}{\psi}\right). \tag{3.7}$$

Proof. By using (2.3), (3.1) and (3.3) we have

$$\Delta = 1 + sQ + (b^2 - s^2)Q' = \frac{\phi}{\psi} + (b^2 - s^2)\frac{\phi\phi''}{\psi^2} = \phi \cdot \frac{\psi + (b^2 - s^2)\phi''}{\psi^2}.$$

From (3.2), (3.3) and the definition of ψ, we get

$$Q - sQ' = \frac{\phi'}{\psi} - s\frac{\phi\phi''}{\psi^2} = \frac{\phi'(\phi - s\phi') - s\phi\phi''}{\psi^2} = \frac{\phi\phi' - s(\phi'^2 + \phi\phi'')}{\psi^2}.$$

Finally, we have

$$\begin{aligned} n\Delta + 1 + sQ &= n\phi\frac{\psi + (b^2 - s^2)\phi''}{\psi^2} + \frac{\phi}{\psi} \\ &= \frac{\phi}{\psi}\left[n\frac{\psi + (b^2 - s^2)\phi''}{\psi^2} + 1\right] = \frac{\phi}{\psi}\left(n + 1 + n\phi''\frac{b^2 - s^2}{\psi}\right) \end{aligned}$$

from (3.1) and (3.5). □

Lemma 3.3. *Equation* (2.2) *can be rewritten as follows*:

$$\begin{aligned} \Phi = &-\frac{\phi}{\psi^4}\left[\phi\phi' - s(\phi'^2 + \phi\phi'')\right]\left[(n+1)\psi + n\phi''(b^2 - s^2)\right] \\ &-\frac{\phi}{\psi^4}(b^2 - s^2)\left[(\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2\right]. \end{aligned} \tag{3.8}$$

Proof. Substituting (3.6), (3.7), (3.1) and (3.4) into (2.2) we have (3.8). □

Lemma 3.4. *Equation* (2.5) *is equivalent to*

$$\begin{aligned} &2k(n+1)\phi^2\left[(b^2 - s^2)\phi''^2 + 2\psi\phi'' + \frac{\psi^2}{b^2 - s^2}\right] \\ = &\left[\phi\phi' - s(\phi'^2 + \phi\phi'')\right]\left[(n+1)\psi + n\phi''(b^2 - s^2)\right] \\ &+ (b^2 - s^2)\left[(\phi'\phi'' + \phi\phi''')\psi + 2s\phi(\phi'')^2\right]. \end{aligned} \tag{3.9}$$

Proof. Plugging (3.5) and (3.8) into (2.5) yields (3.9). □

From Lemma 3.4 we immediately obtain the following:

Lemma 3.5. *Equation* (3.9) *is equivalent to the following normal ODE*:

$$\begin{aligned} \phi''' = \; &2k(n+1)\phi\left[\frac{\phi''^2}{\psi} + \frac{2\phi''}{b^2 - s^2} + \frac{\psi}{(b^2 - s^2)^2}\right] - \frac{\phi'\phi''}{\phi} - \frac{2s\phi''^2}{\psi} \\ &-\left(\frac{n\phi''}{\psi} + \frac{n+1}{b^2 - s^2}\right)\left[\phi' - s\left(\frac{\phi'^2}{\phi} + \phi''\right)\right]. \end{aligned} \tag{3.10}$$

Remark. It is easy to see that $\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s$ (it corresponds the Randers metrics) are not the solution of (3.10).

4. Proof of Theorem 1.1

Now we are going to construct Riemannian metric α and 1-form β satisfy (2.4) with ϵ = constant. If, at a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ and in the direction $y = (y^1, \ldots, y^n) \in T_x\mathbb{R}^n$, Riemannian metric $\alpha = \alpha(x, y)$ and one form $\beta = \beta(x, y)$ are given by

$$\alpha := \sqrt{(y^1)^2 + e^{2x^1}[(y^2)^2 + \cdots + (y^n)^2]}, \qquad \beta := y^1. \tag{4.1}$$

Then

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2x^1} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e^{2x^1} \end{pmatrix}, \tag{4.2}$$

$$b_1 = 1,\ b_2 = \cdots = b_n = 0, \tag{4.3}$$

where $\alpha^2 = a_{ij}y^iy^j$ and $\beta = b_iy^i$. It follows that

$$(a^{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{-2x^1} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e^{-2x^1} \end{pmatrix}. \tag{4.4}$$

By using (4.3) and (4.4), we obtain

$$b = \sqrt{a^{ij}b_ib_j} = \sqrt{a^{11}b_1^2} = 1. \tag{4.5}$$

From (4.3) we have

$$\frac{\partial b_i}{\partial x^j} = 0.$$

It follows that the covariant derivatives of β with respect to α are given by

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k\Gamma^k_{ij} = -b_k\Gamma^k_{ij} = b_{j|i}.$$

Together with (4.3) we get

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}) = b_{i|j} = -b_k\Gamma^k_{ij} = -\Gamma^1_{ij} \tag{4.6}$$

and $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}) = 0$. By (4.2) and (4.4), the Christoffel symbols of α are given by

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}a^{kl}\left(\frac{\partial a_{il}}{\partial x^j} + \frac{\partial a_{jl}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^l}\right) \\ &= \frac{1}{2}a^{kk}\left(\frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k}\right) \\ &= \begin{cases} -e^{2x^1} & \text{if } i = j \neq k = 1, \\ 1 & \text{if } i = k \neq j = 1,\ j = k \neq i = 1, \\ 0 & \text{others.} \end{cases} \end{aligned}$$

Together with (4.2), (4.3), (4.5) and (4.6) we get

$$r_{ij} = b^2 a_{ij} - b_i b_j = \begin{cases} e^{2x^1} & \text{if } i = j = 2, \ldots, n, \\ 0 & \text{others.} \end{cases}$$

Hence α and β satisfy

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0 \tag{4.7}$$

with $\epsilon = b = 1$.

Remark. When $n = 3$, our construction have been studied by Li-Shen [7].

Now we are going to show the existence of regular solution of (2.5) for arbitrary $k \in \mathbb{R}$ when α and β are given by (4.1).

Let k be an arbitrary constant. We consider the solution of (2.5). By Lemma 3.4 and Lemma 3.5, (2.5) is equivalent to 3-order nonlinear ODE (3.10). Put

$$\phi_0 := \phi, \qquad \phi_1 := \phi', \qquad \phi_2 := \phi''.$$

One can express (3.10) in the following form

$$\phi_0' := \phi_1, \qquad \phi_1' := \phi_2,$$

(4.8)

$$\begin{aligned} \phi_2' \;&=\; 2k(n+1)\phi_0 \left[\frac{\phi_2^2}{\phi_0 - s\phi_1} + \frac{2\phi_2}{1-s^2} + \frac{\phi_0 - s\phi_1}{(1-s^2)^2} \right] - \frac{\phi_1\phi_2}{\phi_0} - \frac{2s\phi_2^2}{\phi_0 - s\phi_1} \\ &\quad - \left(\frac{n\phi_2}{\phi_0 - s\phi_1} + \frac{n+1}{1-s^2} \right) \left[\phi_1 - s\left(\frac{\phi_1^2}{\phi_0} + \phi_2 \right) \right] \\ &:= f(s, \phi_0, \phi_1, \phi_2). \end{aligned}$$

Let $\Omega := (-1,\, 1) \times [\frac{N}{2},\, \frac{3N}{2}] \times [0,\, 2\epsilon] \times [0,\, 2\tau]$ where $N > 4\epsilon$. Then

$$\phi_0 - s\phi_1 \geq \frac{N}{2} - 2\epsilon > 0, \qquad \phi_2 \geq 0 \tag{4.9}$$

for $(s,\, \phi_0,\, \phi_1,\, \phi_2) \in \Omega$. Consider the following 3th order system

$$\mathbf{y}' = F(s,\, \mathbf{y}), \tag{4.10}$$

where

$$\mathbf{y} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix}, \qquad F(s, \phi_0, \phi_1, \phi_2) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ f(s, \phi_0, \phi_1, \phi_2) \end{pmatrix}. \tag{4.11}$$

From (4.8) and (4.11), we can expand $F(s, \phi_0, \phi_1, \phi_2)$ into convergence power series of s, $\phi_0 - N$, $\phi_1 - \epsilon$ and $\phi_2 - \tau$. By using the Cauchy theorem, there exists an analytic solution $\mathbf{y}^*(s)$, defined uniquely in Ω which satisfies $\mathbf{y}^*(0) = (N,\, \epsilon,\, \tau)$ (cf. [6]). Put

$$\mathbf{y}^*(s) = \begin{pmatrix} \phi_0^*(s) \\ \phi_1^*(s) \\ \phi_2^*(s) \end{pmatrix}.$$

Then $\phi^*(s) := \phi_0^*(s)$ is an analytic solution of (2.5), which is defined in $(-1, 1)$ and satisfies $\phi^*(0) = N$. Note that

$$\phi^{*\prime}(s) = \phi_1^*(s), \qquad \phi^{*\prime\prime}(s) = \phi_2^*(s)$$

we get $(s, \phi^*(s), \phi^{*\prime}(s), \phi^{*\prime\prime}(s)) \in \Omega$. Together with (4.11) we obtain

$$\phi^*(s) - s\phi^{*\prime}(s) > 0, \qquad \phi^{*\prime\prime}(s) \geq 0, \qquad |s| < 1.$$

It follows that $\alpha\phi^*(\beta/\alpha)$ is an (α, β)-metric where α and β is defined in (4.1). Together with (4.7) and Theorem 2.1 ones get $\alpha\phi^*(\beta/\alpha)$ is an (α, β)-metric with constant S-curvature k which is not Randers type.

5. Counterexamples to Tang's Theorem 1.2

In this section, we are going to manufacture Randers metrics with non-zero S-curvature which have zero H-curvature. For a Finsler manifold (M, F), the flag curvature is a function $K(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is said to be of *scalar curvature* if the flag curvature $K(P, y) = K(x, y)$ is independent of flags P associated with any fixed flagpole y [5]. In particular, F is said to be of *constant flag curvature* if the flag curvature $K(P, y) =$ constant [16]. By a basic result of Arbar-Zadeh [1, 12] for a Finsler metric of scalar flag curvature, the flag curvature is constant on the manifold if and only if $H = 0$.

Theorem 5.1. *Let $h = |y|$ be the Euclidean metric on $\mathbb{R}^n$, and V be a vector field on $\mathbb{R}^n$ given by*

$$V_x := -2cx + xQ + b,$$

where c is a constant, Q is a skew-symmetric matrix and b is a constant vector with $|b| < 1$. Then Finsler metric F is determined by

$$F(x, y) = h(x, y - F(x, y)V_x)$$

is a Randers metric which has the following non-Riemannian curvature properties:

(a) *vanishing H-curvature*

$$\mathbf{H} = 0.$$

(b) *constant S-curvature*

$$\mathbf{S}(x, y) = (n + 1)cF(x, y).$$

Proof. (b) is an immediate conclusion of Theorem 7.3.8 in [5]. On the other hand, Chern-Shen' result tells us F has constant flag curvature. Combining this with Arbar-Zadeh' result yields (a). □

Let us take a look at the special case when $c \neq 0$, F is a Randers metric with non-zero S-curvature which have zero H-curvature. Thus F is a counterexample to Theorem 1.2 in [20].

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