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MAXIMAL PROPERTIES OF SOME SUBSEMIBANDS OF ORDER-PRESERVING FULL TRANSFORMATIONS

Ping Zhao and Mei Yang

Abstract. Let $[n] = \{1, 2, \ldots, n\}$ be ordered in the standard way. The order-preserving full transformation semigroup $\mathcal{O}_n$ is the set of all order-preserving singular full transformations on $[n]$ under composition. For this semigroup we describe maximal subsemibands, maximal regular subsemibands, locally maximal regular subsemibands, and completely obtain their classification.

1. Introduction

A semigroup is called *idempotent-generated* or *semiband* if it is generated by its idempotents. The latter term was introduced by F. Pastijn [7].

Let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way. We denote by $\mathcal{S}ing_n$ the semigroup (under composition) of all singular full transformations on $[n]$. We say that a full transformation α in $\mathcal{S}ing_n$ is *order-preserving* if, for all $x, y \in [n]$, $x \leq y$ implies $x\alpha \leq y\alpha$. We denote by $\mathcal{O}_n$ the subsemigroup of $\mathcal{S}ing_n$ of all order-preserving singular full transformations.

The semigroup $\mathcal{O}_n$ was studied first by Aizenstat [1] and subsequently by many authors (see, for example [2-6], [8-13]). In particular, Howie [5] proved that $\mathcal{O}_n$ is a regular semiband. Garba [2] further proved that each one of the ideals of $\mathcal{O}_n$ is also a regular semiband. Yang [12] classified completely maximal subsemibands and maximal regular subsemibands of $\mathcal{O}_n$. Recently, Xu, Zhao and Li [9] obtained a complete classification of locally maximal subsemibands of $\mathcal{O}_n$. Further, Zhao, Xu and Yang [13] obtained a simpler form of the classification of maximal (regular) subsemibands of $\mathcal{O}_n$, using results of Xu, Zhao and Li [9].

In view of the above work, it is natural to seek a description of the locally maximal regular subsemibands of $\mathcal{O}_n$. In Section 2 we obtain a same simpler form of the classification of maximal (regular) subsemibands of $\mathcal{O}_n$, using a

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different approach from Zhao, Xu and Yang [13]. In Section 3 we obtain a classification of locally maximal regular subsemibands of $\mathcal{O}_n$.

From Gomes and Howie [3], Green's equivalences on $\mathcal{O}_n$ are characterized as:

$$\alpha\mathcal{L}\beta \Leftrightarrow im(\alpha) = im(\beta),$$
$$\alpha\mathcal{R}\beta \Leftrightarrow ker(\alpha) = ker(\beta),$$
$$\alpha\mathcal{J}\beta \Leftrightarrow |im(\alpha)| = |im(\beta)|.$$

Thus $\mathcal{O}_n$ has $n-1$ $\mathcal{J}$-classes: $J_1, J_2, \ldots, J_{n-1}$, where $J_r = \{\alpha \in \mathcal{O}_n : |im(\alpha)| = r\}$.

Gomes and Howie [3] used the notation $[i \to i+1]$ for the increasing idempotent e defined by $ie = i+1$, $xe = x$ $(x \neq i)$, and $[i \to i-1]$ for the decreasing idempotent f defined by $if = i-1, xf = x$ $(x \neq i)$. As usual, we denote by $E(S)$ the set of all idempotents of a subset S of $\mathcal{O}_n$. Let $E^+_{n-1} = \{[i \to i+1] : 1 \leq i \leq n-1\}$ and $E^-_{n-1} = \{[i \to i-1] : 2 \leq i \leq n\}$ be the increasing and decreasing idempotent sets, respectively. Then $E(J_{n-1}) = E^+_{n-1} \cup E^-_{n-1}$.

2. Maximal (regular) subsemibands of $\mathcal{O}_n$

Both maximal subsemibands and maximal regular subsemibands of $\mathcal{O}_n$ were studied by [12]. Zhao, Xu and Yang [13] obtained a simpler form of the classification of maximal (regular) subsemibands of $\mathcal{O}_n$, using results of Xu, Zhao and Li [9]. In this section, we obtain a same simpler form of the classification of maximal (regular) subsemibands of $\mathcal{O}_n$, using a different approach from Zhao, Xu and Yang [13]. For convenience, we introduce the following notation from [8].

Let

$$\mathcal{C}^-_n = \{\alpha \in \mathcal{O}_n : (\forall x \in [n])\ \ x\alpha \leq x\},$$
$$\mathcal{C}^+_n = \{\alpha \in \mathcal{O}_n : (\forall x \in [n])\ \ x\alpha \geq x\},$$

be the semigroups of all singular order-preserving and decreasing full transformations and order-preserving and increasing full transformations on $[n]$, respectively.

As in [4], for any $\alpha \in \mathcal{O}_n$, let

$$x\alpha^- = \begin{cases} x\alpha, & x \in [n]^-_\alpha; \\ x, & \text{otherwise}, \end{cases}$$

$$x\alpha^+ = \begin{cases} x\alpha, & x \in [n]^+_\alpha; \\ x, & \text{otherwise}, \end{cases}$$

where $[n]^-_\alpha = \{x \in [n] : x\alpha \leq x\}$, and $[n]^+_\alpha = \{x \in [n] : x\alpha \geq x\}$. It is obvious that $\alpha^- \in \mathcal{C}^-_n$ and $\alpha^+ \in \mathcal{C}^+_n$. The following lemma was proved by Higgins [4, page 1053].

Lemma 2.1. *Let* $\alpha \in \mathcal{O}_n$. *Then*

$$\alpha = \alpha^+\alpha^- = \alpha^-\alpha^+,$$

with $\alpha^- \in \mathcal{C}_n^-$, $\alpha^+ \in \mathcal{C}_n^+$.

For convenience, we use $[n \to n+1]$ or $[1 \to 0]$ to denote $\emptyset$ (the empty mapping). With this notation, we have:

Lemma 2.2. *Let* $\alpha \in \mathcal{C}_n^+$. *If* $k\alpha = k$ *for some* $1 \le k \le n$, *then*

$$\alpha \in \langle E_{n-1}^+ \backslash \{[k \to k+1]\} \rangle.$$

Proof. Since $\alpha \in \mathcal{C}_n^+ \subseteq \mathcal{O}_n$, we have that the $ker(\alpha)$-classes are convex subsets C of $[n]$, in the sense that

$$x, y \in C \text{ and } x \le z \le y \implies z \in C.$$

Then α can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $A_i = \{a_i, a_i+1, \ldots, a_{i+1}-1\}$ $(1 \le i \le r-1)$, $A_r = \{a_r, a_r+1, \ldots, n\}$, $1 = a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Since $\alpha \in \mathcal{C}_n^+$, we have

$$a_i = \min A_i \le \max A_i \le (\max A_i)\alpha = b_i, \ 1 \le i \le r-1,$$
$$a_r = \min A_r \le \max A_r (= n) \le (\max A_r)\alpha = b_r.$$

Thus

$$b_r = n \ \text{ and } \ \ a_i \le b_i, \ i \in [n].$$

Let e_0 be the identity mapping on $[n]$, and let

$$E^+(i,j) = [i \to i+1] \cdot [i+1 \to i+2] \cdots [j-1 \to j], \ 1 \le i < j \le n,$$
$$E^+(i,i) = e_0, \ i \in [n].$$

Further, let

$$\beta = E^+(a_r, b_r)E^+(a_{r-1}, b_{r-1}) \cdots E^+(a_1, b_1).$$

We claim that $\alpha = \beta$. To prove that $\alpha = \beta$, take any $x \in [n]$. Suppose that $x \in A_s$ $(1 \le s \le r)$. Then

$$x\beta = xE^+(a_r, b_r)E^+(a_{r-1}, b_{r-1}) \cdots E^+(a_1, b_1) = b_s = x\alpha.$$

Note that $[n \to n+1] = \emptyset$ (the empty mapping). If $k = n$, then $\alpha = \beta \in \langle E_{n-1}^+ \cup \{e_0\} \rangle = \langle E_{n-1}^+ \backslash \{[n \to n+1]\} \rangle \cup \{e_0\}$. Since $\alpha \in \mathcal{C}_n^+ \subseteq \mathcal{O}_n \subseteq Sing_n$, we have $\alpha \in \langle E_{n-1}^+ \backslash \{[n \to n+1]\} \rangle$. If $1 \le k \le n-1$. Note that $b_r = n$. Since $k\alpha = k$, there exists $1 \le j \le r-1$ such that $k \in A_j$ and $b_j = k$. Since $\alpha \in \mathcal{C}_n^+$, we have $b_j = \max A_j = k$ and so $a_{j+1} = \min A_{j+1} = k+1$. Thus

$$(2.1) \quad [k \to k+1] \notin \{[a_i \to a_i+1], [a_i+1 \to a_i+2], \ldots, [b_i-1 \to b_i]\}, \ a_i < b_i.$$

Note that $E^+(a_i, b_i) = [a_i \to a_i+1][a_i+1 \to a_i+2] \cdots [b_i-1 \to b_i]$ if $a_i < b_i$; $E^+(a_i, b_i) = e_0$ if $a_i = b_i$. It follows immediately from (2.1) that

$$E^+(a_i, b_i) \in \langle E_{n-1}^+ \backslash \{[k \to k+1]\} \cup \{e_0\} \rangle, \ 1 \le i \le r.$$

Then $\alpha = \beta \in \langle E^+_{n-1} \backslash \{[k \to k+1]\} \cup \{e_0\}\rangle$. It is obvious that $\langle E^+_{n-1} \backslash \{[k \to k+1]\} \cup \{e_0\}\rangle = \langle E^+_{n-1} \backslash \{[k \to k+1]\}\rangle \cup \{e_0\}$. Since $\alpha \in \mathcal{C}^+_n \subseteq \mathcal{O}_n \subseteq \mathcal{S}ing_n$, we have

$$\alpha \in \langle E^+_{n-1} \backslash \{[k \to k+1]\}\rangle. \qquad \square$$

Similarly, we can prove:

Lemma 2.3. *Let $\alpha \in \mathcal{C}^-_n$. If $k\alpha = k$ for some $1 \leq k \leq n$. Then*

$$\alpha \in \langle E^-_{n-1} \backslash \{[k \to k-1]\}\rangle.$$

The following lemma is immediate by definition of α^+, α^-:

Lemma 2.4. *For any $\alpha \in \mathcal{O}_n$, we have*
(i) *If $k\alpha \leq k$ for some $1 \leq k \leq n$, then $k\alpha^+ = k$.*
(ii) *If $k\alpha \geq k$ for some $1 \leq k \leq n$, then $k\alpha^- = k$.*

For any $s, t \in [n]$, let

$$M_{st} = \{\alpha \in \mathcal{O}_n : s\alpha \leq s, t\alpha \geq t\}. \tag{2.2}$$

With above notation, we have:

Lemma 2.5. *Let $n \geq 3$. Then*

$$M_{st} = \langle E(J_{n-1}) \backslash \{[s \to s+1], [t \to t-1]\}\rangle, \ s, t \in [n].$$

Proof. Let $P_{st} = \langle E(J_{n-1}) \backslash \{[s \to s+1], [t \to t-1]\}\rangle$. It is easy to prove that M_{st} is a subsemigroup of $\mathcal{O}_n$. It is obvious that $E(J_{n-1}) \backslash \{[s \to s+1], [t \to t-1]\} \subseteq M_{st}$. Then $P_{st} = \langle E(J_{n-1}) \backslash \{[s \to s+1], [t \to t-1]\}\rangle \subseteq M_{st}$.

It remains to prove that $M_{st} \subseteq P_{st}$. Let $\alpha \in M_{st} \subseteq \mathcal{O}_n$. By Lemmas 2.1 and 2.4, we have

$$\alpha = \alpha^+\alpha^- = \alpha^-\alpha^+, \quad \alpha^- \in \mathcal{C}^-_n, \quad \alpha^+ \in \mathcal{C}^+_n,$$

and $s\alpha^+ = s$, $t\alpha^- = t$. Note that $E(J_{n-1}) = E^+_{n-1} \cup E^-_{n-1}$. Thus, by Lemmas 2.2 and 2.3,

$$\begin{aligned} \alpha = \alpha^+\alpha^- &\in \langle E^+_{n-1} \backslash \{[s \to s+1]\}\rangle \cdot \langle E^-_{n-1} \backslash \{[t \to t-1]\}\rangle \\ &\subseteq \langle E(J_{n-1}) \backslash \{[s \to s+1], [t \to t-1]\}\rangle = P_{st}. \end{aligned} \qquad \square$$

A subsemiband S of $\mathcal{O}_n$ is called *maximal subsemiband* if for an arbitrary subsemiband T of $\mathcal{O}_n$ such that $S \subset T$, then $T = \mathcal{O}_n$. Combining [12, Theorem 2.1 and Lemma 2.3], we obtain the following.

Lemma 2.6. *Let $n \geq 3$. Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Then each maximal subsemiband of $\mathcal{O}_n$ must be one of the following forms*:
(C) $C_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1]\rangle$, $s = 1, 2, \ldots, n-1$.
(D) $D_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s-1]\rangle$, $s = 2, 3, \ldots, n$.

Now, it is easy to prove one of the main results of this section:

Theorem 2.7. *Let $n \geq 3$. Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Then each maximal subsemiband of $\mathcal{O}_n$ must be one of the following forms*:

(A) $A_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \leq s\}$, $s = 1, 2, \ldots, n-1$.

(B) $B_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \geq s\}$, $s = 2, 3, \ldots, n$.

Proof. Let M_{st} be as defined in (2.2). Then $M_{s1} = \{\alpha \in \mathcal{O}_n : s\alpha \leq s\}$ and $M_{ns} = \{\alpha \in \mathcal{O}_n : s\alpha \geq s\}$. Note that $[1 \to 0] = [n \to n+1] = \emptyset$ (the empty mapping). Thus, by Lemma 2.5,

$$A_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \leq s\} = I_{n-2} \cup M_{s1}$$

$$= I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1]\rangle = C_s,$$

$$B_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \geq s\} = I_{n-2} \cup M_{ns}$$

$$= I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s-1]\rangle = D_s.$$

Hence Theorem 2.7 holds by Lemma 2.6. □

A regular subsemiband S of $\mathcal{O}_n$ is called *maximal regular subsemiband* if for an arbitrary regular subsemiband T of $\mathcal{O}_n$ such that $S \subset T$, then $T = \mathcal{O}_n$. Note that $[1 \to 0] = [n \to n+1] = \emptyset$ (the empty mapping). Combining [12, Lemma 2.3 and Theorem 4.1], we obtain the following.

Lemma 2.8. *Let $n \geq 4$. Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Then each maximal regular subsemiband of $\mathcal{O}_n$ must be the following forms*:

(F) $F_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\}\rangle$, $s = 1, 2, \ldots, n$.

(G) $G_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s+1 \to s]\}\rangle$, $s = 2, 3, \ldots, n-2$.

Using Lemma 2.5 and Lemma 2.8, the other main result of this section is now established:

Theorem 2.9. *Let $n \geq 4$. Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Then each maximal regular subsemiband of $\mathcal{O}_n$ must be the following forms*:

(A) $A_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha = s\}$, $s = 1, 2, \ldots, n$.

(B) $B_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \leq s, (s+1)\alpha \geq s+1\}$, $s = 2, 3, \ldots, n-2$.

Proof. Let M_{st} be as defined in (2.2). Then $M_{ss} = \{\alpha \in \mathcal{O}_n : s\alpha = s\}$ and $M_{s(s+1)} = \{\alpha \in \mathcal{O}_n : s\alpha \leq s, (s+1)\alpha \geq s+1\}$. Note that $[1 \to 0] = [n \to n+1] = \emptyset$ (the empty mapping). Thus, by Lemma 2.5,

$$A_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha = s\} = I_{n-2} \cup M_{ss}$$

$$= I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\}\rangle = F_s,$$

$$B_s = I_{n-2} \cup \{\alpha \in \mathcal{O}_n : s\alpha \leq s, (s+1)\alpha \geq s+1\} = I_{n-2} \cup M_{s(s+1)}$$

$$= I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s+1 \to s]\}\rangle = G_s.$$

Hence Theorem 2.9 holds by Lemma 2.8. □

3. Locally maximal regular subsemibands of $\mathcal{O}_n$

Let I be a subset of $E(J_{n-1})$. A subsemiband $\langle I\rangle$ of $\mathcal{O}_n$ is called a *locally maximal regular subsemiband* if $\langle I\rangle$ is regular, and any regular subsemiband $\langle J\rangle$ ($J \subseteq E(J_{n-1})$) of $\mathcal{O}_n$ properly containing $\langle I\rangle$ must be $\mathcal{O}_n$. In this section, we obtain a classification of locally maximal regular subsemibands of $\mathcal{O}_n$.

The main result of this section is:

Theorem 3.1. *Let $n \geq 4$. Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Then each locally maximal regular subsemiband of $\mathcal{O}_n$ must be the following forms*:
(A) $A_s = \{\alpha \in \mathcal{O}_n : s\alpha = s\},\ s = 1, 2, \ldots, n$.
(B) $B_s = \{\alpha \in \mathcal{O}_n : s\alpha \leq s, (s+1)\alpha \geq s+1\},\ s = 2, 3, \ldots, n-2$.

To prove Theorem 3.1, we begin by establishing a series of lemmas. Combining [5, Lemmas 1.2 and 1.3], we know that $\mathcal{O}_n$ is generated by $E(J_{n-1})$. Note that $|E(J_{n-1})| = 2n-2$. From the result [3, Theorem 2.8] that the rank of $\mathcal{O}_n$ is $2n-2$, we immediately deduce:

Lemma 3.2. *Let $n \geq 4$. Then*

$$\mathcal{O}_n = \langle E(J_{n-1})\rangle \textit{ and no proper subset of } E(J_{n-1}) \textit{ can generate } \mathcal{O}_n.$$

It is well known that the characterized forms of the Green's relations in $\mathcal{S}ing_n$ are the same as in $\mathcal{O}_n$ (see Section 1). $\mathcal{S}ing_n$ has $n-1$ $\mathcal{J}$-classes: $SJ_r = \{\alpha \in \mathcal{S}ing_n : |im(\alpha)| = r\},\ r = 1, 2, \ldots, n-1$. Let

$$SI_r = \{\alpha \in \mathcal{S}ing_n : |im(\alpha)| \leq r\},\ r = 1, 2, \ldots, n-1.$$

Then the sets SI_r are two-sided ideal of $\mathcal{S}ing_n$. As usual, we denote by $E(S)$ the set of all idempotents of a subset S of $\mathcal{S}ing_n$. Let I be nonempty subsets of $E(SJ_{n-1})$. It is obvious that $I \subseteq E(\langle I\rangle \cap SJ_{n-1})$. In general, $E(\langle I\rangle \cap SJ_{n-1}) \subseteq I$ is false. For example, let

$$f = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n,1\} \\ 2 & 3 & \cdots & n-1 & 1 \end{pmatrix},\ g = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n,1\} \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix},$$

then $f, g \in E(SJ_{n-1})$. Let $\eta = f \cdot [n-1 \to n] \cdot\ [n-2 \to n-1] \cdots [1 \to 2]$, then

$$\eta = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n,1\} \\ 3 & 4 & \cdots & n & 2 \end{pmatrix}$$

and so $\eta^{n-1} = g$. Clearly, $E^+_{n-1} \subseteq E(SJ_{n-1})$. Let $I = E^+_{n-1} \cup \{f\}$. Then $g = \eta^{n-1} \in \langle I\rangle$ and so $g \in E(\langle I\rangle \cap SJ_{n-1})$. Clearly, $g \notin I$. Thus $E(\langle I\rangle \cap SJ_{n-1}) \nsubseteq I$. However, using Lemma 3.2, we have the following.

Lemma 3.3. *Let I be a subset of $E(J_{n-1})$. Then*

$$E(\langle I\rangle \cap J_{n-1}) = I.$$

Proof. Clearly, $I \subseteq E(\langle I\rangle \cap J_{n-1})$. Now, we need to prove that $E(\langle I\rangle \cap J_{n-1}) \subseteq I$. Note that $I \subseteq E(\langle I\rangle \cap J_{n-1}) \subseteq \langle I\rangle$. Then $\langle I\rangle \subseteq \langle E(\langle I\rangle \cap J_{n-1})\rangle \subseteq \langle\langle I\rangle\rangle = \langle I\rangle$

and so $\langle E(\langle I\rangle \cap J_{n-1})\rangle = \langle I\rangle$. Let $I^* = E(\langle I\rangle \cap J_{n-1})\backslash I$. Then $I \subseteq E(J_{n-1})\backslash I^*$ and so $\langle I^*\rangle \subseteq \langle E(\langle I\rangle \cap J_{n-1})\rangle = \langle I\rangle \subseteq \langle E(J_{n-1})\backslash I^*\rangle$. Thus

$$E(J_{n-1}) = I^* \cup (E(J_{n-1})\backslash I^*) \subseteq \langle I^*\rangle \cup \langle E(J_{n-1})\backslash I^*\rangle = \langle E(J_{n-1})\backslash I^*\rangle$$

and so $\langle E(J_{n-1})\rangle \subseteq \langle E(J_{n-1})\backslash I^*\rangle \subseteq \mathcal{O}_n$. It follows immediately form Lemma 3.2 that

$$E(J_{n-1})\backslash I^* = E(J_{n-1}).$$

Then $I^* = \emptyset$ (the empty set) and so $E(\langle I\rangle \cap J_{n-1}) \subseteq I$. □

Further, we have:

Lemma 3.4. *Let $I_{n-2} = \{\alpha \in \mathcal{O}_n : |im(\alpha)| \leq n-2\}$. Let I and J be nonempty subsets of $E(J_{n-1})$. Then*
(i) $I \subseteq J \Leftrightarrow \langle I\rangle \subseteq \langle J\rangle \Leftrightarrow I_{n-2} \cup \langle I\rangle \subseteq I_{n-2} \cup \langle J\rangle$.
(ii) $I \subset J \Leftrightarrow \langle I\rangle \subset \langle J\rangle \Leftrightarrow I_{n-2} \cup \langle I\rangle \subset I_{n-2} \cup \langle J\rangle$.

Proof. (i) Clearly,

$$I \subseteq J \Rightarrow \langle I\rangle \subseteq \langle J\rangle \Rightarrow I_{n-2} \cup \langle I\rangle \subseteq I_{n-2} \cup \langle J\rangle.$$

To prove that

$$I \subseteq J \Leftarrow \langle I\rangle \subseteq \langle J\rangle \Leftarrow I_{n-2} \cup \langle I\rangle \subseteq I_{n-2} \cup \langle J\rangle.$$

It suffices to prove that

$$I_{n-2} \cup \langle I\rangle \subseteq I_{n-2} \cup \langle J\rangle \Rightarrow I \subseteq J.$$

Suppose that $I_{n-2} \cup \langle I\rangle \subseteq I_{n-2} \cup \langle J\rangle$. Then $\langle I\rangle \cap J_{n-1} = (I_{n-2} \cup \langle I\rangle) \cap J_{n-1} \subseteq (I_{n-2} \cup \langle J\rangle) \cap J_{n-1} = \langle J\rangle \cap J_{n-1}$. Thus, by Lemma 3.3,

$$I = E(\langle I\rangle \cap J_{n-1}) \subseteq E(\langle J\rangle \cap J_{n-1}) = J.$$

(ii) By (i), we easily deduce that

$$I = J \Leftrightarrow \langle I\rangle = \langle J\rangle \Leftrightarrow I_{n-2} \cup \langle I\rangle = I_{n-2} \cup \langle J\rangle.$$

It follows immediately that

$$I \subset J \Leftrightarrow \langle I\rangle \subset \langle J\rangle \Leftrightarrow I_{n-2} \cup \langle I\rangle \subset I_{n-2} \cup \langle J\rangle.$$ □

Now, we can use Lemmas 2.5, 2.8, 3.2 and 3.4 to obtain the following.

Lemma 3.5. *For $n \geq 4$ and $s \in [n]$, let M_{ss} be as defined in* (2.2). *Then M_{ss} is a locally maximal regular subsemiband of $\mathcal{O}_n$.*

Proof. Recall that $M_{ss} = \{\alpha \in \mathcal{O}_n : s\alpha = s\}$. Let $\alpha \in M_{ss}$. If $|im(\alpha)| = 1$, then clearly $\alpha = \alpha\alpha$ and so α is regular. If $|im(\alpha)| \geq 2$, suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \in M_{ss},$$

where $a_1 < a_2 < \cdots < a_r$, $\min A_i > \max A_{i-1}$, $i = 2, 3, \ldots, r$. Since $\alpha \in M_{ss}$, there exists $k \in \{1, 2, \ldots, r\}$ such that $s \in A_k$ and $a_k = s$. Let

$$\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $b_k = s \in A_k$, $b_i = \min A_i$, $i \neq k$, $B_1 = \{1, 2, \ldots, a_1\}$, $B_s = \{x \in [n] : a_{s-1} < x \leq a_s\}$, $s = 2, 3, \ldots, r-1$, and $B_r = \{a_{r-1}+1, \ldots, n\}$, then $\alpha = \alpha\beta\alpha$ and $\beta \in M_{ss}$ (since $s = a_k \in B_k$ and $b_k = s$). Then α is regular and by Lemma 2.5, we have

$$M_{ss} = \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\} \rangle. \tag{3.1}$$

Thus M_{ss} is a regular subsemiband.

For some $J \subseteq E(J_{n-1})$, let $\langle J \rangle$ be a regular subsemiband of $\mathcal{O}_n$ properly containing M_{ss}, see (3.1). Then, by Lemma 3.4(ii),

$$E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\} \subset J. \tag{3.2}$$

Let $T = I_{n-2} \cup \langle J \rangle$ and let F_s be as defined in Lemma 2.8, i.e., $F_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\} \rangle$, see (3.2). Then, by Lemma 3.4(ii), $F_s \subset T$, and since $\langle J \rangle$ is regular, I_{n-2} is a regular semiband (see [2]) and also an ideal of $\mathcal{O}_n$, we deduce that T is a regular subsemiband of $\mathcal{O}_n$. Thus, by maximality of F_s (by Lemma 2.8) and $F_s \subset T$, $T = I_{n-2} \cup \langle J \rangle = \mathcal{O}_n$. It now follows immediately that $E(J_{n-1}) \subseteq \langle J \rangle$ and so $\langle E(J_{n-1}) \rangle \subseteq \langle J \rangle$. Thus, by Lemma 3.2, $\langle J \rangle = \mathcal{O}_n$. □

Also, using Lemmas 2.5, 2.8, 3.2 and 3.4, we have:

Lemma 3.6. *For $n \geq 4$ and $2 \leq s \leq n-2$, let $M_{s(s+1)}$ be as defined in* (2.2). *Then $M_{s(s+1)}$ is a locally maximal regular subsemiband of $\mathcal{O}_n$.*

Proof. Recall that $M_{s(s+1)} = \{\alpha \in \mathcal{O}_n : s\alpha \leq s, (s+1)\alpha \geq s+1\}$. Note that for any $\alpha \in M_{s(s+1)}$, $|im(\alpha)| \geq 2$ (since $s\alpha \leq s$ and $(s+1)\alpha \geq s+1$). Consider a typical element

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \in M_{s(s+1)},$$

where $a_1 < a_2 < \cdots < a_r$, $\min A_i > \max A_{i-1}$, $i = 2, \ldots, r$. Since $\alpha \in M_{s(s+1)}$, there exist $k \in \{1, 2, \ldots, r-1\}$ such that $s \in A_k$, $s+1 \in A_{k+1}$ and $a_k \leq s < s+1 \leq a_{k+1}$. Let $c_i = a_i$ $(i \neq k)$ and $c_k = s$, then $c_1 < c_2 < \cdots < c_r$. Let

$$\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $b_{k+1} = s+1 \in A_{k+1}$, $b_i = \min A_i$, $i \neq k+1$, $B_1 = \{1, 2, \ldots, c_1\}$, $B_i = \{x \in [n] : c_{i-1} < x \leq c_i\}$, $i = 2, 3, \ldots, r-1$ and $B_r = \{c_{r-1}+1, \ldots, n\}$. Clearly, $\beta \in \mathcal{O}_n$. Note that $s = c_k \in B_k$ and $s+1 \in B_{k+1}$ (since $c_k = s < s+1 \leq a_{k+1} = c_{k+1}$). It follows that $s\beta = B_k\beta = b_k = \min A_k \leq s$ (since $s \in A_k$) and $(s+1)\beta = B_{k+1}\beta = b_{k+1} = s+1$. Thus $\beta \in M_{s(s+1)}$. Note that $c_i = a_i$ $(i \neq k)$ and $a_k \leq s = c_k < s+1 \leq a_{k+1}$. It follows that $a_i \in B_i$

$(i = 1, 2, \ldots, r)$ and so $\alpha = \alpha\beta\alpha$. Thus α is regular and by Lemma 2.5, we have

$$M_{s(s+1)} = \langle E(J_{n-1})\backslash\{[s \to s+1], [s+1 \to s]\}\rangle. \tag{3.3}$$

Thus $M_{s(s+1)}$ is a regular subsemiband.

For some $J \subseteq E(J_{n-1})$, let $\langle J\rangle$ be a regular subsemiband of $\mathcal{O}_n$ properly containing $M_{s(s+1)}$, see (3.3). Then, by Lemma 3.4(ii),

$$E(J_{n-1})\backslash\{[s \to s+1], [s+1 \to s]\} \subset J. \tag{3.4}$$

Let $T = I_{n-2} \cup \langle J\rangle$ and let G_s be as defined in Lemma 2.8, i.e., $G_s = I_{n-2} \cup \langle E(J_{n-1})\backslash\{[s \to s+1], [s+1 \to s]\}\rangle$. Then, by Lemma 3.4(ii), $G_s \subset T$. Since $\langle J\rangle$ is regular, I_{n-2} is a regular semiband (see [2]) and also an ideal of $\mathcal{O}_n$, we deduce that T is a regular subsemiband of $\mathcal{O}_n$. Thus, by maximality of G_s (by Lemma 2.8) and $G_s \subset T$, $T = I_{n-2} \cup \langle J\rangle = \mathcal{O}_n$. It follows immediately that $E(J_{n-1}) \subseteq \langle J\rangle$ and so $\langle E(J_{n-1})\rangle \subseteq \langle J\rangle$. Thus, by Lemma 3.2, $\langle J\rangle = \mathcal{O}_n$. □

The following lemma gives a necessary condition for a locally regular subsemiband of $\mathcal{O}_n$ to be maximal.

Lemma 3.7. *Let I be a nonempty set of $E(J_{n-1})$. If $\langle I\rangle$ is a locally maximal regular subsemiband of $\mathcal{O}_n$, then $T = I_{n-2} \cup \langle I\rangle$ is a maximal regular subsemiband of $\mathcal{O}_n$.*

Proof. Suppose that $\langle I\rangle$ is a locally maximal regular subsemiband of $\mathcal{O}_n$. Let M be a regular subsemiband of $\mathcal{O}_n$ properly containing T. Since $M = \langle E(M)\rangle$ and $I_{n-2} \subseteq M$ (since $T \subset M$), we have $M = I_{n-2} \cup M = I_{n-2} \cup \langle E(M \cap J_{n-1})\rangle$ and so

$$I_{n-2} \cup \langle I\rangle = T \subset M = I_{n-2} \cup \langle E(M \cap J_{n-1})\rangle.$$

Note that $E(M \cap J_{n-1}) \subseteq E(J_{n-1})$. Then, by Lemma 3.4(ii), $\langle I\rangle \subset \langle E(M \cap J_{n-1})\rangle$ and so, by the locally maximality of $\langle I\rangle$, $\langle E(M \cap J_{n-1})\rangle = \mathcal{O}_n$. Thus $M = \mathcal{O}_n$ and so $T = I_{n-2} \cup \langle I\rangle$ is a maximal regular subsemiband of $\mathcal{O}_n$. □

Now, we can prove Theorem 3.1.

Proof Theorem 3.1. Let M_{ss} and $M_{s(s+1)}$ be defined earlier. It is obvious that

$$A_s = \{\alpha \in \mathcal{O}_n : s\alpha = s\} = M_{ss}, \tag{3.4}$$

$$B_s = \{\alpha \in \mathcal{O}_n : s\alpha \le s, (s+1)\alpha \ge s+1\} = M_{s(s+1)}. \tag{3.5}$$

Thus, by Lemmas 3.5 and 3.6, A_s and B_s are locally maximal regular subsemibands of $\mathcal{O}_n$.

Conversely, we shall prove that each locally maximal regular subsemiband of $\mathcal{O}_n$ must be of the form A_s or B_s. For some $I \subseteq E(J_{n-1})$, let $\langle I\rangle$ is a locally maximal regular subsemiband of $\mathcal{O}_n$. Then, by Lemma 3.7, $T = I_{n-2} \cup \langle I\rangle$ is a maximal regular subsemiband of $\mathcal{O}_n$. Thus, by Lemma 2.8, there exists $s \in [n]$ such that $T = F_s = I_{n-2} \cup \langle E(J_{n-1})\backslash\{[s \to s+1], [s \to s-1]\}\rangle$ or there exists

$s \in \{2, 3, \ldots, n-2\}$ such that $T = G_s = I_{n-2} \cup \langle E(J_{n-1}) \backslash \{[s \to s+1], [s+1 \to s]\}\rangle$. It follows immediately from Lemmas 3.4 that

$$\langle I \rangle = \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\}\rangle \text{ or}$$
$$\langle I \rangle = \langle E(J_{n-1}) \backslash \{[s \to s+1], [s+1 \to s]\}\rangle.$$

Thus, by Lemma 2.5 and (3.4), (3.5),

$$\langle I \rangle = \langle E(J_{n-1}) \backslash \{[s \to s+1], [s \to s-1]\}\rangle = M_{ss} = A_s \text{ or}$$
$$\langle I \rangle = \langle E(J_{n-1}) \backslash \{[s \to s+1], [s+1 \to s]\}\rangle = M_{s(s+1)} = B_s.$$

□

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References

[1] A. Ya. Aizenstat, *The defining relations of the endomorphism semigroup of a finite linearly ordered set*, Sibirsk. Mat. **3** (1962), 161–169.

[2] G. U. Garba, *On the idempotent ranks of certain semigroups of order-preserving transformations*, Portugal. Math. **51** (1994), no. 2, 185–204.

[3] G. M. S. Gomes and J. M. Howie, *On the ranks of certain semigroups of order-preserving transformations*, Semigroup Forum **45** (1992), no. 3, 272–282.

[4] P. M. Higgins, *Idempotent depths in semigroups of order-preserving mappings*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 5, 1045–1058.

[5] J. M. Howie, *Products of idempotents in certain semigroups of transformation*, Proc. Edinburgh Math. Soc. **17** (1971), 223–236.

[6] A. Laradji and A. Umar, *Combinatorial results for semigroups of order-preserving full transformations*, Semigroup Forum **62** (2006), no. 1, 51–62.

[7] F. Pastijin, *Embedding semigroups in semibands*, Semigroup Forum **14** (1977), no. 3, 247–263.

[8] A. Umar, *On the semigroups of order-decreasing finite full transformations*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), no. 1-2, 129–142.

[9] B. Xu, P. Zhao, and J. Y. Li, *Locally maximal idempotent-generated subsemigroups of singular order-preserving transformation semigroups*, Semigroup Forum **72** (2006), no. 3, 488–492.

[10] X. L. Yang, *On the nilpotent ranks of the principal factors of order-preserving transformation semigroups*, Semigroup Forum **57** (1998), no. 3, 331–340.

[11] ______, *Products of idempotents of defect 1 in certain semigroups of transformations*, Comm. Algebra **27** (1999), no. 7, 3557–3568.

[12] X. L. Yang and C. H. Lu, *Maximal properties of some subsemigroups in finite order-preserving transformation semigroups*, Comm. Algebra **28** (2000), no. 7, 3125–3135.

[13] P. Zhao, B. Xu, and M. Yang, *A note on maximal properties of some subsemigroups of order-preserving transformation semigroups*, Comm. Algebra **40** (2012), no. 3, 1116–1121.

Ping Zhao
School of Mathematics and Computer Science
GuiZhou Normal University
Guiyang, GuiZhou Province 550001, P. R. China
and
Mathematics Teaching & Research Section
Guiyang Medical College
Guiyang, GuiZhou Province 550004, P. R. China
E-mail address: zhaoping731108@hotmail.com, peaceguorong@yahoo.com

Mei Yang
Cadre Proppants, Houston, Texas, 77079, USA
E-mail address: mei.yang@cadreproppants.com